Constraint propagation and constraint-damping in the C^2-adjusted formulation

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Abstract

In order to perform accurate and stable long-term numerical calculations, we construct new sets of ADM and BSSN evolution equations by adjusting constraints to their right-hand-sides. We apply a method suggested by Fiske (2004), which adds functional derivative of the norm of constraints, C^2. We derive their constraint propagation equations (evolution equations of constraints) in flat spacetime, which show that C^2 itself evolve decaying. We also perform numerical tests with the polarized Gowdy-wave testbed, and show that the constraint-damping appears. The life-times of the standard ADM and BSSN simulations are improved as about twice as longer.

1 Introduction

In numerical relativity, the standard way to integrate the Einstein equations is 3+1 splitting of spacetime. The fundamental spacetime splitting is the Arnowitt-Deser-Misner (ADM) formulation [1]. However, it is known that the ADM formulation is not suitable formulation to perform long-term simulations in strong gravitational fields [2]. To perform simulations such as coalescences of binary neutron stars and/or black holes, many formulations are suggested, one of the most commonly used among the numerical relativists is the so-called Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation [3].

The current formulations use the constraint-damping technique which is obtained by adding the constraint equation to the evolution equations. Fiske [4] suggested a method of constructing a set of evolution equations which we call "C^2-adjusted system". We apply his system to the ADM and BSSN formulations. To see the effect of the adjustments, we analyze the constraint propagation of the C^2-adjusted ADM and BSSN formulations, and we also perform some numerical tests to confirm the constraint-damping behaviors.

2 General Idea of C^2-adjusted system

Suppose dynamical variables u^i obeys a set of evolution equations with constraint equations, C^a;

$$\partial_t u^i = f(u^i, \partial_j u^i, \dots), \tag{1}$$
$$C^a = g(u^i, \partial_j u^i, \dots). \tag{2}$$

Fiske [4] proposed an adjustment of the evolution equations in the way of

$$\partial_t u^i = f(u^i, \partial_j u^i, \cdots) - \kappa^{ij} \left(\frac{\delta C^2}{\delta u^j} \right), \tag{3}$$

where κ^{ij} is a positive-definite constant coefficient, and C^2 is the norm of constraints which is defined as $C^2 \equiv \int C_a C^a d^3x$. The term $(\delta C^2 / \delta u^j)$ is the functional derivative of C^2 with u^j. We call the set of (3) with (2) as "C^2-adjusted formulation". The associated constraint propagation equation becomes

$$\partial_t C^2 = h(C^a, \partial_i C^a, \cdots) - \int d^3x \left(\frac{\delta C^2}{\delta u^i} \right) \kappa^{ij} \left(\frac{\delta C^2}{\delta u^j} \right). \tag{4}$$

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If we set κ^{ij} so as the second term in the RHS of (4) becomes dominant than the first term, then $\partial_t C^2$ becomes negative, which indicates that constraint violations are expected to decay to zero.

3 Applications

Now, we apply the idea of C^2-adjusted system to the ADM and BSSN formulations.

3.1 C^2-adjusted ADM Formulation

3.1.1 Evolution Equations

We apply C^2-adjusted system to the ADM formulation. The evolution equations are formally written as

$$\partial_t \gamma_{ij} = [\textbf{Original Terms}] - \kappa_{\gamma ijmn}\left(\frac{\delta(C^{\mathrm{ADM}})^2}{\delta\gamma_{mn}}\right), \tag{5}$$

$$\partial_t K_{ij} = [\textbf{Original Terms}] - \kappa_{Kijmn}\left(\frac{\delta(C^{\mathrm{ADM}})^2}{\delta K_{mn}}\right), \tag{6}$$

where $(C^{\mathrm{ADM}})^2$ is the norm of the constraints, which we set

$$(C^{\mathrm{ADM}})^2 \equiv \int \{(\mathcal{H}^{\mathrm{ADM}})^2 + \gamma^{ij}\mathcal{M}_i^{\mathrm{ADM}}\mathcal{M}_j^{\mathrm{ADM}}\} d^3x, \tag{7}$$

and both coefficients, $\kappa_{\gamma ijmn}$ and κ_{Kijmn}, are supposed to be positive definite. The adjusted terms, $(\delta(C^{\mathrm{ADM}})^2/\delta\gamma_{mn})$ and $(\delta(C^{\mathrm{ADM}})^2/\delta K_{mn})$, are explicitly written as eqs. (A1) and (A2) in [5], respectively.

3.1.2 Constraint Propagation Equations

In order to investigate the effect of the constraint-damping due to the adjusted terms in (5)-(6), we show the constraint propagation equations in the flat spacetime:

$$\partial_t \mathcal{H}^{\mathrm{ADM}} = [\textbf{Original Terms}] - 2\kappa_\gamma \Delta^2 \mathcal{H}^{\mathrm{ADM}}, \tag{8}$$

$$\partial_t \mathcal{M}_i^{\mathrm{ADM}} = [\textbf{Original Terms}] + \kappa_K \Delta \mathcal{M}_i^{\mathrm{ADM}} + 3\kappa_K \partial_i \partial_j (\mathcal{M}^{\mathrm{ADM}})^j, \tag{9}$$

where, we set the coefficients as $\kappa_{\gamma ijmn} = \kappa_\gamma \delta_{im}\delta_{jn}$ and $\kappa_{Kijmn} = \kappa_K \delta_{im}\delta_{jn}$. In both equations (8)-(9), we see the diffusion terms, $-2\kappa_\gamma\Delta^2\mathcal{H}^{\mathrm{ADM}}$ and $\kappa_K\Delta\mathcal{M}_i^{\mathrm{ADM}}$, respectively. These terms contribute to the damping of the constraint violations.

3.2 C^2-adjusted BSSN Formulation

3.2.1 Evolution Equations

Next, we apply the idea to the BSSN formulation. The evolution equations are formally written as

$$\partial_t \varphi = [\textbf{Original Terms}] - \lambda_\varphi \left(\frac{\delta(C^{\mathrm{BSSN}})^2}{\delta\varphi}\right), \tag{10}$$

$$\partial_t K = [\textbf{Original Terms}] - \lambda_K \left(\frac{\delta(C^{\mathrm{BSSN}})^2}{\delta K}\right), \tag{11}$$

$$\partial_t \widetilde{\gamma}_{ij} = [\textbf{Original Terms}] - \lambda_{\widetilde{\gamma}ijmn} \left(\frac{\delta(C^{\mathrm{BSSN}})^2}{\delta\widetilde{\gamma}_{mn}}\right), \tag{12}$$

$$\partial_t \widetilde{A}_{ij} = [\textbf{Original Terms}] - \lambda_{\widetilde{A}ijmn} \left(\frac{\delta(C^{\mathrm{BSSN}})^2}{\delta\widetilde{A}_{mn}}\right), \tag{13}$$

$$\partial_t \widetilde{\Gamma}^i = [\textbf{Original Terms}] - \lambda_{\widetilde{\Gamma}}^{ij} \left(\frac{\delta(C^{\mathrm{BSSN}})^2}{\delta\widetilde{\Gamma}^j}\right), \tag{14}$$

where all the coefficients λ_φ, λ_K, $\lambda_{\tilde{\gamma}ijmn}$, $\lambda_{\tilde{A}ijmn}$, and $\lambda_{\tilde{\Gamma}}^{ij}$ are positive definite. $(C^{\rm BSSN})^2$ is a function of the constraints $\mathcal{H}^{\rm BSSN}$, $\mathcal{M}_i^{\rm BSSN}$, $\mathcal{G}^i$, $\mathcal{A}$, and $\mathcal{S}$, which we set as

$$(C^{\rm BSSN})^2 = \int \{(\mathcal{H}^{\rm BSSN})^2 + \gamma^{ij}\mathcal{M}_i^{\rm BSSN}\mathcal{M}_j^{\rm BSSN} + c_G\gamma_{ij}\mathcal{G}^i\mathcal{G}^j + c_A\mathcal{A}^2 + c_S\mathcal{S}^2\}d^3x, \tag{15}$$

where, c_G, c_A, and c_S are Boolean parameters (0 or 1). These three parameters are introduced to prove the necessity of the algebraic constraint terms in (15). The adjusted terms in (10)-(14) are then written down explicitly, as shown as eqs. (A1)-(A5) in [6], respectively.

3.2.2 Constraint Propagation Equations

In order to see the effect of the adjusted terms in (10)-(14), we derive the constraint propagation equations in the flat spacetime:

$$\begin{aligned}\partial_t\mathcal{H}^{\rm BSSN} &= [\textbf{Original Terms}] + \left(-128\lambda_\varphi\Delta^2 - (3/2)\lambda_{\tilde{\gamma}}\Delta^2 + 2\lambda_{\tilde{\Gamma}}\Delta\right)\mathcal{H}^{\rm BSSN} \\ &\quad + c_G\left(-(1/2)\lambda_{\tilde{\gamma}}\Delta\partial_m - 2\lambda_{\tilde{\Gamma}}\partial_m\right)\mathcal{G}^m + 3c_S\lambda_{\tilde{\gamma}}\Delta\mathcal{S}, \end{aligned}\tag{16}$$

$$\partial_t\mathcal{M}_a^{\rm BSSN} = [\textbf{Original Terms}] + \left\{(8/9)\lambda_K\delta^{bc}\partial_a\partial_b + \lambda_{\tilde{A}}\Delta\delta_a{}^c + \lambda_{\tilde{A}}\delta^{bc}\partial_a\partial_b\right\}\mathcal{M}_c^{\rm BSSN} - 2c_A\lambda_{\tilde{A}}\partial_a\mathcal{A}, \tag{17}$$

$$\begin{aligned}\partial_t\mathcal{G}^a &= [\textbf{Original Terms}] + \delta^{ab}\left((1/2)\lambda_{\tilde{\gamma}}\partial_b\Delta + 2\lambda_{\tilde{\Gamma}}\partial_b\right)\mathcal{H}^{\rm BSSN} \\ &\quad + c_G\left(\lambda_{\tilde{\gamma}}\Delta\delta^a{}_b + (1/2)\lambda_{\tilde{\gamma}}\delta^{ac}\partial_c\partial_b - 2\lambda_{\tilde{\Gamma}}\delta^a{}_b\right)\mathcal{G}^b - \lambda_{\tilde{\gamma}}c_S\delta^{ab}\partial_b\mathcal{S}, \end{aligned}\tag{18}$$

$$\partial_t\mathcal{A} = [\textbf{Original Terms}] + 2\lambda_{\tilde{A}}\delta^{ij}(\partial_i\mathcal{M}_j^{\rm BSSN}) - 6c_A\lambda_{\tilde{A}}\mathcal{A}, \tag{19}$$

$$\partial_t\mathcal{S} = [\textbf{Original Terms}] + 3\lambda_{\tilde{\gamma}}\Delta\mathcal{H}^{\rm BSSN} + c_G\lambda_{\tilde{\gamma}}\partial_\ell\mathcal{G}^\ell - 6c_S\lambda_{\tilde{\gamma}}\mathcal{S}. \tag{20}$$

where we set the coefficient parameters, $\lambda_{\tilde{\gamma}ijmn} = \lambda_{\tilde{\gamma}}\delta_{im}\delta_{jn}$, $\lambda_{\tilde{A}ijmn} = \lambda_{\tilde{A}}\delta_{im}\delta_{jn}$ and $\lambda_{\tilde{\Gamma}}{}^{ij} = \lambda_{\tilde{\Gamma}}\delta^{ij}$ for simplicity. In the above equations, we see the appearances of diffusion terms $(-128\lambda_\varphi\Delta^2 - (3/2)\lambda_{\tilde{\gamma}}\Delta^2 + 2\lambda_{\tilde{\Gamma}}\Delta)\mathcal{H}^{\rm BSSN}$, $\lambda_{\tilde{A}}\Delta\mathcal{M}_a^{\rm BSSN}$, $c_G(\lambda_{\tilde{\gamma}}\Delta - 2\lambda_{\tilde{\Gamma}})\mathcal{G}^a$, $-6c_A\lambda_{\tilde{A}}\mathcal{A}$ and $-6c_S\lambda_{\tilde{\gamma}}\mathcal{S}$, respectively. These terms contribute to the damping of the violations. If we set the parameters, c_G, c_A and c_S, are zero, equations (18)-(20) turn not to include diffusion terms. Therefore, $(C^{\rm BSSN})^2$ should include $\mathcal{G}^i$, $\mathcal{A}$ and $\mathcal{S}$.

4 Numerical Tests

We perform simulations of the polarized Gowdy wave which is one of the testbeds for comparing formulatiuons [7]. The numerical parameters are the same with those in [7].

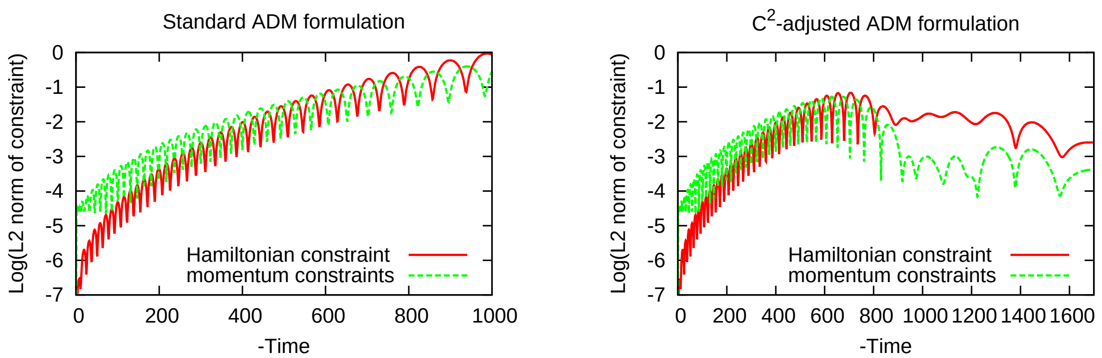

Figure 1: L2 norm of each constraint in the polarized Gowdy wave test with the ADM formulations. The L2 norm of the constrains are shown as a function of the backward time. We adopt the parameters same with the line (c) in Fig.2 of [5].

In Figure 1, we show that the constraint violations with the ADM formulaitons. We see that the adjusted cases decrease the violation, and the lifetime of simulations becomes about 1.7 times longer than that of the standard case. Next, in Fig.2, we show that the constraint violations with the BSSN

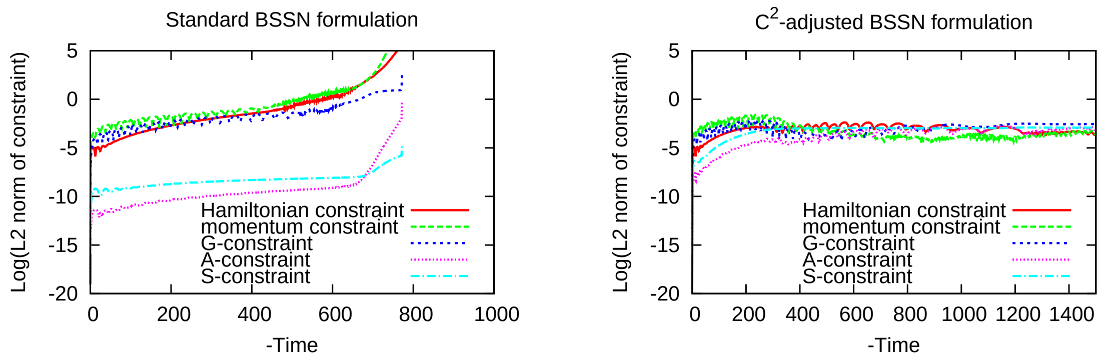

Figure 2: L2 norm of each constraint with the BSSN formulations. We adopt the parameters same with the line (C) in Fig.8 of [6].

formulations. We see the similar effects, and the lifetime of simulations of the adjusted case becomes more than twice longer than that of the standard case.

5 Summary

In this report, we reviewed the idea of the C^2-adjusted system and applied the system to the ADM and BSSN formulations. We see the effect of constraint-damping due to the adjusted terms by showing the constraint propagation equations. We performed the simulations with these formulations in the Gowdy wave testbed and confirmed the constraint-damping behaviors and the life time of the simulations becomes longer than those of the standard cases. Please refer [6] for more details.

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