電気磁気学Ｉ　第７回

電荷と電界（２）　追加資料

電子情報システム工学科
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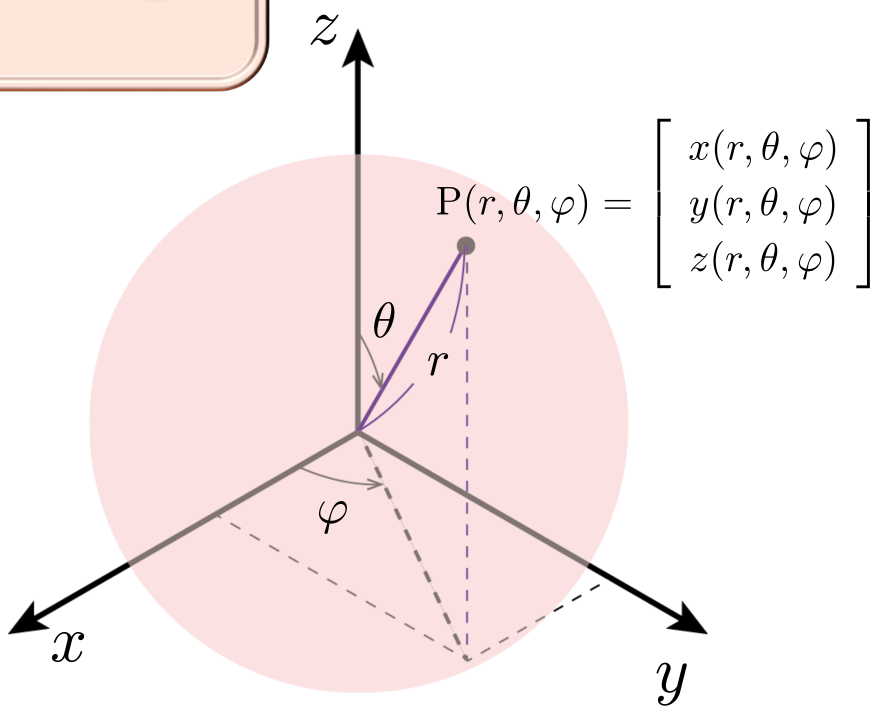
教科書P. 46 問1

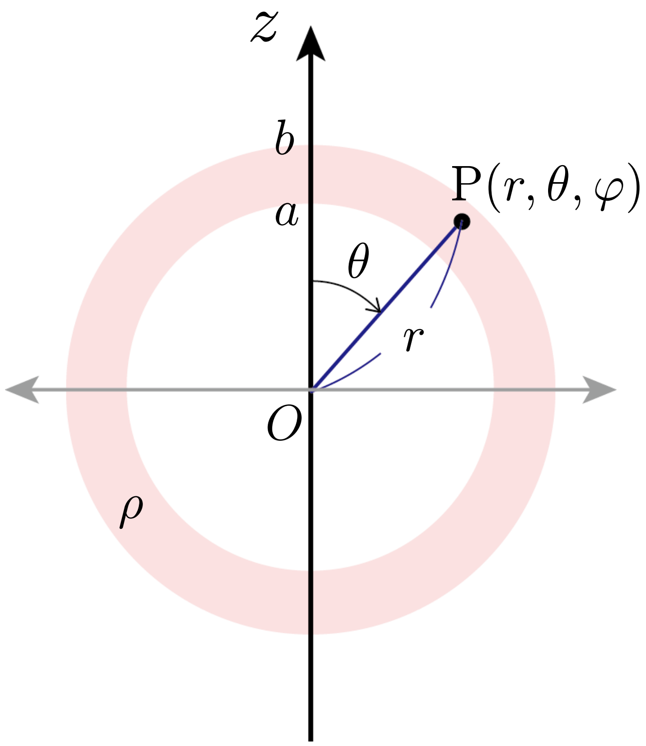
図のような半径 r が $a < r < b$ の球殻内に一様の電荷密度 ρ が分布しており，$r < a$, $r > b$ では電荷はない．このときの電荷の強さを求めよ．

球殻の対称性に注意すること．

球座標系表示

$$\mathrm{P}(r,\theta,\varphi) = \begin{bmatrix} x(r,\theta,\varphi) \\ y(r,\theta,\varphi) \\ z(r,\theta,\varphi) \end{bmatrix} = \begin{bmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{bmatrix}$$



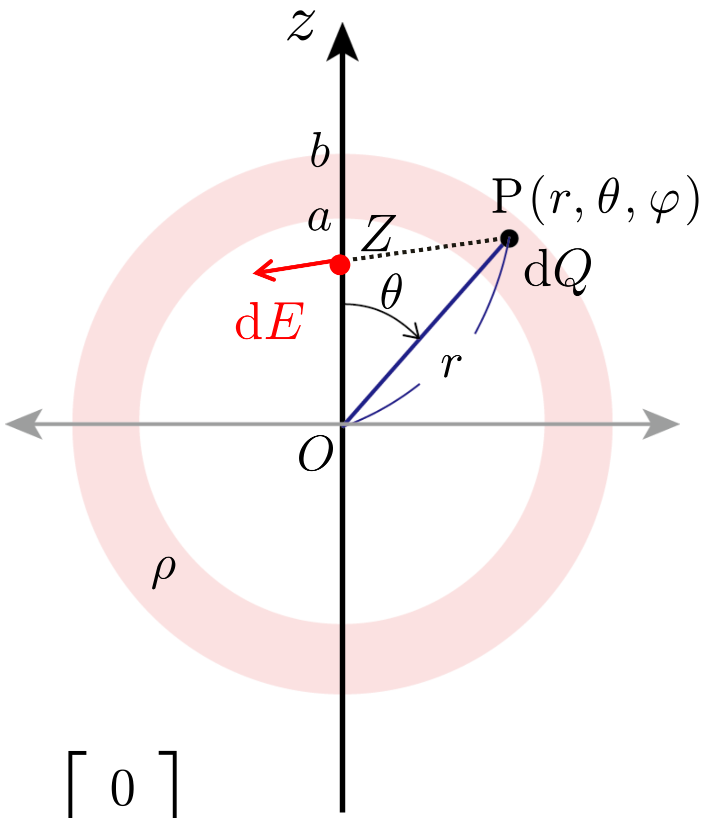


別解

球殻内の任意の点 P 近傍の微小領域における電荷量 $\mathrm{d}Q$ は

$$\mathrm{d}Q = \rho r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\varphi$$

と表せる．球殻の対称性に注意して，正の z 軸上の点の電界および電位について考えれば十分である．z 軸上の任意の点 $Z = \begin{bmatrix} 0 & 0 & z \end{bmatrix}^T$ における $\mathrm{d}Q$ による電界 $\mathrm{d}E$ および電位 $\mathrm{d}V$ は



$$\mathrm{d}E = \frac{\mathrm{d}Q}{4\pi\varepsilon_0 \left\{ r^2 \sin^2\theta + (z - r\cos\theta)^2 \right\}^{3/2}} \begin{bmatrix} -r\sin\theta\cos\varphi \\ -r\sin\theta\sin\varphi \\ z - r\cos\theta \end{bmatrix}$$

$$\mathrm{d}V = -\frac{\mathrm{d}Q}{4\pi\varepsilon_0} \int_\infty^z \frac{1}{\left\{ r^2 \sin^2\theta + (\zeta - r\cos\theta)^2 \right\}^{3/2}} \begin{bmatrix} -r\sin\theta\cos\varphi \\ -r\sin\theta\sin\varphi \\ \zeta - r\cos\theta \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{d}\zeta$$

別解（つづき）

$$
\begin{aligned}
\mathrm{d}V &= -\frac{\mathrm{d}Q}{4\pi\varepsilon_0}\int_\infty^z \frac{1}{\left\{r^2\sin^2\theta+(\zeta-r\cos\theta)^2\right\}^{3/2}}\begin{bmatrix}-r\sin\theta\cos\varphi\\-r\sin\theta\sin\varphi\\\zeta-r\cos\theta\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}\mathrm{d}\zeta\\
&= -\frac{\mathrm{d}Q}{4\pi\varepsilon_0}\int_\infty^z \frac{\zeta-r\cos\theta}{\left\{r^2\sin^2\theta+(\zeta-r\cos\theta)^2\right\}^{3/2}}\mathrm{d}\zeta\\
&= -\frac{\mathrm{d}Q}{4\pi\varepsilon_0}\int_\infty^z \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{-1}{\left\{r^2\sin^2\theta+(\zeta-r\cos\theta)^2\right\}^{1/2}}\right)\mathrm{d}\zeta\\
&= \frac{\mathrm{d}Q}{4\pi\varepsilon_0}\left[\frac{1}{\left\{r^2\sin^2\theta+(\zeta-r\cos\theta)^2\right\}^{1/2}}\right]_\infty^z\\
&= \frac{\rho r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi}{4\pi\varepsilon_0\left\{r^2\sin^2\theta+(z-r\cos\theta)^2\right\}^{1/2}}
\end{aligned}
$$

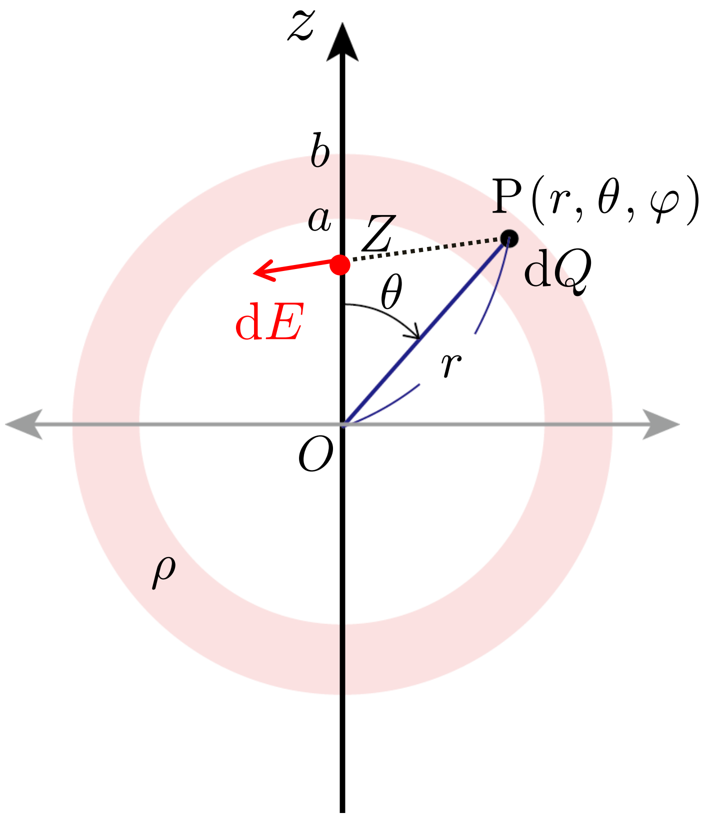
別解（つづき）

微小電荷による電位 $\mathrm{d}V$ を球殻 S で体積分する．ここで

$$S = \{P(r,\theta,\varphi) \mid a \le r \le b,\ 0 \le \theta \le \pi,\ 0 \le \varphi < 2\pi\}$$

つまり，

$$\begin{aligned} V(z) &= \int_S \mathrm{d}V \\ &= \frac{\rho}{4\pi\varepsilon_0}\int_a^b\int_0^\pi\int_0^{2\pi}\frac{r^2\sin\theta}{\left\{r^2\sin^2\theta+(z-r\cos\theta)^2\right\}^{1/2}}\mathrm{d}\varphi\mathrm{d}\theta\mathrm{d}r \\ &= \frac{\rho}{2\varepsilon_0}\int_a^b\int_0^\pi\frac{r^2\sin\theta}{\left\{r^2\sin^2\theta+(z-r\cos\theta)^2\right\}^{1/2}}\mathrm{d}\theta\mathrm{d}r \\ &= \frac{\rho}{2\varepsilon_0 z}\int_a^b\int_0^\pi\frac{\mathrm{d}}{\mathrm{d}\theta}\left(r\left(r^2+z^2-2zr\cos\theta\right)^{\frac{1}{2}}\right)\mathrm{d}\theta\mathrm{d}r \end{aligned}$$



別解（つづき）

$$V(z) = \frac{\rho}{2\varepsilon_0 z}\int_a^b \int_0^\pi \frac{\mathrm{d}}{\mathrm{d}\theta}\left(r\left(r^2+z^2-2zr\cos\theta\right)^{\frac{1}{2}}\right)\mathrm{d}\theta\mathrm{d}r$$

$$= \frac{\rho}{2\varepsilon_0 z}\int_a^b \left[r\left(r^2+z^2-2zr\cos\theta\right)^{\frac{1}{2}}\right]_0^\pi \mathrm{d}r$$

$$= \frac{\rho}{2\varepsilon_0 z}\int_a^b r\left(|r+z|-|r-z|\right)\mathrm{d}r$$

(1) 球殻の内側 $0 \leq z < a$

$$V(z) = \frac{\rho}{2\varepsilon_0 z}\int_a^b r\left((r+z)-(r-z)\right)\mathrm{d}r$$

$$= \frac{\rho}{2\varepsilon_0 z}\int_a^b 2zr\mathrm{d}r = \frac{\rho}{2\varepsilon_0 z}\left[zr^2\right]_a^b = \frac{\rho(b^2-a^2)}{2\varepsilon_0}$$

電位$V(z)$はzに依らず一定値，
つまり球殻内側は等電位である．
このことから，電界については，

$$E = -\mathrm{grad}V = 0$$

別解（つづき）

$$V(z) = \frac{\rho}{2\varepsilon_0 z}\int_a^b r\left(|r+z| - |r-z|\right)\mathrm{d}r$$

(2) 球殻中　　$a \leq z < b$

$$\begin{aligned}
V(z) &= \frac{\rho}{2\varepsilon_0 z}\int_a^b r\left(|r+z| - |r-z|\right)\mathrm{d}r \\
&= \frac{\rho}{2\varepsilon_0 z}\int_a^z r\left((r+z) - (z-r)\right)\mathrm{d}r + \frac{\rho}{2\varepsilon_0 z}\int_z^b r\left((r+z) - (r-z)\right)\mathrm{d}r \\
&= \frac{\rho}{2\varepsilon_0 z}\int_a^z 2r^2\mathrm{d}r + \frac{\rho}{2\varepsilon_0 z}\int_z^b 2rz\mathrm{d}r = \frac{\rho}{2\varepsilon_0 z}\left[\frac{2r^3}{3}\right]_a^z + \frac{\rho}{2\varepsilon_0 z}\left[r^2 z\right]_z^b \\
&= \frac{\rho}{2\varepsilon_0 z}\left(\frac{2z^3}{3} - \frac{2a^3}{3}\right) + \frac{\rho}{2\varepsilon_0 z}\left(b^2 z - z^3\right) = \frac{\rho z^2}{3\varepsilon_0} - \frac{\rho a^3}{3\varepsilon_0 z} + \frac{\rho b^2}{2\varepsilon_0} - \frac{\rho z^2}{2\varepsilon_0} \\
&= \frac{\rho b^2}{2\varepsilon_0} - \frac{\rho a^3}{3\varepsilon_0 z} - \frac{\rho z^2}{6\varepsilon_0}
\end{aligned}$$

$$E = -\mathrm{grad}V = -\frac{\partial V}{\partial x}\boldsymbol{i} - \frac{\partial V}{\partial y}\boldsymbol{j} - \frac{\partial V}{\partial z}\boldsymbol{k} = \left(\frac{\rho z}{3\varepsilon_0} - \frac{\rho a^3}{3\varepsilon_0 z^2}\right)\boldsymbol{k}$$

別解（つづき）

$$V(z) = \frac{\rho}{2\varepsilon_0 z}\int_a^b r\left(|r+z| - |r-z|\right)\mathrm{d}r$$

(3) 球殻の外側　$b \leq z$

$$V(z) = \frac{\rho}{2\varepsilon_0 z}\int_a^b r\left(|r+z| - |r-z|\right)\mathrm{d}r = \frac{\rho}{2\varepsilon_0 z}\int_a^b r\left((r+z) - (z-r)\right)\mathrm{d}r$$

$$= \frac{\rho}{2\varepsilon_0 z}\int_a^b 2r^2\mathrm{d}r = \frac{\rho}{2\varepsilon_0 z}\left[\frac{2r^3}{3}\right]_a^b = \frac{\rho}{3\varepsilon_0 z}\left(b^3 - a^3\right)$$

$$E = -\mathrm{grad}V = -\frac{\partial V}{\partial x}\boldsymbol{i} - \frac{\partial V}{\partial y}\boldsymbol{j} - \frac{\partial V}{\partial z}\boldsymbol{k} = \frac{\rho}{3\varepsilon_0 z^2}(b^3 - a^3)\boldsymbol{k}$$