Controlling Constraint Violation using Adjusted ADM Systems

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Formulation of the Einstein equations is one of the necessary implements for realizing long-term stable and accurate numerical simulations. We demonstrate our re-formulation scheme, adjusted ADM systems, that intends to construct a dynamical system which evolves toward the constraint surface as the attractor, by adjusting ADM evolution equations with constraints. Our 3-dimensional numerical evolution of Teukolsky wave data under periodic boundary condition shows the life-time of simulation can be four-times longer than that of the standard ADM system.

In 2005-2006, several groups independently announced the success of the inspiral black-hole binary merger.[1)–3)] There are many implements for their successes, such as gauge conditions, coordinate selections, boundary treatments, singularity treatments, numerical discretization, and mesh refinements, together with the re-formulation of the Einstein equations which we discuss here.

There are many approaches to re-formulate the Einstein equations for obtaining a long-term stable and accurate numerical evolution (e.g. see references in[4)]). In a series of our works,[5)–8)] we have proposed to construct a system that has its constraint surface as the attractor. By applying eigenvalue analysis of constraint propagation equations, we showed that there *is* a constraint-violating mode in the standard Arnowitt-Deser-Misner (ADM) evolution system,which has been used for simulations over 20 years, when it is applied to a single non-rotating black-hole space-time.[7)]

Our basic idea can be described in general form as follows. Suppose we have a dynamical system of variables $u^a(x^i, t)$, which has evolution equations,

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots), \qquad (0\cdot1)$$

and the (first class) constraints,

$$C^\alpha(u^a, \partial_i u^a, \cdots) \approx 0. \qquad (0\cdot2)$$

We propose to investigate the evolution equation of C^α (constraint propagation)

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots), \qquad (0\cdot3)$$

for evaluating violation features of constraints.

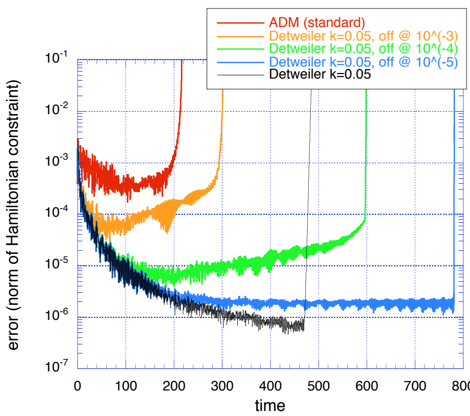

Fig. 1. Comparisons of numerical evolutions of adjusted ADM systems, using Teukolsky wave propagation.

The character of constraint propagation, (0·3), will change when we modify the original evolution equations. Suppose we modify (adjust) (0·1) using constraints

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots) + F(C^\alpha, \partial_i C^\alpha, \cdots), \quad (0\cdot4)$$

with a function $F(C^\alpha, \cdots)$, then (0·3) will also be modified as

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots) + G(C^\alpha, \partial_i C^\alpha, \cdots), \quad (0\cdot5)$$

which implies the adjustment $F(C^\alpha, \cdots)$ changes the nature of (0·3).

We have proceeded an eigenvalue analysis of the whole RHS in (0·3) and (0·5) after a suitable homogenization,

$$\partial_t \hat{C}^\alpha = \hat{g}(\hat{C}^\alpha) = M^\alpha{}_\beta \hat{C}^\beta, \text{ where } \quad C(x,t)^\alpha = \int \hat{C}(k,t)^\alpha \exp(ik \cdot x) d^3k, \quad (0\cdot6)$$

and conjectured that the system is more stable, if the eigenvalues of $M^\alpha{}_\beta$ has a *negative real-part* or *non-zero imaginary-part*.[5)–8)]

For the ADM system,

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \quad (0\cdot7)$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + 2(\nabla_{(i} \beta^k) K_{j)k} + \beta^k \nabla_k K_{ij} \quad (0\cdot8)$$

we[7)] investigated effective adjustments systematically using the Hamiltonian constraint, $\mathcal{H} := R^{(3)} + K^2 - K_{ij}K^{ij}$, and the momentum constraint, $\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K$. Among then a combination proposed by Detweiler[9)] passes above criteria.

In Fig.1, we demonstrate numerical evolutions of ADM-Detweiler system;

$$\begin{aligned} \partial_t \gamma_{ij} &= \text{eq.}(0\cdot7) - \kappa_1 \alpha^3 \gamma_{ij} \mathcal{H} \\ \partial_t K_{ij} &= \text{eq.}(0\cdot8) + \kappa_1 \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_2 \alpha \gamma_{ij} \gamma^{kl} \partial_k \mathcal{M}_l \\ &\quad + \kappa_1 \alpha^2 [3(\partial_{(i} \alpha) \delta^k_{j)} - (\partial_l \alpha) \gamma_{ij} \gamma^{kl}] \mathcal{M}_k + \kappa_1 \alpha^3 [\delta^k_{(i} \delta^l_{j)} - (1/3) \gamma_{ij} \gamma^{ki}] (\nabla_k \mathcal{M}_l). \end{aligned}$$

We plot the violation of $\mathcal{H}$ versus time for Teukolsky wave evolution with harmonic slicing, and with periodic boundary condition in our 3-dimensional code. We see that the evolution by the ADM-Detweiler system with tuned κ_1 can realize more than four-times longer stable evolution than that of the standard ADM system. (We simply cut off the adjusted term at a certain absolute value of $\mathcal{H}$.)

The similar results are obtained also in the adjusted BSSN formulation,[8)] which is reported by Kiuchi and Shinkai.[10)]

References

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