Towards the dynamics in Einstein-Gauss-Bonnet gravity: Initial Value Problem[1]

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Abstract

Towards the investigation of the full dynamics in higher-dimensional and/or stringy gravitational model, we present the basic equations of the Einstein-Gauss-Bonnet gravity theory. We show $(N+1)$-dimensional version of the Arnowitt-Deser-Misner decomposition including the Gauss-Bonnet term, which shall be the standard approach to treat the space-time as a Cauchy problem. Due to the quasi-linear property of the Gauss-Bonnet gravity, we find that the evolution equations can be in a treatable form in numerics. We also show the conformally-transformed constraint equations for constructing an initial data. Our equations can be used both for timelike and spacelike foliations.

1 Introduction

General relativity (GR) has been tested with many experiments and observations both in the strong and weak gravitational field regimes, and none of them are contradictory to GR. However, the theory also predicts the appearance of the spacetime singularities under natural conditions, which also indicates that GR is still incomplete as a physics theory that describes whole of the gravity and the spacetime structure.

We expect that the true fundamental theory will resolve these theoretical problems. Up to now, several quantum theories of gravity have been proposed. Among them superstring/M-theory, formulated in higher dimensional spacetime, is the most promising candidate. The Gauss-Bonnet (GB) term is the next leading order of the α'-expansion (α' is the inverse string tension) of type IIB superstring theory[2, 3], and has nice properties such that it is ghost-free combinations[4] and does not give higher derivative equations but an ordinary set of equations with up to second derivative in spite of the higher curvature combinations.

The models with the GB term and/or other higher curvature terms have been intensively studied in the high energy physics, in the contexts both in string cosmology and in black hole physics (see references in [1]). All the analysis so far are performed on the assumption of highly symmetric spacetime because the system is much more complicated than that in GR. To obtain deeper understanding of the early stage of the universe, singularity, and/or black holes, we should consider less symmetric and/or dynamical spacetime; the analyses require the direct numerical integration of the equations. None of the fully dynamical simulations in GB gravity has been performed.

In this article, we present the basic equations of the Einstein-GB gravity theory. We show $(N+1)$-dimensional version of the ADM decomposition, which is the standard approach to treat the spacetime as a Cauchy problem. The topic was first discussed by Choquet-Bruhat [5], but the full set of equations and the methodology have not yet been presented. Therefore, as the first step, we in this paper just present the fundamental space-time decomposition of the GB equations, focusing on the GB term.

The ADM decomposition is supposed to construct the spacetime with foliations of the constant-time hypersurfaces. This method can be also applied to study the brane-world model. We think these expressions are useful for future dynamical investigations.

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2 $(N+1)$-decomposition in Einstein-Gauss-Bonnet gravity

We start from the Einstein-Gauss-Bonnet action in $(N+1)$-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$ which is described as [4]:

$$S = \int_{\mathcal{M}} d^{N+1}X\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(\mathcal{R} - 2\Lambda + \alpha_{GB}\mathcal{L}_{GB}\right) + \mathcal{L}_{\text{matter}}\right], \tag{1}$$

with $\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$,where κ^2 is the $(N+1)$-dimensional gravitational constant, $\mathcal{R}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}_{\mu\nu\rho\sigma}$ and $\mathcal{L}_{\text{matter}}$ are the $(N+1)$-dimensional scalar curvature, Ricci tensor, Riemann curvature and the matter Lagrangian, respectively. This action reproduces the standard $(N+1)$-dimensional Einstein gravity, if we set the coupling constant α_{GB} (≥ 0) equals to zero.

The action (1) gives the gravitational equation as

$$\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu} = \kappa^2\,\mathcal{T}_{\mu\nu}\,, \tag{2}$$

$$\text{where}\;\; \mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + g_{\mu\nu}\Lambda, \quad \mathcal{T}_{\mu\nu} = -2\frac{\delta\mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\text{matter}}, \tag{3}$$

$$\mathcal{H}_{\mu\nu} = 2\Big(\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}_{\ \nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}^{\ \alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}\Big) - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB}. \tag{4}$$

In order to investigate the space-time structure as the foliations of the N-dimensional (spacelike or timelike) hypersurface Σ, we introduce the projection operator to Σ as

$$\perp_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu, \tag{5}$$

where n_μ is the unit-normal vector to Σ with $n_\mu n^\mu = \varepsilon$, with which we define n_μ is timelike (if $\varepsilon = -1$) or spacelike (if $\varepsilon = 1$). Therefore, Σ is spacelike (timelike) if n_μ is timelike (spacelike).

The projections of the gravitational equation (2) give the following three equations:

$$\big(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu}\big)\, n^\mu\, n^\nu = \kappa^2\,\mathcal{T}_{\mu\nu}\, n^\mu\, n^\nu = \kappa^2\rho, \tag{6}$$

$$\big(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu}\big)\, n^\mu \perp^{\nu}_{\ \rho} = \kappa^2\,\mathcal{T}_{\mu\nu}\, n^\mu \perp^{\nu}_{\ \rho} = -\kappa^2 J_\rho, \tag{7}$$

$$\big(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu}\big) \perp^{\mu}_{\ \rho} \perp^{\nu}_{\ \sigma} = \kappa^2\,\mathcal{T}_{\mu\nu} \perp^{\mu}_{\ \rho} \perp^{\nu}_{\ \sigma} = \kappa^2 S_{\rho\sigma}, \tag{8}$$

where we defined the components of the energy-momentum tensor as $\mathcal{T}_{\mu\nu} = \rho n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}$, and we also define $\mathcal{T} = \varepsilon\rho + S^{\alpha}_{\ \alpha}$ for later convenience.

Following the standard procedure of the ADM formulation, we find that the equations (6)–(8) correspond to (a) the Hamiltonian constraint equation:

$$M + \alpha_{GB}\big(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}\big) = -2\varepsilon\kappa^2\rho_H + 2\Lambda\,, \tag{9}$$

(b) the momentum constraint equation:

$$N_i + 2\alpha_{GB}\big(MN_i - 2M_i^{\ a}N_a + 2M^{ab}N_{iab} - M_i^{\ cab}N_{abc}\big) = \kappa^2 J_i\,, \tag{10}$$

and (c) the evolution equations for γ_{ij}:

$$\begin{aligned} &M_{ij} - \frac{1}{2}M\gamma_{ij} - \varepsilon\big(-K_{ia}K^a_{\ j} + \gamma_{ij}K_{ab}K^{ab} - \pounds_n K_{ij} + \gamma_{ij}\gamma^{ab}\pounds_n K_{ab}\big) \\ &\quad + 2\alpha_{GB}\Big[H_{ij} + \varepsilon\big(M\pounds_n K_{ij} - 2M_i^{\ a}\pounds_n K_{aj} - 2M_j^{\ a}\pounds_n K_{ai} - W_{ij}^{\ \ ab}\pounds_n K_{ab}\big)\Big] = \kappa^2 S_{ij} - \gamma_{ij}\Lambda\,, \end{aligned} \tag{11}$$

respectively, where

$$M_{ijkl} = R_{ijkl} - \varepsilon(K_{ik}K_{jl} - K_{il}K_{jk}), \tag{12}$$

$$N_{ijk} = D_i K_{jk} - D_j K_{ik}, \tag{13}$$

[4]The Greek indices move $0, 1, \cdots, N$, while the Latin indices move $1, \cdots, N$.

$$
\begin{aligned}
H_{ij} = \quad & MM_{ij} - 2(M_{ia}M^a_{\ j} + M^{ab}M_{iajb}) + M_{iabc}M_j^{\ abc} \\
& -2\varepsilon\Big[-K_{ab}K^{ab}M_{ij} - \frac{1}{2}MK_{ia}K^a_{\ j} + K_{ia}K^a_{\ b}M^b_{\ j} + K_{ja}K^a_{\ b}M^b_{\ i} + K^{ac}K_c^{\ b}M_{iajb} \\
& \qquad + N_iN_j - N^a(N_{aij} + N_{aji}) - \frac{1}{2}N_{abi}N^{ab}_{\ \ j} - N_{iab}N_j^{\ ab}\Big] \\
& -\frac{1}{4}\gamma_{ij}\big(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}\big) \\
& -\varepsilon\gamma_{ij}\big(K_{ab}K^{ab}M - 2M_{ab}K^{ac}K_c^{\ b} - 2N_aN^a + N_{abc}N^{abc}\big), && (14) \\
W_{ij}^{\ \ kl} = \quad & M\gamma_{ij}\gamma^{kl} - 2M_{ij}\gamma^{kl} - 2\gamma_{ij}M^{kl} + 2M_{iajb}\gamma^{ak}\gamma^{bl}. && (15)
\end{aligned}
$$

and these contracted variables; $M_{ij} = \gamma^{ab}M_{iajb}$, $M = \gamma^{ab}M_{ab}$, and $N_i = \gamma^{ab}N_{aib}$. We remark that the terms of $\pounds_n K_{ij}$ appear only in the linear form in (11). This is due to the quasi-linear property of the GB gravity.

3 Conformal Approach to solve the Constraints

In order to prepare an initial data for dynamical evolution, we have to solve two constraints, (9) and (10). The standard approach is to apply a conformal transformation on the initial hypersurface [6]. The idea is that introducing a conformal factor ψ between the initial trial metric $\hat{\gamma}_{ij}$ and the solution γ_{ij}, as

$$
\gamma_{ij} = \psi^{2m}\hat{\gamma}_{ij}, \;\; \gamma^{ij} = \psi^{-2m}\hat{\gamma}^{ij}, \tag{16}
$$

where m is a constant, and solve for ψ so as to the solution satisfies the constraints.

Regarding to the extrinsic curvature, we decompose K_{ij} into its trace part, $K = \gamma^{ij}K_{ij}$, and the traceless part, $A_{ij} = K_{ij} - \frac{1}{N}\gamma_{ij}K$, and assume the conformal transformation [5] as $A_{ij} = \psi^{\ell}\hat{A}_{ij}, A^{ij} = \psi^{\ell-4m}\hat{A}^{ij}, K = \psi^{\tau}\hat{K}$,where ℓ and τ are constants. For the matter terms, we also assume the relations $\rho = \psi^{-p}\hat{\rho}$ and $J^i = \psi^{-q}\hat{J}^i$, where p and q are constants, while we regard the cosmological constant is common to the both flames, $\Lambda = \hat{\Lambda}$.

Up to here, the powers of conformal transformation, ℓ, m, τ, p and q are not yet specified. Note that in the standard three-dimensional initial-data construction cases, the combination of $m = 2$, $\ell = -2$, $\tau = 0$, $p = 5$ and $q = 10$ is preferred since this simplifies the equations. We also remark that if we chose $\tau = \ell - 2m$, then the extrinsic curvature can be transformed as $K_{ij} = \psi^{\ell}\hat{K}_{ij}$ and $K^{ij} = \psi^{\ell-4m}\hat{K}^{ij}$.

- **Hamiltonian constraint:** Using these equations, (9) turns to be

$$
\begin{aligned}
& 2(N-1)m\hat{D}_a\hat{D}^a\psi - (N-1)\big[2-(N-2)m\big]m(\hat{D}\psi)^2\psi^{-1} \\
& \quad = \hat{R}\psi - \frac{N-1}{N}\varepsilon\psi^{2m+2\tau+1}\hat{K}^2 + \varepsilon\psi^{-2m+2\ell+1}\hat{A}_{ab}\hat{A}^{ab} + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} + \alpha_{GB}\hat{\Theta}\psi^{2m+1}.
\end{aligned}
$$

  The explicit form of the GB part $\hat{\Theta} = M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}$ is shown in [1].

- **Momentum constraint:** By introducing the transverse traceless part and the longitudinal part of $\hat{A}^{ij}$ as $\hat{D}_j\hat{A}^{ij}_{TT} = 0, \hat{A}^{ij}_L = \hat{A}^{ij} - \hat{A}^{ij}_{TT}$, respectively, then (10), can be written as

$$
\psi^{\ell-2m}\hat{D}_a\hat{A}_{i\,L}^{\ a} + \big[\ell + (N-2)m\big]\psi^{\ell-2m-1}\hat{A}_{i\,L}^{\ a}\hat{D}_a\psi - \frac{N-1}{N}\hat{D}_i(\psi^{\tau}\hat{K}) + 2\alpha_{GB}\hat{\Xi}_i = \kappa^2\psi^{2m-q}\hat{J}_i
$$

  The explicit form of the GB part $\hat{\Xi}_i$ is shown in [1].

In [1], we discussed how the equations turn to be in two sets of parameter choices, together with the version of momentarily static situation.

[5] In the strict sense this is not the conformal transformation but just the relation between the values with and without a caret.

4 Dynamical equations

The Einstein evolution equation in general N-dimensional ADM version is presented in [7]. With the GB terms, the evolution equation (11) cannot be expressed explicitly for each $\pounds_n K_{ij}$. That is, (11) is rewritten as

$$\begin{aligned}(1+2\alpha_{GB}M)\pounds_n K_{ij} - (\gamma_{ij}\gamma^{ab} + 2\alpha_{GB}W_{ij}{}^{ab})\pounds_n K_{ab} - 8\alpha_{GB}M_{(i}{}^{a}\pounds_n K_{|a|j)} \\ = -\varepsilon\left(M_{ij} - \frac{1}{2}M\gamma_{ij}\right) - K_{ia}K^{a}_{\ j} + \gamma_{ij}K_{ab}K^{ab} + \varepsilon\kappa^2 S_{ij} - \varepsilon\gamma_{ij}\Lambda - 2\varepsilon\alpha_{GB}H_{ij}\,, \qquad (17)\end{aligned}$$

and the second and the third terms in r.h.s include the linearly-coupled terms between $\pounds_n K_{ij}$. Therefore, in an actual simulation, we have to extract each evolution equation of K_{ij} using a matrix form of Eq. (17) like

$$\mathbf{k} = A\mathbf{k} + \mathbf{b} \qquad (18)$$

where $\mathbf{k} = (\pounds_n K_{11}, \pounds_n K_{12}, \cdots, \pounds_n K_{NN})^T$ and A, $\mathbf{b}$ are appropriate matrix and vector derived from Eq. (17).

The procedure of the inverting the matrix $(1-A)$ is technically available, but the invertibility of the matrix is not generally guaranteed at this moment. In the case of the standard ADM foliation in four-dimensional Einstein equations, the continuity of the time evolutions depends on the models and the choice of gauge conditions for the lapse function and shift vectors. If the combination is not appropriate, then the foliation hits the singularity which stops the evolution. The similar obstacle may exist also for the GB gravity. Actually, Deruelle and Madore [8] gave an explicit example in a simple cosmological model where the equation corresponding to (18) is not invertible. We expect that in the most cases Eq. (18) is invertible for K_{ij} but we cannot deny the pathological cases which depend on the models and gauge conditions. Such a study must be done together with actual numerical integrations in the future.

5 Discussion

In summary, we show the $(N+1)$-dimensional decomposition of the basic equations, in order to treat the space-time as a Cauchy problem. The equations can be separated to the constraints (the Hamiltonian constraint and the momentum constraint) and the evolution equations.

Two constraints should be solved for constructing an initial data, and we show how the actual equations turn to be. If we have the GB term, however, the equations still remain in a complicated style.

For the evolution equations, we find that $\pounds_n K_{ij}$ components are coupled. However, this mixture is only up to the linear order due to the quasi-linear property of the GB term, so that the equations can be in a treatable form in numerics.

We are now developing our numerical code and hope to present some results elsewhere near future.

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