$N + 1$ formalism in Einstein-Gauss-Bonnet gravity

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Abstract

With the aim of numerical investigations of spacetime dynamics in higher curvature models, we present the basic equations of the Einstein-Gauss-Bonnet gravity theory. We show $(N + 1)$-dimensional version of the ADM decomposition including Gauss-Bonnet terms, and also show conformally-transformed constraint equations for obtaining an initial data.

1 Introduction

One of the most surprising achievements in studies of general relativity (GR) is the singularity theorems established by Hawking and Penrose in 1960s. It states that the spacetime singularities inevitably occur (or occurred) under natural conditions within the framework of GR. This fact implies that GR cannot describe whole of the spacetime structure, and GR itself is incomplete as a physics theory since an appearance of singularity makes the future unpredictable.

One of the remedy of this paradox is the cosmic censorship conjecture proposed by Penrose. The conjecture states that any singularity is hidden inside an event horizon in the process of gravitational collapse, and is causally disconnected from our side of spacetime. However, it is also true that this censorship does not essentially solve the break-down of GR at the singularity, and also that the initial singularity at the birth of the universe, which is the consequence of the standard Big-Bang scenario, can not be resolved. Therefore, we expect that the true fundamental theory will resolve this problematic singularity treatment.

Up to now, several quantum theories of gravity have been proposed. Among them superstring/M-theory, formulated in higher dimensional spacetime, is the most promising candidate. We are still far from understanding the non-perturbative aspects of the theory, but perturbative treatments of string effects to classical gravity theory begin revealing new features of the spacetime.

One of the typical string effects can be seen in a series of studies of cosmological models, which is called string cosmology [1] or pre-Big-Bang scenario [2]. Although these analysis show that the singularity problem has not been resolved yet, there are some cosmological solutions which do not start from an initial singularity.

Another attractive proposal is the brane-world model of the Universe [3]; a picture that we live on a four-dimensional timelike hypersurface embedded in higher-dimensional bulk spacetime. Since the fundamental scale of the brane-world model could be around TeV scale, the model is thought to be tested using the large hadron collider (LHC) by monitoring the creations and evaporations of tiny black holes[4].

Along to such a theoretical developments, we are planning to promote a direct numerical approach to investigate non-linear dynamics in higher-dimensional and/or higher curvature gravitational models both for singularity structure and cosmological models. This article is the first step; we rewrite the fundamental equations into a suitable form for future numerical treatments.

The standard numerical approach is to treat the spacetime as a Cauchy problem. We therefore apply the ADM formalism of GR for the $(N + 1)$-dimensional Einstein-Gauss-Bonnet gravity theory. The Gauss-Bonnet terms are the next leading order of the α'-expansion of type IIB superstring theory, where α' is the inverse string tension [5], so that the first model to be investigated. In §2.1, we show that the set of equations are divided into two constraints and evolution equations along to the standard procedure.

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In §2.2, we present the conformal approach to solve the constraints which shall be used for preparing an initial data. All the details will be reported elsewhere[6].

2 Equations in Einstein-Gauss-Bonnet gravity

2.1 Equations to solve

We consider $(N+1)$-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$ which is described by the Einstein-Gauss-Bonnet action: *

$$S = \int_{\mathcal{M}} d^{N+1}X\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(\mathcal{R} - 2\Lambda + \alpha_{GB}\mathcal{L}_{GB}\right) + \mathcal{L}_{\rm matter}\right], \tag{1}$$

$$\text{with } \mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}, \tag{2}$$

where κ^2 is the $(N+1)$-dimensional gravitational constant, $\mathcal{R}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}_{\mu\nu\rho\sigma}$ and $\mathcal{L}_{\rm matter}$ are the $(N+1)$-dimensional scalar curvature, Ricci tensor, Riemann curvature and the matter Lagrangian, respectively.

The action (1) gives the gravitational equation as

$$\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu} = \kappa^2\,\mathcal{T}_{\mu\nu}\,, \tag{3}$$

$$\text{where } \mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu}, \tag{4}$$

$$\mathcal{H}_{\mu\nu} = 2\left[\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}_{\ \nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}^{\ \alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}\right] - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB}, \tag{5}$$

$$\text{and } \mathcal{T}_{\mu\nu} = -2\frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\rm matter}. \tag{6}$$

We define the projection operator to N-dimensional (spacelike or timelike) hypersurface, Σ, $\perp_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu$, where n_μ is the unit-normal vector to Σ with $n_\mu n^\mu = \varepsilon$, with which we define n_μ is timelike (if $\varepsilon = -1$) or spacelike (if $\varepsilon = 1$). Σ is spacelike (timelike) if n_μ is timelike (spacelike). We define the induced N-dimensional metric γ_{ij} as $\gamma_{ij} = \perp_{ij}$, and the extrinsic curvature K_{ij} as $K_{ij} = -\perp^{\alpha}_{\ i}\perp^{\beta}_{\ j}\nabla_\alpha n_\beta$.

The projections of the gravitational equation can be the following three:

$$\big(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu}\big)\, n^\mu\, n^\nu = \kappa^2\, T_{\mu\nu}\, n^\mu\, n^\nu =: \kappa^2\rho_H, \tag{7}$$

$$\big(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu}\big)\, n^\mu \perp^{\nu}_{\ \rho} = \kappa^2\, T_{\mu\nu}\, n^\mu \perp^{\nu}_{\ \rho} =: -\kappa^2 J_\rho, \tag{8}$$

$$\big(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu}\big) \perp^{\mu}_{\ \rho} \perp^{\nu}_{\ \sigma} = \kappa^2\, T_{\mu\nu} \perp^{\mu}_{\ \rho} \perp^{\nu}_{\ \sigma} =: \kappa^2 S_{\rho\sigma}, \tag{9}$$

where we defined $T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}$, which gives $T = -\rho_H + S^\ell_{\ \ell}$.

Following the standard procedure of the ADM formulation, we find the equations, eq. (7)-(9), correspond to (a) the Hamiltonian constraint equation:

$$M + \alpha_{GB}\big(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}\big) = -2\varepsilon\kappa^2\mathcal{T}_{\mu\nu}n^\mu n^\nu\,, \tag{10}$$

(b) the momentum constraint equation:

$$N_i + 2\alpha_{GB}\left(MN_i - 2M_i^{\ a}N_a + 2M^{ab}N_{iab} - M_i^{\ cab}N_{abc}\right) = -\kappa^2\mathcal{T}_{\mu\nu}n^\mu\gamma^\nu_{\ i}\,, \tag{11}$$

and (c) the evolution equations for γ_{ij}:

$$M_{ij} - \frac{1}{2}M\gamma_{ij} - \varepsilon\big(-K_{ia}K^a_{\ j} + \gamma_{ij}K_{ab}K^{ab} - \pounds_n K_{ij} + \gamma_{ij}\gamma^{ab}\pounds_n K_{ab}\big) + 2\alpha_{GB}\Big[H_{ij} + \varepsilon\big(M\pounds_n K_{ij} - 2M_i^{\ a}\pounds_n K_{aj} - 2M_j^{\ a}\pounds_n K_{ai} - W_{ij}^{\ \ ab}\pounds_n K_{ab}\big)\Big] = \kappa^2\mathcal{T}_{\mu\nu}\gamma^\mu_{\ i}\gamma^\nu_{\ j}\,, \tag{12}$$

*The Greek indices move $1, \cdots, N+1$, while the Latin indices move $1, \cdots, N$.

respectively, where

$$\begin{aligned}
M_{ijkl} &= R_{ijkl} - \varepsilon(K_{ik}K_{jl} - K_{il}K_{jk}), && (13)\\
N_{ijk} &= D_iK_{jk} - D_jK_{ik}, && (14)\\
H_{ij} &= MM_{ij} - 2(M_{ia}M^a_{\ j} + M^{ab}M_{iajb}) + M_{iabc}M_j^{\ abc}\\
&\quad -2\varepsilon\Big[-K_{ab}K^{ab}M_{ij} - \frac{1}{2}MK_{ia}K^a_{\ j} + K_{ia}K^a_{\ b}M^b_{\ j} + K_{ja}K^a_{\ b}M^b_{\ i} + K^{ac}K_c^{\ b}M_{iajb}\\
&\qquad + N_iN_j - N^a(N_{aij} + N_{aji}) - \frac{1}{2}N_{abi}N^{ab}_{\ \ j} - N_{iab}N_j^{\ ab}\Big]\\
&\quad -\frac{1}{4}\gamma_{ij}\left[M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}\right]\\
&\quad -\varepsilon\gamma_{ij}\left[K_{ab}K^{ab}M - 2M_{ab}K^{ac}K_c^{\ b} - 2N_aN^a + N_{abc}N^{abc}\right], && (15)\\
W_{ij}^{\ \ kl} &= M\gamma_{ij}\gamma^{kl} - 2M_{ij}\gamma^{kl} - 2\gamma_{ij}M^{kl} + 2M_{iajb}\gamma^{ak}\gamma^{bl}\,, && (16)
\end{aligned}$$

and these contracted variables; $M_{ij} = \gamma^{ab}M_{iajb}$, $M = \gamma^{ab}M_{ab}$, and $N_i = \gamma^{ab}N_{aib}$. Note that the terms of $\pounds_nK_{\mu\nu}$ appear only in the linear form. This is due to the quali-linear property of the Gauss-Bonnet gravity.

2.2 Conformal Approach to solve the Constraints

In order to prepare an initial data for numerical evolution, we have to solve two constraints, (10) and (11). The standard approach [7] is to apply conformal transformation between the initial trial metric $\hat{\gamma}_{ij}$ and the solution γ_{ij}, as

$$\gamma_{ij} = \psi^{2m}\hat{\gamma}_{ij}, \tag{17}$$

and solve for ψ. (We generalized the power to $2m$ here.) For N-dimensional spacetime, Ricci scalar is transformed as

$$\tilde{R} = \psi^{-2m}R - 2(N-1)(\Delta\psi)\psi^{-2m-1} - (N-1)\big[2-(N-2)m\big]m(\nabla\psi)^2\psi^{-2m-2},$$

from which we specify $m = 2/(N-2)$ for simplifying the equation.

Regarding to the extrinsic curvature, we decompose K_{ij} into its trace part, $K = \gamma^{ij}K_{ij}$, and traceless part, $A_{ij} = K_{ij} - \frac{1}{N}\gamma_{ij}K$, and assume the conformal transformation as

$$A_{ij} = \psi^{\ell}\hat{A}_{ij}. \tag{18}$$

The conformal transformation of the divergence D_jA^{ij} becomes

$$D_jA^{ij} = \psi^{-4m+\ell}\hat{D}_j\hat{A}^{ij} + \psi^{-4m+\ell-1}\big[\ell + m(N-2)\big]\hat{A}^{ij}\hat{D}_j\psi, \tag{19}$$

which indicates to set $\ell = -m(N-2) = -2$ for simplifying the equation.

We introduce the longitudinal part of $\hat{A}^{ij}$, $\hat{A}^{ij}_L = \hat{A}^{ij} - \hat{A}^{ij}_{TT}$, where $\hat{D}_j\hat{A}^{ij}_{TT} = 0$, and express $\hat{A}^{ij}_L$ with a vector potential $\hat{A}^{ij}_L = \hat{D}^iW^j + \hat{D}^jW^i - \frac{2}{N}\hat{\gamma}^{ij}\hat{D}_kW^k$.

We also assume the conformal transformation of matter terms as $\rho = \psi^{-n}\hat{\rho}$ and $J^i = \psi^{-4m+\ell}\hat{J}^i$, and assume $K = \hat{K}$, then two constraints, (10) and (11), can be written as

$$\begin{aligned}
4\frac{N-1}{N-2}\hat{\Delta}\psi &= \hat{R}\psi - (\hat{A}_{ij}\hat{A}^{ij})\psi^{(-3N+2)/(N-2)} + \left[\frac{N-1}{N}K^2 - 2\Lambda - 16\pi G\hat{\rho}\psi^{-n}\right]\psi^{(N+2)/(N-2)}\\
&\quad + \alpha_{GB}\left(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}\right)\psi^{(N+2)/(N-2)} && (20)
\end{aligned}$$

and

$$\begin{aligned}
\hat{\Delta}W^i + \frac{N-2}{N}\hat{D}^i\hat{D}_kW^k + \hat{R}^i_{\ k}W^k &= \frac{N-1}{N}\psi^{2N/(N-2)}\hat{D}^i\hat{K} + 8\pi G\hat{J}^i\\
&\quad -2\alpha_{GB}\left(MN^i - 2M^{ia}N_a + 2M^{ab}N^i_{\ ab} - M^{iabc}N_{bca}\right). && (21)
\end{aligned}$$

Note that we do not transformed Gauss-Bonnet terms in these expression, since they produce higher-power terms in ψ. Therefore we have to proceed iterative schemes for solving both (20) and (21) updating the trial metric as $\hat{\gamma}_{ij}|_{\rm new} = \psi_{\rm old}^{4N/(N-2)}\hat{\gamma}_{ij}|_{\rm old}$. Although there is no proof to guarantee the existence of a solution in such a system, our numerical code obtains converged solutions. We will report details elsewhere.

2.3 Evolution equations

The Einstein evolution equation in general N-dimensional ADM version is presented in [8]. With the Gauss-Bonnet terms, the evolution equation, (12), cannot be expressed explicitly for each $\pounds_n K_{ij}$. That is, eq. (12) is rewritten as

$$\begin{aligned}&(1+2\alpha_{GB}M)\pounds_n K_{ij} - (h_{ij}h^{ab} + 2\alpha_{GB}{W_{ij}}^{ab})\pounds_n K_{ab} - 8\alpha_{GB}{M_{(i}}^{a}\pounds_n K_{|a|j)} \\ &\quad = -\varepsilon\left(M_{ij} - \frac{1}{2}Mh_{ij}\right) - K_{ia}K^{a}{}_{j} + h_{ij}K_{ab}K^{ab} + \varepsilon\kappa^2\mathcal{T}_{\mu\nu}h^{\mu}{}_{i}h^{\nu}{}_{j} - 2\varepsilon\alpha_{GB}H_{ij}\,, \end{aligned} \qquad (22)$$

and the second and third terms in RHS include the mixing terms between $\pounds_n K_{ij}$. Therefore, in an actual simulation, we have to evolve γ_{ij} and K_{ij} in each step simultaneously using a matrix form of (22) like

$$\begin{pmatrix} \pounds_n\gamma_{11} \\ \pounds_n\gamma_{12} \\ \pounds_n\gamma_{13} \\ \vdots \\ \pounds_n K_{11} \\ \pounds_n K_{12} \\ \pounds_n K_{13} \\ \vdots \end{pmatrix} = \begin{pmatrix} O & O \\ & \\ O & \text{Mixing} \end{pmatrix} \begin{pmatrix} \pounds_n\gamma_{11} \\ \pounds_n\gamma_{12} \\ \pounds_n\gamma_{13} \\ \vdots \\ \pounds_n K_{11} \\ \pounds_n K_{12} \\ \pounds_n K_{13} \\ \vdots \end{pmatrix} + \begin{pmatrix} K_{11} \\ K_{12} \\ K_{13} \\ \vdots \\ \\ \text{Source} \\ \\ \end{pmatrix}.$$

We are now developing our numerical code and hope to present some results elsewhere near future.

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