Rotationally Symmetric Relations of Standard Normal Distribution Using Right Triangle, Circle, and Square

First Author, Presenter, and Designer about these researches: © Shingo NAKANISHI (Osaka Inst. of Tech.).

Co-author: Masamitsu OHNISHI (Osaka Univ.)

Key Points:
The Aspect ratio which is 1.0 informs us of a lot of geometric characterizations and symmetric relations from now.

Cox also confirmed the clustering about 3 groups of normal distribution on $\lambda$.

**Karl Pearson’s finding probability point is**

$\lambda = 0.612003.$

Its cumulative probability is

$\Phi(-\lambda) = 0.2702678.$

From these values, Kelley proposed

$\phi(\lambda) = 2\lambda \Phi(-\lambda) = 0.3308.$
Integral form of cumulative distribution is $h_P(u) = \phi(u) + u\Phi(-k)$.

Modified Intercept Forms of Linear Equations:

- $-\frac{\phi(k)}{\Phi(-k)} u + \frac{1}{\phi(k)} v = 1$
- $-\frac{1}{\phi(k)} u + \frac{1}{\phi(k)} v = 1$

References
ORSJ (Chiba Institute of Tech.)
SETA2019 (Osaka Univ.)
RSS2019(ICC Belfast)

From the ancient Egyptian drawing styles, we can create the harmonies between standard normal distribution, inverse Mills ratio, and Linear Intercept Forms based on the Greek Pythagorean theorem.
Symmetric Beauty between Squares and Standard Normal Distribution at 0.612003

Each cumulative probability is about 27 percent.

Truncated cumulative probability is 100 percent.

Relations of Squares about Truncated Normal Distribution With $\lambda = 0.612003$ and Bernoulli Differential Equations.

Differential Equations about Inverse Mills Ratio are Bernoulli Differential Equations.

References.
My Doctoral Thesis, March 2015, No.17777 (Osaka Univ.) about Square
ORSJ 2016 Spring (Keio Univ.) about Square
EURO2016 (Poznan Univ. of Tech.) about Square
ORSJ 2017 Fall (Kansai Univ.) about Square
RIMS2078 (Kyoto Univ.) about Bernoulli Differential Equations
SETA2019 (Osaka Univ.) about Bernoulli Differential Equations
by © Shingo Nakanishi

Refs.
Hald, A., 1949, about the Bernoulli differential equation of inverse Mills Ratio,
Isa, K., 2011 in Japan about illustrations of inverse Mills ratio
Second order linear differential equations and these tangent lines

First derivative is the probability function: $\Phi(u)$.
Second derivative is the density probability function: $\phi(u)$.
These curves are combinations of both them.

Refs. in Japan by
Fumio Hashimoto et al., 1985 about integrals of CDF,
And Takahiro Nagashima, 2005 about ordinally differential equations.
Several important probability points of our previous works.
Ex. $x = 0.0$, $x = 0.30263084$, $x = 0.506054$, and ⋯.

![Probability points diagram]

The circle is widely spreading little by little according to the probability point from $0$ to $\infty$. Of course, $0.612003$ is also the most important probability point.

### Probability Points:

1. $k = \pm 0.0 \therefore \Phi(-k) = \Phi(k) = 1/2$
2. $k = \pm 0.30263084 \therefore \phi(k)/\Phi(-k) = 1$
3. $k = \pm 0.506054 \therefore k = \phi(k)/\Phi(k)$
4. $k = \pm 0.612003 \therefore \phi(k) = 2k\Phi(-k)$
5. $k = \pm 0.67449 \therefore \Phi(-k) = 1/4, \Phi(k) = 3/4$
What are $\frac{1}{2}$ and $\frac{1}{4}$ about standard normal distribution? Are only half and quantile points?

Probabilities of standard normal distribution are connected to the right triangle by Pythagorean theorem geometrically.

Probability point: $x = 0$

$\Phi(x) = 1 - \Phi(-x) = 0.5$

$\Phi(-x) = 0.5$

Proportion of right triangle $= 1:2:\sqrt{5}$

$\Phi(-x) : \Phi(x) = 0.5 : 0.5 = 1:1$

© Shingo NAKANISHI at Osaka Institute of Technology in Japan
Masamitsu OHNISHI at Osaka University in Japan, Feb. 15, 2019

Probability point is $0.0$ as $\frac{1}{2}$.

The idea about the folds of distance from the advice by Prof. Kohji Kamejima at OIT

Probability point is $0.67449$ as $\frac{1}{4}$.

The idea about the small difference with beauty from the advice by Prof. Hidemasa Yoshimura at OIT

Refs.
RIMS2078 (2017) as the First Presentation (Kyoto Univ.)
RIMS Modified Version 2078-10 (2018) as Second Comments at my OIT website (Osaka Inst. of Tech.)
ORSJ (National Grad. Inst. for Policy Studies) (Chiba Inst. of Tech.) (Tokyo Inst. of Tech.) (Higashi Hiroshima)
SETA2019 (Osaka Univ. & Osaka Inst. of Tech.)
Integrals of CDF

Negative Inverse Mills Ratio

Truncated Normal Distribution

Egyptian drawing styles show us the symmetric relations between right triangles, squares, circles, and standard normal distributions.

\[ \Phi(-\lambda) + \Phi(\lambda) = 1 \]
1. $k = \pm 0.0$
2. $k = \pm 0.30263084$
3. $k = \pm 0.506054$
4. $k = \pm 0.612003 (= \lambda)$
5. $k = \pm 0.67449$

**k = Probability points**

**Integrals of CDF**

**Negative Inverse Mills Ratio**

**Truncated Normal Distribution**

**Modified intercept forms of linear equations according to k**

$$-\frac{1}{\phi(k)} u + \frac{1}{\Phi(k)} v = 1$$

$$-\frac{1}{\phi(k)} u + \frac{1}{\Phi(-k)} v = 1$$

**Inverse Mills Ratio**

With two tangent lines of the green solid lines on two probability points.

**Refs.**
ORSJ (Chiba Inst. of Tech.)
(Tokyo Inst. of Tech.)
SETA2019 (Osaka Univ.)
RSS2019 (ICC Belfast, UK)
1. \( k = \pm 0.0 \)
2. \( k = \pm 0.30263084 \)
3. \( k = \pm 0.506054 \)
4. \( k = \pm 0.612003 (= \lambda) \)
5. \( k = \pm 0.67449 \)

Inverse Mills Ratio

Truncated Normal Distribution

With two groups of parallel lines between golden dashed lines and silver dashed lines based on the circle.
$k = \text{Probability points}$

1. $k = \pm 0.0$
2. $k = \pm 0.30263084$
3. $k = \pm 0.506054$
4. $k = \pm 0.612003 (= \lambda)$
5. $k = \pm 0.67449$

The proportion : $\Phi(-k): \Phi(k)$

- Negative Inverse Mills Ratio
- Truncated Normal Distribution

Modified intercept forms of linear equations according to $k$

$$-\frac{1}{\Phi(-k)} u + \frac{1}{\Phi(k)} v = 1$$

Inverse Mills Ratio

With a circle and a square. The bottom line is located on the horizontal axis at the probability point $k = 0.612003$. 
$k =$ Probability points

1. $k = \pm 0.0$
2. $k = \pm 0.30263084$
3. $k = \pm 0.506054$
4. $k = \pm 0.612003 (= \lambda)$
5. $k = \pm 0.67449$

Inverse Mills Ratio
Truncated Normal Distribution

Refs.
ORSJ (Chiba Inst. of Tech.)
(Tokyo Inst. of Tech.)
SETA2019 (Osaka Univ.)
RSS2019 (ICC Belfast, UK)
\( \phi(k) = \min \Phi(k)h_P(u) + \Phi(-k)h_N(u) \)

\( k \) = Probability points

1. \( k = \pm 0.0 \)
2. \( k = \pm 0.30263084 \)
3. \( k = \pm 0.506054 \)
4. \( k = \pm 0.612003 (= \lambda) \)
5. \( k = \pm 0.67449 \)

Ref. RSS2019(ICC Belfast)

Special case as the geometric characterizations and rotationally symmetric relations between winners, losers, and their banker based on the condition: \( \lambda = 0.612003 \) and \( \Phi(-\lambda) = 0.2702678 \).

With two groups of parallel lines between cyan, golden, and silver dashed lines based on the circle.
The idea based on squaring the circle from $\pi/4$ to $\pi/2$.

The probability points based on $k$ get the optimal values $\phi(k)$.
And these tendencies are related to the Squaring the Circle.
The important probability points $k$ and $0$ are shown the Rotationally Symmetric Relation:

$$
\Phi(-k)g_P(k) + \Phi(k)g_N(k) = 2(\Phi(k)h_P(k) + \Phi(-k)h_N(k)).
$$
The important probability point $k$ is shown the Rotationally Symmetric Relation.

$k =$ Probability points

1. $k = \pm 0.0$
2. $k = \pm 0.30263084$
3. $k = \pm 0.506054$
4. $k = \pm 0.612003 = \lambda$
5. $k = \pm 0.67449$

The right terminal point of $k$ is 0.

**Boundary conditions:**

\[ \Phi(k)h_P(x) + \Phi(-k)h_N(x) - \phi(k) = 0, \]
\[ \Phi(k)h'_P(x) + \Phi(-k)h'_N(x) = 0 \]

Modified intercept forms of linear equations according to $k$

\[ -\frac{1}{\phi(k)}u + \frac{1}{\Phi(k)}v = 1 \]

Refs.
ORSJ (Higashi Hiroshima.)
RSS2019 (ICC Belfast, UK)
The important probability points $k$ and $0$ are shown the Rotationally Symmetric Relation.

$k =$Probability points

1. $k = \pm 0.0$
2. $k = \pm 0.30263084$
3. $k = \pm 0.506054$
4. $k = \pm 0.612003 = \lambda$
5. $k = \pm 0.67449$

The right terminal point of $k$ is $\phi(0)$.

Negative
Inverse Mills Ratio
Truncated
Normal Distribution

Boundary conditions:

$\Phi(k)h_P(0) + \Phi(-k)h_N(0) - \phi(0) = 0,$
$\Phi(k)h_P(x) + \Phi(-k)h_N(x) - \phi(0) = 0.$

$\frac{1}{\phi(k)} u + \frac{1}{\phi(k)} v = 1$

Refs.
ORSJ (Higashi Hiroshima.)
RSS2019 (ICC Belfast, UK)
From the cross as squaring the circle at \( k = 0.05056989451 \) to the equilateral triangles at \( k = 0.6435087 \) as diamonds, we can imagine a cross, a flower, a four-leaf clover, and five RIMS or Olympic Track according to the probability points \( k \). And from the visual animation, we can also imagine the fire works in the sky.

From the concept based on squaring the circle and fixed length of the radius of circle, we can show you the two slopes as cumulative probabilities at the probability points.

The area of the large cyan square is \( \frac{4}{\pi} \).
The area of the small cyan square is \( \frac{2}{\pi} \).
From the cross as squaring the circle at $k = 0.05056989451$ to the equilateral triangles at $k = 0.6435087$ as diamonds, we can imagine a cross, a flower, a four-leaf clover, and five RIMS or Olympic Track according to the probability points $k$. And from the visual animation, we can also imagine the fireworks in the sky.

From the concept based on squaring the circle and fixed length of the radius of circle, we can show you the two slopes as cumulative probabilities at the probability points.

The area of the large cyan square is $\frac{4}{\pi}$. The area of the small cyan square is $\frac{2}{\pi}$. 

$$\frac{1}{\sqrt{2\pi}} \quad \frac{1}{\sqrt{\pi}}$$
From the cross as squaring the circle at $k = 0.05056989451$ to the equilateral triangles at $k = 0.6435087$ as diamonds, we can imagine a cross, a flower, a four-leaf clover, and five RIMS or Olympic Track according to the probability points $k$. And from the visual animation, we can also imagine the fire works in the sky.

\[
\Phi(k)\sqrt{1 + \Phi(k)^2} = 1, \quad \therefore k = 0.7931383 \text{ and } 1 + \Phi(k)^2 = \text{olden Ratio} = 1.6180339, \Phi(k)^{-2} = 1.6180339
\]
Concluding Remarks about my Researches

1. Several rotationally weighted balances of integrals of Standard Normal Distribution are much more important than we thought.

2. So their inverse Mills ratios are.

3. Many geometrically interesting probability points can be found and firstly illustrated.

4. There might be some of the most emphasized facts between historically, geometrically, and mathematically attractive truth and beauty on earth in the future. …???

5. Finally, Right Triangles, Squares, and Circles are certainly related to the Standard Normal Distribution.

Original Ref: Vitruvian Man
https://en.wikipedia.org/wiki/Vitruvian_Man

CIE2007 in Egypt
http://www.oit.ac.jp/center/~nakanishi/english/
Acknowledgments

This work was supported by the Research Institute for Mathematical Sciences (RIMS), an International Joint Usage/Research Center located in Kyoto University. The first author and presenter, Shingo NAKANISHI, would like to show my grateful to the following many people and societies in Japanese. And, especially,


Oral presentation

Title: From Mistakes to Modifications

ORJS2017 (Kansai Univ.) & My Presentation of Prof. Ohnishi’s Lab. Seminar (Osaka Univ., July 6th, 2017)

Title: From Mistakes to Modifications