Autoregressive Approach to Extract Ring-down Gravitational Wave of Black-hole Merger

Hisa-aki Shinkai*, Shun Yamamoto

Department of Information Systems, Osaka Institute of Technology, Hirakata City, Osaka 573-0196, Japan
**E-mail: hisaaki.shinkai@oit.ac.jp*

We apply an autoregressive (AR) model for identifying the ring-down part of gravitational wave of binary black-hole mergers. This approach enables us to extract signals without templates, and is effective for short-period data. We obtain the ring-down frequency of the remnant black-hole of GW150914 as $305.94^{+18.68}_{-27.82}$ Hz (Hanford) and $300.02^{+17.49}_{-27.21}$ Hz (Livingston), and the damping frequency of it as $43.55^{+13.00}_{-17.99}$ Hz (Hanford) and $44.94^{+12.88}_{-18.30}$ Hz (Livingston). These results indicate the mass and spin of the black-hole, both of which are consistent with the values reported originally.

Keywords: Data Analysis of Gravitational Wave; Black hole

1. Introduction

After the announcement of the direct detection of gravitational wave, GW150914 [1] by LIGO/Virgo group, the astrophysics and astronomy turned into the new era of researches. During the past two years, we received four events and one candidate event of the mergers of binary black holes (BHs) (GW150914 [1], LVT151012, GW151226, GW170104, GW170814), and one event of the merger of binary neutron stars (GW170817).

Information from gravitational-wave signals gives us not only the kinetic parameters of the sources, but also the nature of strong gravity, high-energy physical phenomena, and cosmological implications. Among them, one of the most important topic which is not available from other observations is the test of general relativity. BHs are the product of strong gravity, and their properties can be only directly searched using gravitational wave.

Waveform of the mergers of binary compact stars is divided into three parts; inspiral, merger, and ring-down. The fundamental waveform of the inspiral part is given by the post-Newtonian approach of the Einstein equation, and its precise form is given from the combination with numerically simulated data. The waveform of the ring-down part is given by the perturbative approach of the Einstein equation. Both precise waveforms can be used for testing the theories of strong gravity. Especially, the ring-down part is the apparent result of the existence of a BH, so that we expect the differences from general relativity, if it exists, will be detected from this part.

Once a candidate of gravitational wave is detected, then the data is analyzed with the templates of the expected gravitational waveforms and the signal is identified with matched-filtering method. However, finding out the ring-down mode of BH from noisy data is quite challenging task. This is because it decays quite rapidly for a typical BH. For example, a BH with mass $M = 60M_\odot$ and angular momentum (normalized Kerr parameter) $a = 0.75$, the mode ($\ell = m = 2$) has frequency $f_R = 300$ Hz and decaying time $\tau = 3.7$ ms, which indicate that we have to detect decaying one-cycle wave with 15 data points (for the data with sampling rate 4096).

If the theory of gravity differs from general relativity, we have to prepare the templates of gravitational wave from other theories. There are plenty of candidates of gravity theories which have survived from solar-system experiments and inspiral part of binary coalescences, and most of them are not yet ready for preparing waveform.

Concerning the background above, we here propose to apply a new method for extracting gravitational wave signals; auto-regressive (AR) approach. AR method is well-known time-sequence analysis which has been used in e.g. acoustic signal processing. The key idea of AR method is to find a fitting function from a segment of data (with noise), and analyze the fitting function for extracting its frequencies and damping rates. It does not require templates of signals in advance, and it can be applied only for a short segment of data.

In this article, we briefly introduce our approach of applying AR model for extracting gravitational wave.

2. Autoregressive approach

2.1. *Basic idea*

Let a discrete data $x_n \equiv x(t_n)$, where $t_n = n\Delta t$, forms a segment with N-data points, $(x_1, x_2, \cdots, x_n, \cdots, x_N)$. We assume that the signal is expressed with linear combination of its previous M-data,

$$x_n = \sum_{j=1}^{M} a_j x_{n-j} + \varepsilon, \tag{1}$$

where a_j, M is the coefficients and the order of AR model, respectively, and ε is the remained noise of this modeling. The key idea of AR approach is to find the expression (1), and apply this expression to data analysis. We know a simple oscillating function, such as trigonometric or damped (or increasing) oscillation, is expressed with $M = 2$. Even if the original data includes unknown noises, we expect that they will be sweeped out from the expression (1) by taking a certain length of N.

Various methods are proposed to determine a_j and M. We used the Burg method for finding a_j and FPE (final prediction error) method for determining M. That is, for a given segment of data, we first find the best-fit coefficients a_js for a particular M by applying the least-square method to eq.(1) back and forth in a segment. Then we change M and find the M which gives the smallest error, ε.

Once the model (1) is fixed, we then reconstruct wave signal from (1) and analyze it. By setting $z(f) = e^{i2\pi f\Delta t}$, the power spectrum of the wave signal can be expressed as

$$p(f) = \sigma^2 \left| 1 - \sum_{j=1}^{M} a_j z^{-j} \right|^{-2}, \tag{2}$$

where σ is the variance of ε. The resolution of frequency in (2) is not related with the original data set, so that AR method is expected to identify signal frequency precisely than the standard Fourier decomposition.

If we transform data x_n to $X(z) = \sum_{j=1}^{\infty} x_j z^{-j}$ (z-transformation), from (1), the (local) maximums of the spectrum, $p(f)$, are given at

$$F(z) \equiv 1 - \sum_{j=1}^{M} a_j z^j \approx 0. \tag{3}$$

This is a M-th order polynomial equation. The solutions of the characteristic equation

$$F(z) = 0, \tag{4}$$

then, express the fundamental modes which consist the data segment. Suppose $z_k = e^{-i2\pi f_k \Delta t}$ ($k = 1, \cdots, M$), then the real part of f_k expresses a frequency, and the imaginary part of f_k indicates a damping term. (Actually, $|z_k| \leq 1$ is expected to see the expression (1) is stable.) Therefore AR method can determine the frequencies and damping rates of quasi-normal modes from the data themselves.

2.2. *Methods and Results*

We apply AR method for GW150914. We downloaded the data from the Ligo site[2], of its 32-second data of 4096 sampling-rate taken at both Hanford and Livingston observatories. We used the data from $t = 8$ to $t = 24$. We applied a bandpass filter for 100 Hz $\sim$ 400 Hz, and made a segment of 1/64 second (64 data points) shifting them with 1/512 second (8 data points).

At every segment, we obtained the coefficients a_j and the order M of eq. (1), and found that at most segments M is less than four, and max is M=10. See Fig.1 for example.

We then get the power-spectrum $p(f)$ from eq. (2) at each segment, and the number of its local maximum are less than 3 for the data $t = 16.4$ to $t = 17.4$. We name the frequencies $f_1, f_2, \cdots$ which give the local maximum of the spectrum. We also solve (4) at each segment (which is at most 10-th order polynomial equation), and identify the solution z_k of which real part of frequency is mostly close to the one obtained as $f_1, f_2, \cdots$. We list the solutions z_k of each segment which are candidates for ring-down modes, and check whether these candidates are within a close value over several segments. We found that sometimes a segment is full of noises and

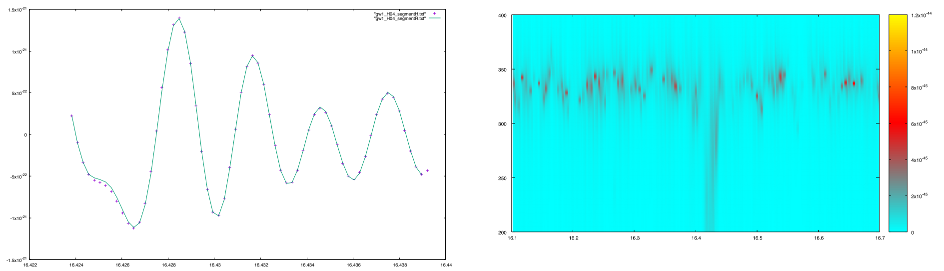

Fig. 1. (left) Example of AR fitting; $\times$ is the actual data, line is the fitted one. The order of M in this segment was $M = 4$. (right) Spectrogram of Hanford data.

shows quite different numbers from continual segments. We excluded such data if it shows the peak frequency is one-sigma apart from the list.

In result, we conclude that 6 segments of Hanford data (from $t = 16.4258$ to 16.4312), and 8 segments of Livingston data (from $t = 16.4121$ to 16.4473) have a consistent frequency and damping rate. We obtain the ring-down frequency of the remnant BH of GW150914 as in Table 1. The results indicate the mass and spin of the BH, which are consistent with the values reported originally [1], as $62.2^{+3.7}_{-3.4}\ M_\odot$ and $a/M = 0.68^{+0.05}_{-0.06}$.

Table 1. Results of frequency and damping rate of ring-down gravitational wave of GW150914.

data	$f_{\rm real}$[Hz]	$f_{\rm imag}$[Hz]	mass $(M/M_\odot)$	Kerr parameter a/M
Hanford	$305.94^{+18.68}_{-27.82}$	$43.55^{+13.00}_{-17.99}$	$58.74^{+16.03}_{-9.37}$	$0.75^{+0.18}_{-0.27}$
Livingston	$300.02^{+17.49}_{-27.21}$	$44.94^{+12.88}_{-18.30}$	$58.15^{+16.49}_{-9.53}$	$0.71^{+0.20}_{-0.30}$

3. Conclusion

We developed a new method for extracting gravitational-wave signals using autoregressive method. It does not require templates, and can be applied to a short-period data. We applied this method for extracting ring-down mode of black-hole merger, intending our future applications for testing gravity theories. The results we obtained in this brief note was for GW150914, and are consistent with the reported values originally estimated from inspiral part of the signal. We are preparing more detail report, which treats other observed events.

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References

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