Controlling Constraint Violations in General Relativity
— Asymptotically Constrained Systems via Constraint Propagation Analysis —

Hisaaki Shinkai
Computataional Science Div., RIKEN Institute, Japan
hshinkai@riken.jp
http://atlas.riken.go.jp/∼shinkai/

Outline

- **Introduction to Numerical Relativity**
- **Why mathematically equivalent eqs produce different numerical stability?**
- **Three approaches: (1) ADM/BSSN, (2) hyperbolic form. (3) attractor systems**
- **Proposals : A unified treatment as Adjusted Systems**

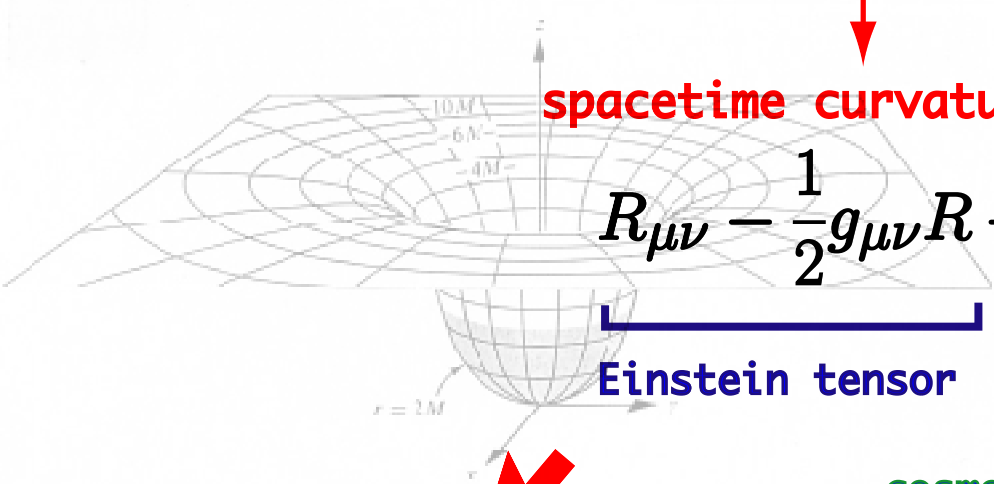
Ref:

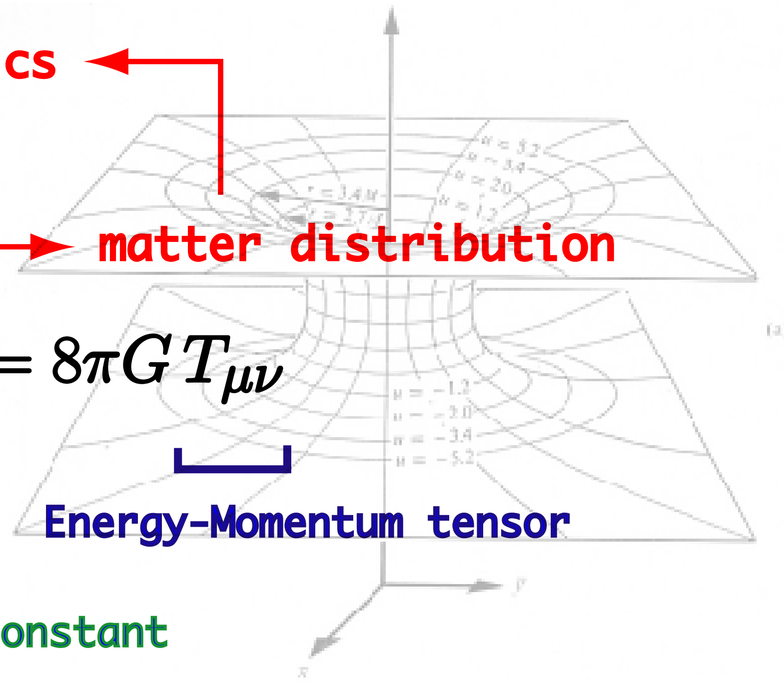
http://xxx.lanl.gov/abs/gr-qc/0209111
a review article, in print. (Nova Science Publ.)

at Minisymposium "Numerical Methods for PDEs with Constraints"
in The 5th International Congress on Industrial and Applied Mathematics, Sydney, 2003 July.

1. **Introduction: General Relativity and Numerical Relativity**
2. **Formulation problem of Numerical Relativity**
   (0) Arnowitt-Deser-Misner
   (1) Baumgarte-Shapiro-Shibata-Nakamura formulation
   (2) Hyperbolic formulations
   (3) Attractor systems
3. **"Adjusted Systems"**
   Asymptotically constrained system by adjusting evolution eqs.
   based on Constraint Propagation analysis
4. **Summary and Future Issues**

The Einstein equation





geodesics

spacetime curvature → matter distribution

$$\underbrace{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R}_{\text{Einstein tensor}} + \Lambda g_{\mu\nu} = 8\pi G \underbrace{T_{\mu\nu}}_{\text{Energy-Momentum tensor}}$$

cosmological constant

Solve for metric
$g_{\mu\nu}$(t,x,y,z)
(10 components)

flat spacetime (Minkowskii spacetime):

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\ &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned}$$

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu}dx^\mu dx^\nu := g_{\mu\nu}dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ & g_{xx} & g_{xy} & g_{xz} \\ & & g_{yy} & g_{yz} \\ sym. & & & g_{zz} \end{pmatrix}$$

The Einstein equations (General Relativity): $R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$
$=$ Physics of strong gravity.

- gravitational waves from colliding black holes, neutron stars, supernovae, ...
- cosmology, higher-dimensional models, singularity, ...
- relativistic phenomena at active galactic nuclei, ...

What are the difficulties?

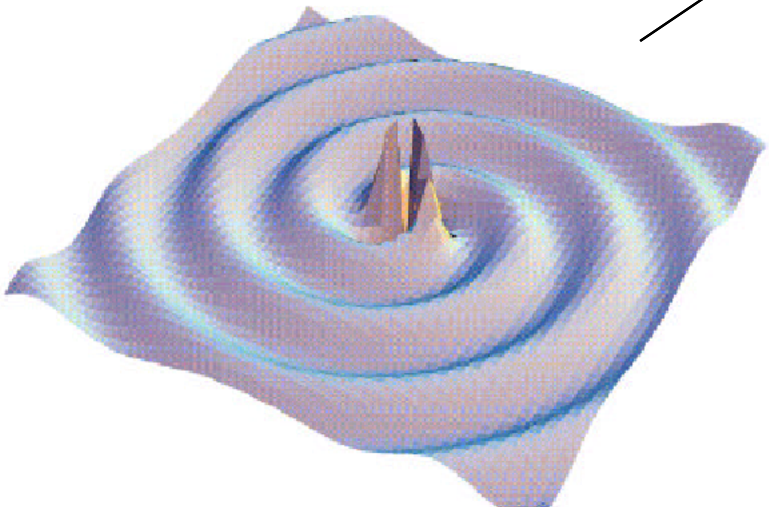
- for 10-component metric, highly nonlinear PDEs. (4 elliptic/6 hyperbolic in 3+1)
- free to choose cooordinates, gauge conditions, and even for decomposition of the space-time (3+1 or 2+2 or whatever).
- has singularity in its nature.

Solve it using computers, anyway !

Numerical Relativity $=$ Solve the Einstein equations numerically.
over 20 years history, but still looking for a recipe ...

Toward Direct Detection of Gravitational Wave

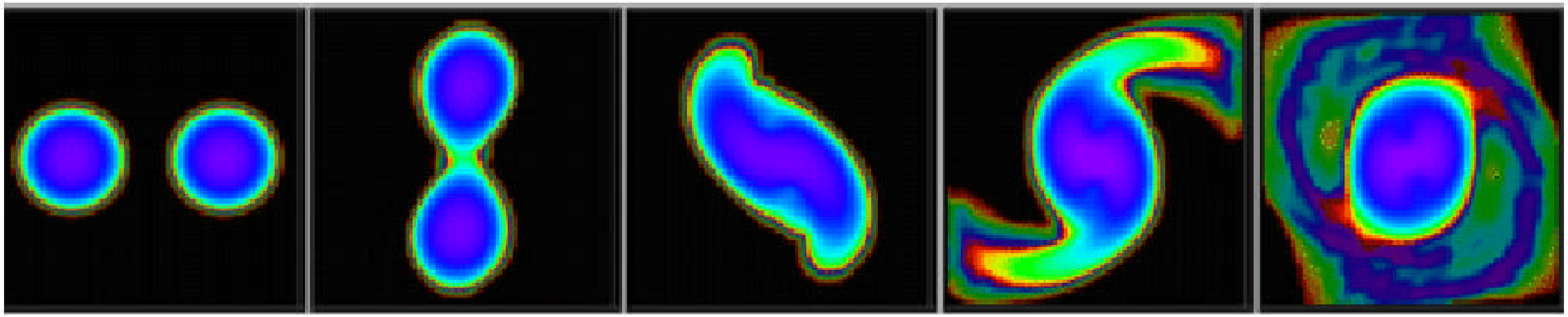
GW is produced by coalescing Black-holes and/or Neutron Stars

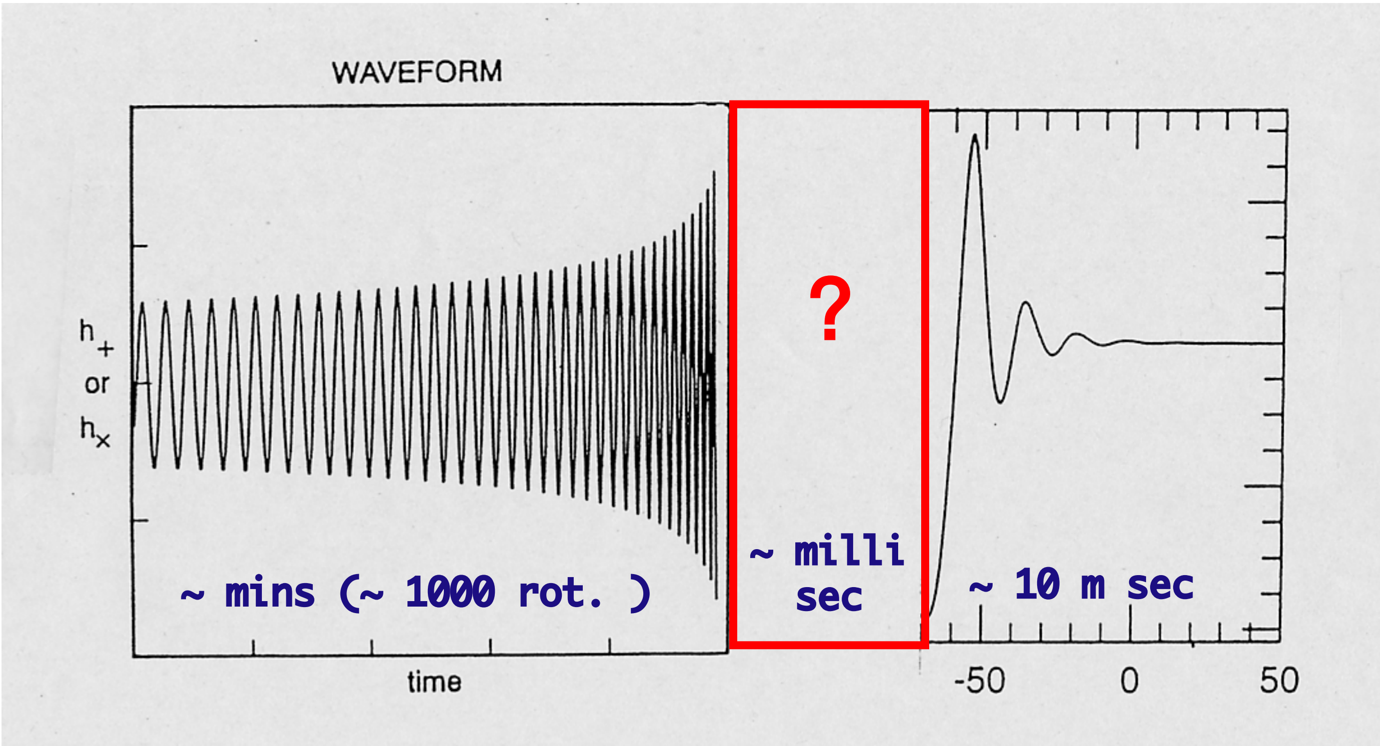


Laser Interferometers

JAPAN	300m	2000-
USA	4Km/2Km	2002-
GermanyUK	600m	2002-
ItalyFrance	3Km	2003-

- Neutron star – neutron star (Centrella et al.)





INSPIRAL → COALESCE → BLACKHOLE FORMATION

Innermost Stable
Circular Orbit?

Post Newtonian Approx.

→ Numerical Relativity

→ BH. Perturbation

Numerical Relativity – open issues

0. How to foliate space-time

Cauchy $(3+1)$, Hyperboloidal $(3+1)$, characteristic $(2+2)$, or combined?

$\Rightarrow$ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data

Theoretical:	Proper formulation for solving constraints? How to prepare realistic initial data? Effects of background gravitational waves? Connection to the post-Newtonian approximation?
Numerical:	Techniques for solving coupled elliptic equations? Appropriate boundary conditions?

2. How to evolve the data

Theoretical:	Free evolution or constrained evolution? Proper formulation for the evolution equations? $\Rightarrow$ THIS TALK! Suitable slicing conditions (gauge conditions)?
Numerical:	Techniques for solving the evolution equations? Appropriate boundary treatments? Singularity excision techniques? Matter and shock surface treatments? Parallelization of the code?

3. How to extract the physical information

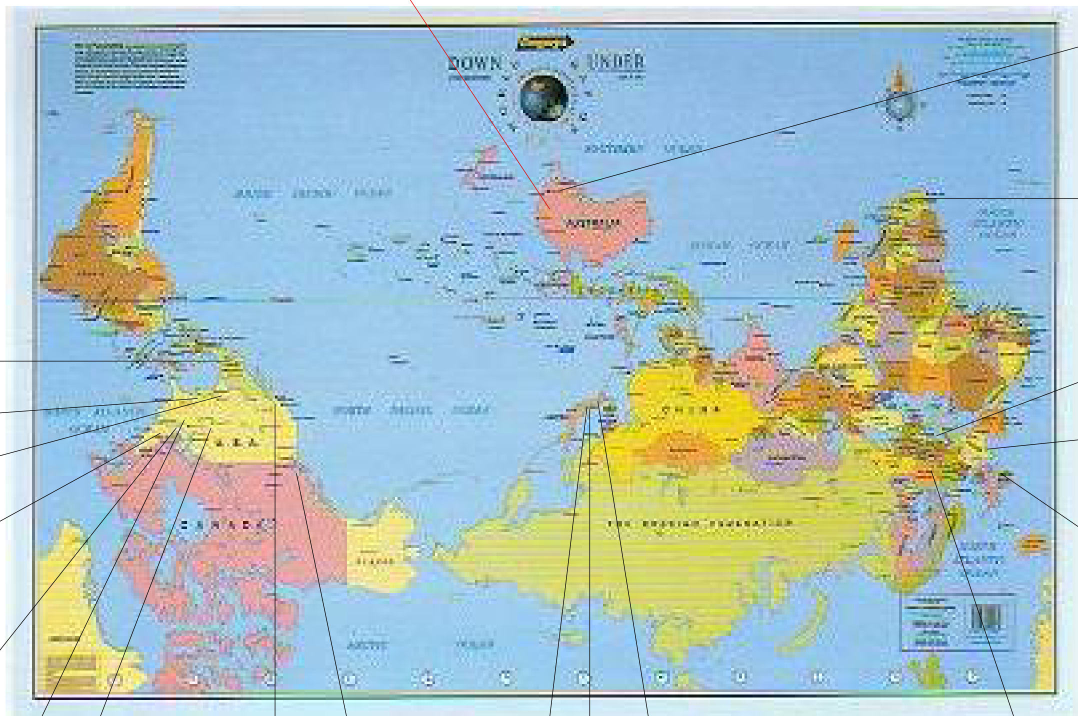
Theoretical:	Gravitational wave extraction? Connection to other approximations?
Numerical:	Identification of black hole horizons? Visualization of simulations?

Numerical Relativity groups around the world

We are here

Monash, Australia

SouthAfrica



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PennState

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Illinois

Caltech

Vancouver

SISSA, Italy

Meudon, France

Southampton, UK

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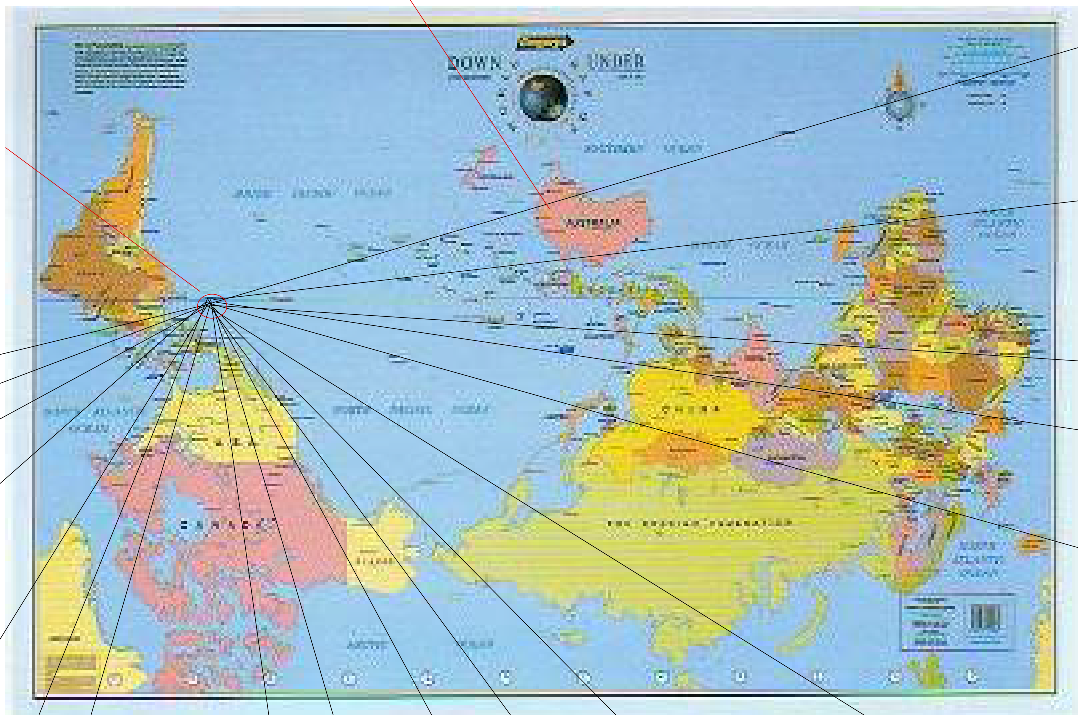

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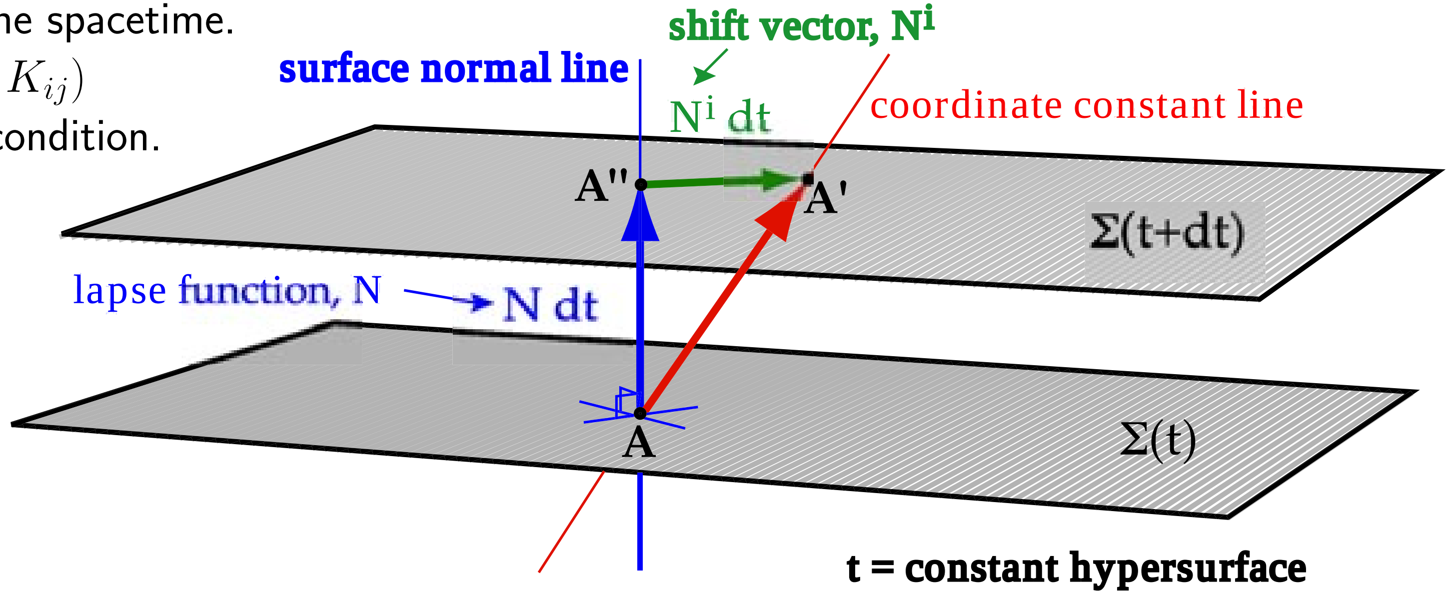
strategy 0 The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

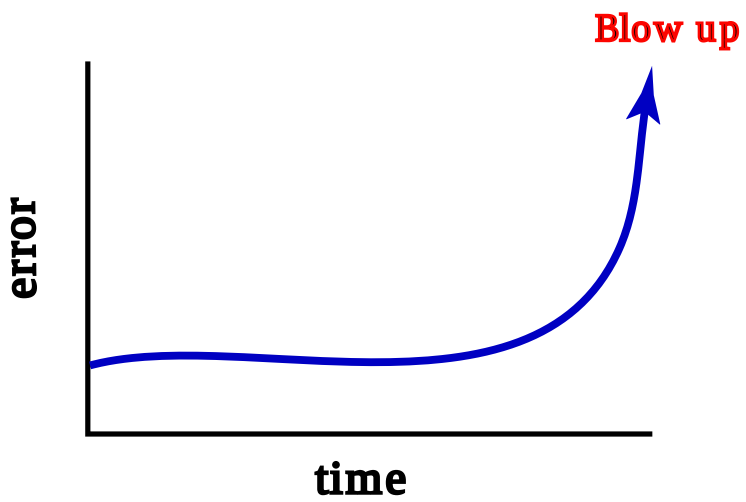
3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.

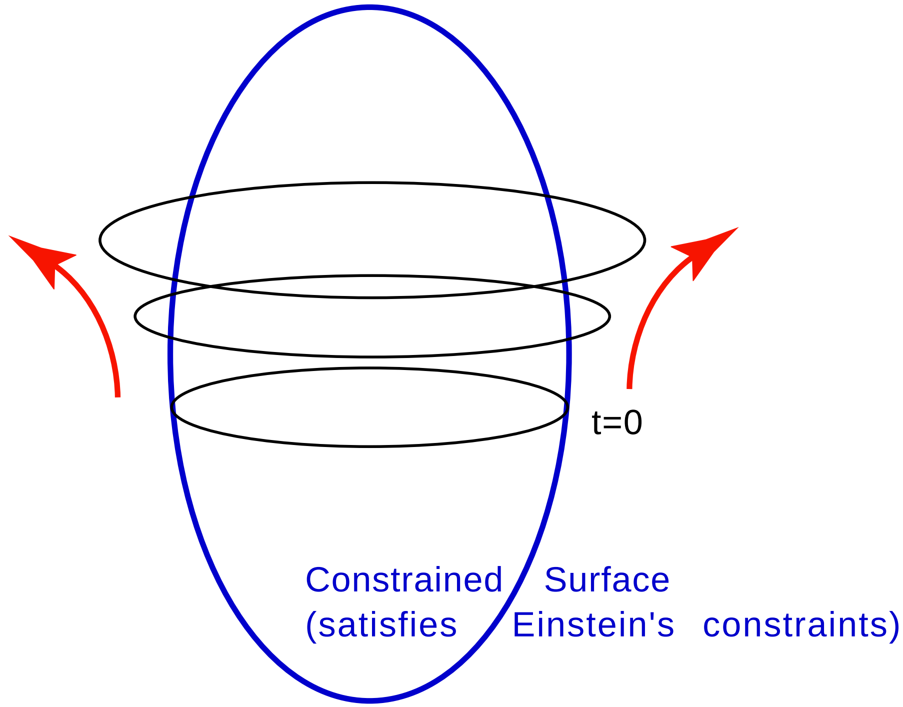


	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$ div $\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} =$ rot $\mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -$rot $\mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

Best formulation of the Einstein eqs. for long-term stable & accurate simulation?

Many (too many) trials and errors, not yet a definit recipe.





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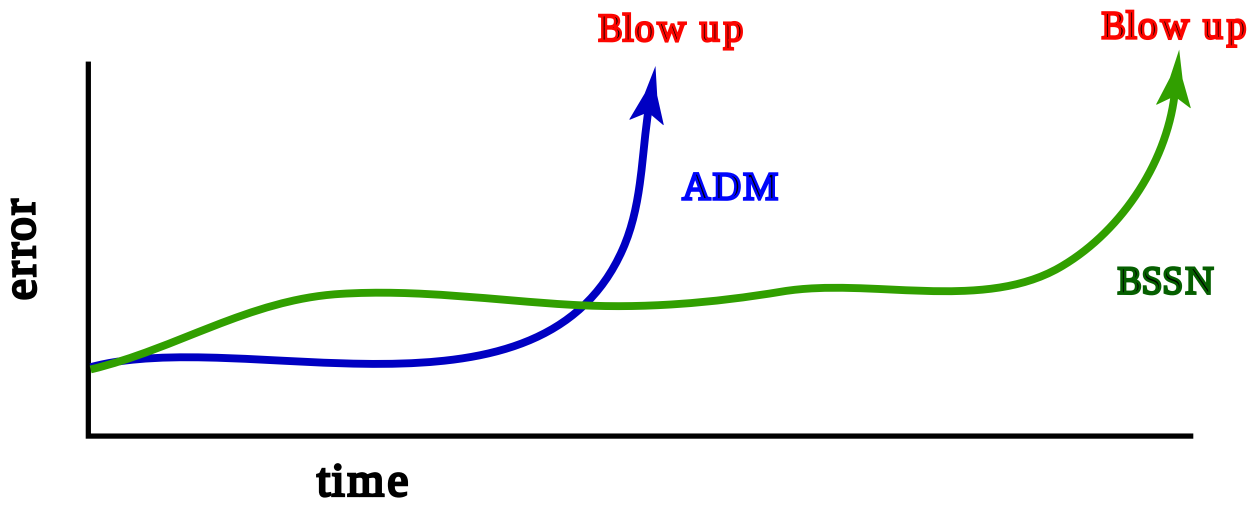


Mathematically equivalent formulations, but differ in its stability!

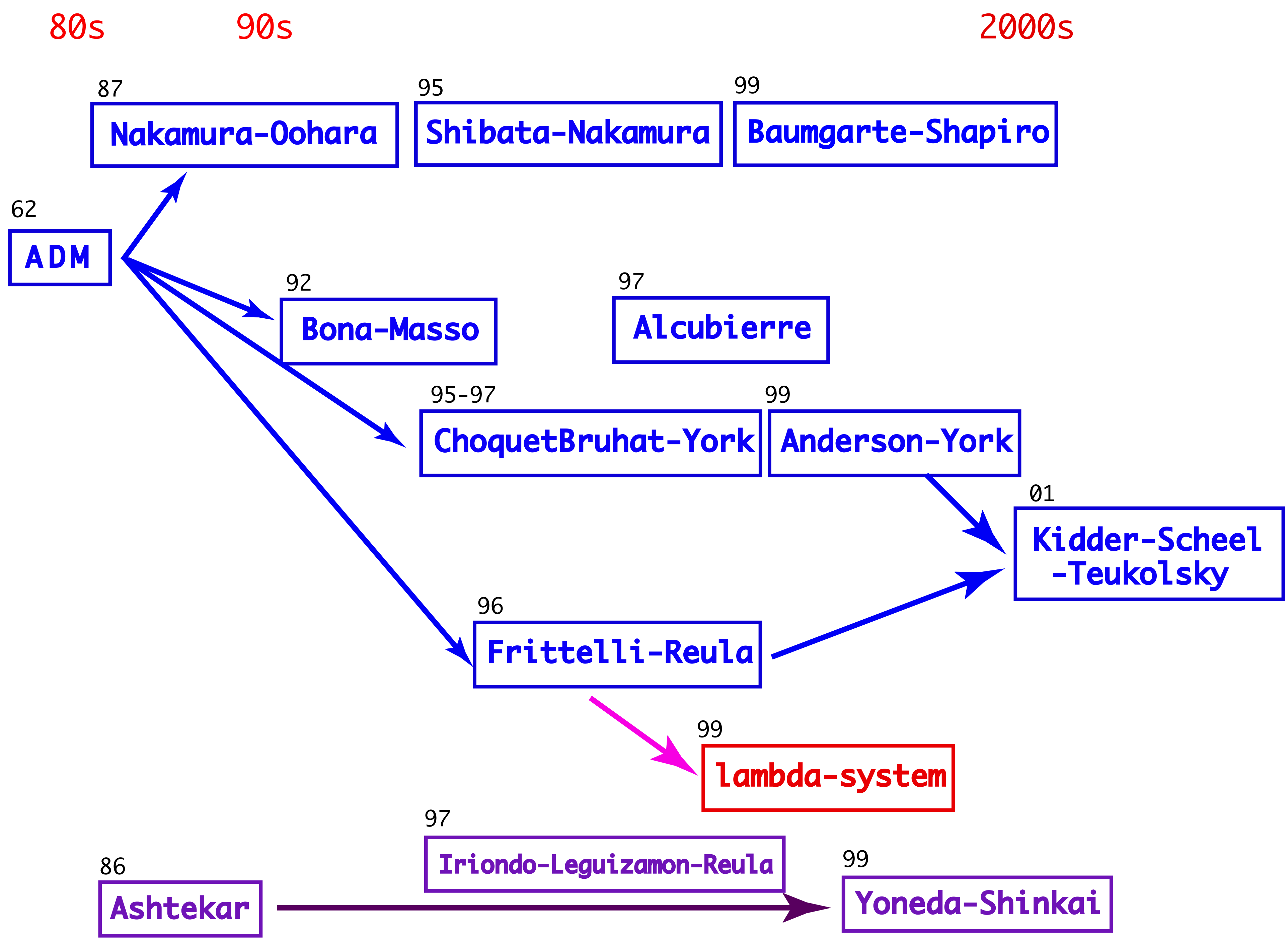
strategy 0: Arnowitt-Deser-Misner formulation

strategy 1: Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

strategy 2: Apply a formulation which reveals a hyperbolicity explicitly

strategy 3: Formulate a system which is "asymptotically constrained" against a violation of constraints

By adding constraints in RHS, we can kill error-growing modes
$\Rightarrow$ How can we understand the features systematically?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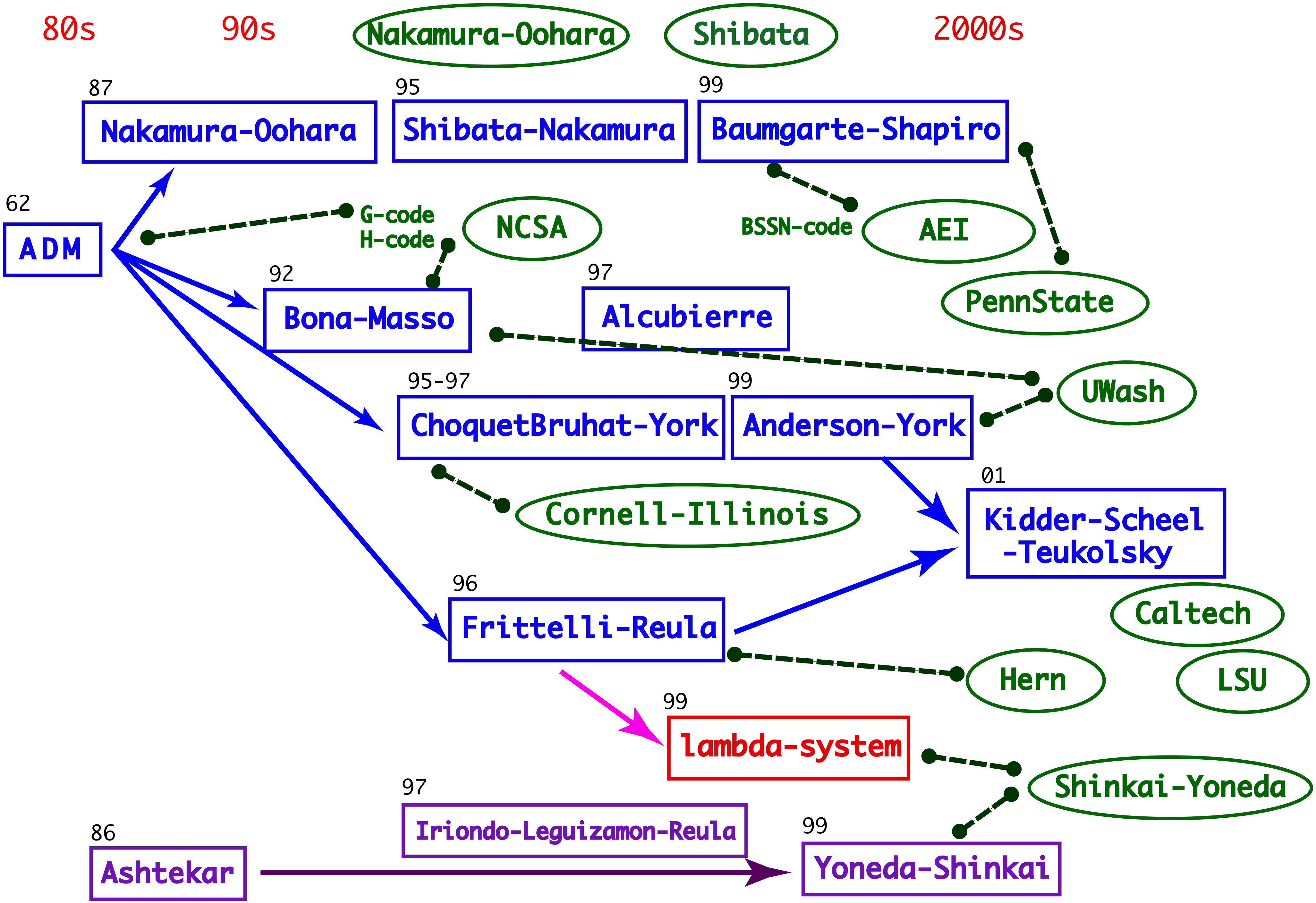
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strategy 1 Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

– define new variables ($\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i$), instead of the ADM's (γ_{ij}, K_{ij}) where

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},$$

use momentum constraint in Γ^i-eq., and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

– The set of evolution equations become

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

$$\begin{aligned}
R_{ij} &= \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^\phi_{ij} \\
R^\phi_{ij} &= -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi) \\
\tilde{R}_{ij} &= -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}lm\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}
\end{aligned}$$

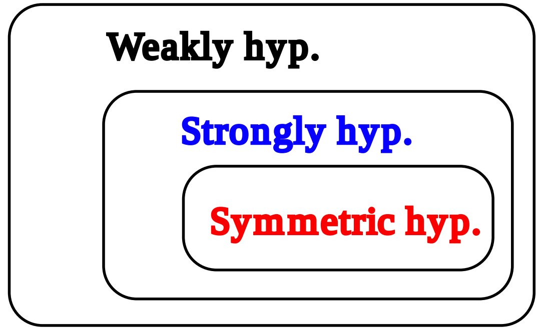
– No explicit explanations why this formulation works better.
AEI group (2000): the replacement by momentum constraint is essential.

strategy 2 Apply a formulation which reveals a hyperbolicity explicitly.

For a first order partial differential equations on a vector u,

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{characteristic part}} + \underbrace{B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{lower order part}}$$

	if the eigenvalues of A are
weakly hyperbolic	all real.
strongly hyperbolic	all real and $\exists$ a complete set of eigenvalues.
symmetric hyperbolic	if A is real and symmetric (Hermitian).



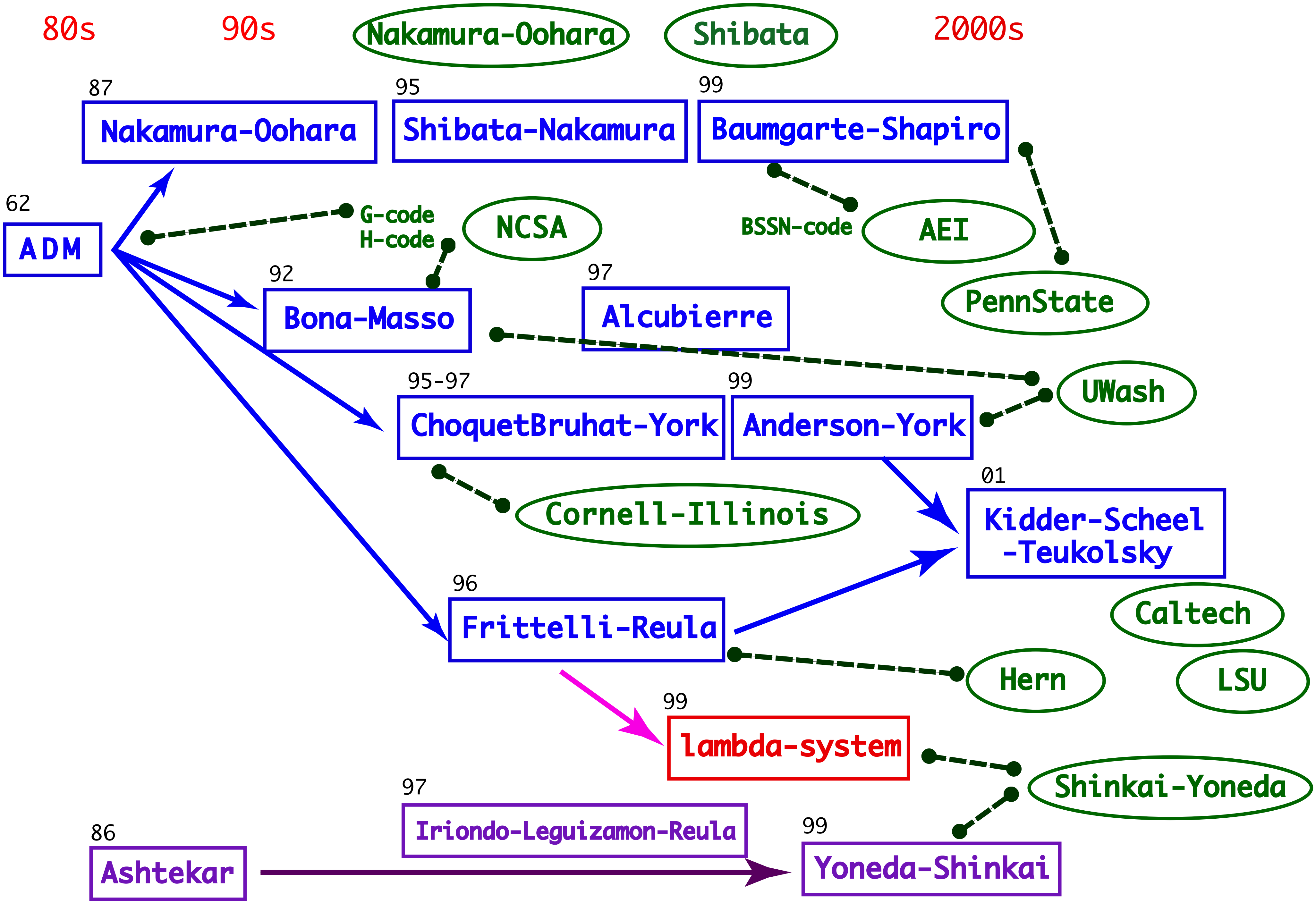
Expectations

– Wellposed behaviour

symmetric hyperbolic system $\Longrightarrow$ WELL-POSED , $||u(t)|| \leq e^{\kappa t}||u(0)||$

– Better boundary treatments $\Longleftarrow$ $\exists$ characteristic field.

– known numerical techniques in Newtonian hydrodynamics.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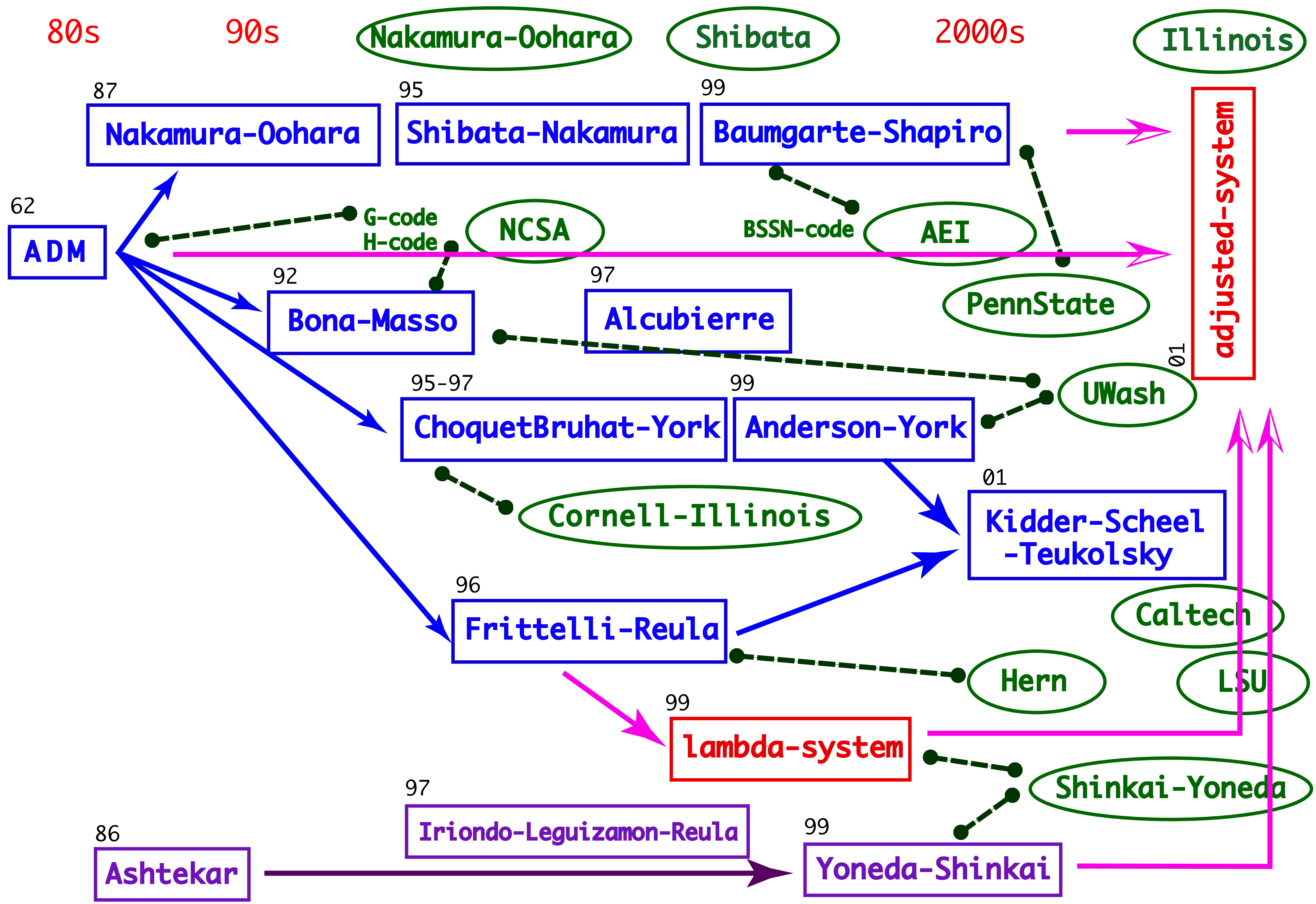
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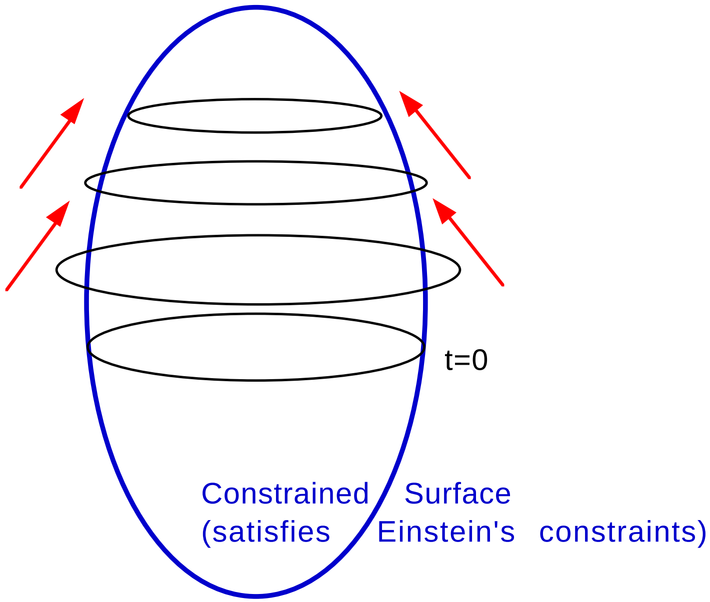
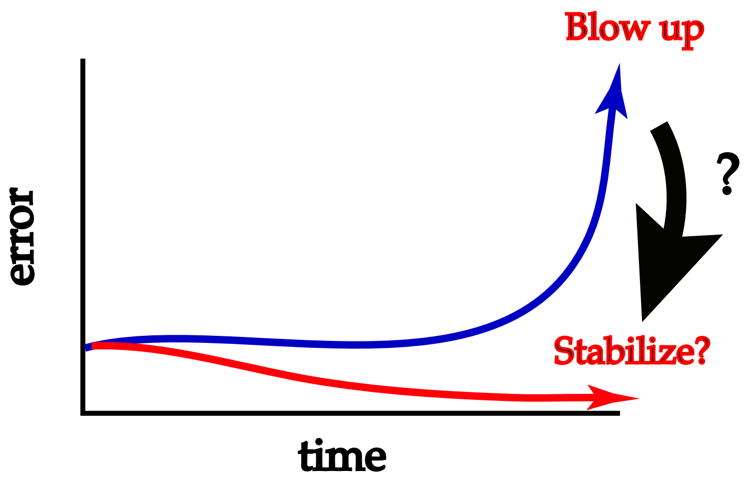
strategy 3 Formulate a system which is "asymptotically constrained" against a violation of constraints

"Asymptotically Constrained System"– Constraint Surface as an Attractor





method 1: λ-system (Brodbeck et al, 2000)

- Add aritificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (HS-Yoneda, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may prodict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. $\Rightarrow$ for the ADM/BSSN formulation, too!!

Idea of λ-system

Brodbeck, Frittelli, Hübner and Reula, JMP40(99)909

We expect a system that is robust for controlling the violation of constraints

Recipe

1. Prepare a symmetric hyperbolic evolution system $\qquad \partial_t u = J\partial_i u + K$
2. Introduce λ as an indicator of violation of constraint which obeys dissipative eqs. of motion $\qquad \partial_t \lambda = \alpha C - \beta \lambda \quad (\alpha \neq 0, \beta > 0)$
3. Take a set of (u, λ) as dynamical variables $\qquad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$
4. Modify evolution eqs so as to form a symmetric hyperbolic system $\qquad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$

Remarks

- BFHR used a sym. hyp. formulation by Frittelli-Reula [PRL76(96)4667]
- The version for the Ashtekar formulation by HS-Yoneda [PRD60(99)101502] for controlling the constraints or reality conditions or both.
- Succeeded in evolution of GW in planar spacetime using Ashtekar vars. [CQG18(2001)441]
- Do the recovered solutions represent true evolution? by Siebel-Hübner [PRD64(2001)024021]

Idea of "Adjusted system" and Our Conjecture

CQG18 (2001) 441, PRD 63 (2001) 120419, CQG 19 (2002) 1027

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) \underbrace{+F(C^a, \partial_b C^a, \cdots)}$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$
   to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \cdots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) \underbrace{+G(C^a, \partial_b C^a, \cdots)}$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = \underbrace{A(\hat{C}^a)}\, \hat{C}^k$

Conjecture: Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs.
If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

The Adjusted system (essentials):

Purpose:	Control the violation of constraints by reformulating the system so as to have a constrained surface an attractor.
Procedure:	Add constraints to evolution eqs, and adjust its multipliers.
Theoretical support:	Eigenvalue analysis of the constraint propagation equations.
Advantages:	Available even if the base system is not a symmetric hyperbolic.
Advantages:	Keep the number of the variable same with the original system.

Conjecture on Constraint Amplification Factors (CAFs):

$$\partial_t \begin{pmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{pmatrix} = \begin{pmatrix} \text{Constraint} \\ \text{Propagation} \\ \text{Matrix} \end{pmatrix} \begin{pmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{pmatrix},$$

Eigenvalues = CAFs

We see more stable evolution, if CAFs have

(A) negative real-part (the constraints are forced to be diminished), or

(B) non-zero imaginary-part (the constraints are propagating away).

Example: the Maxwell equations

Yoneda HS, CQG 18 (2001) 441

Maxwell evolution equations.

$$
\begin{aligned}
\partial_t E_i &= c\epsilon_i{}^{jk}\partial_j B_k + P_i C_E + Q_i C_B, \\
\partial_t B_i &= -c\epsilon_i{}^{jk}\partial_j E_k + R_i C_E + S_i C_B, \\
& C_E = \partial_i E^i \approx 0, \quad C_B = \partial_i B^i \approx 0,
\end{aligned}
\qquad
\begin{cases}
\text{sym. hyp} & \Leftrightarrow \quad P_i = Q_i = R_i = S_i = 0, \\
\text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\
\text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0
\end{cases}
$$

Constraint propagation equations

$$
\begin{aligned}
\partial_t C_E &= (\partial_i P^i)C_E + P^i(\partial_i C_E) + (\partial_i Q^i)C_B + Q^i(\partial_i C_B), \\
\partial_t C_B &= (\partial_i R^i)C_E + R^i(\partial_i C_E) + (\partial_i S^i)C_B + S^i(\partial_i C_B), \\
& \begin{cases}
\text{sym. hyp} & \Leftrightarrow \quad Q_i = R_i, \\
\text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\
\text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0
\end{cases}
\end{aligned}
$$

CAFs?

$$
\begin{aligned}
\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} &= \begin{pmatrix} \partial_i P^i + P^i k_i & \partial_i Q^i + Q^i k_i \\ \partial_i R^i + R^i k_i & \partial_i S^i + S^i k_i \end{pmatrix} \partial_l \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} \approx \begin{pmatrix} P^i k_i & Q^i k_i \\ R^i k_i & S^i k_i \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} =: T \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} \\
&\Rightarrow \ \text{CAFs} = (P^i k_i + S^i k_i \pm \sqrt{(P^i k_i + S^i k_i)^2 + 4(Q^i k_i R^j k_j - P^i k_i S^j k_j)})/2
\end{aligned}
$$

Therefore CAFs become negative-real when

$$
P^i k_i + S^i k_i < 0, \qquad \text{and} \qquad Q^i k_i R^j k_j - P^i k_i S^j k_j < 0
$$

1. Introduction: General Relativity and Numerical Relativity
2. Formulation problem of Numerical Relativity
   (0) Arnowitt-Deser-Misner
   (1) Baumgarte-Shapiro-Shibata-Nakamura formulation
   (2) Hyperbolic formulations
   (3) Attractor systems
3. "Adjusted Systems"
   Asymptotically constrained system by adjusting evolution eqs.
   based on Constraint Propagation analysis
4. Summary and Future Issues

3 Adjusted ADM systems

We adjust the standard ADM system using constraints as:

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, && (1)\\
&\quad + P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (2)\\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, && (3)\\
&\quad + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (4)
\end{aligned}
$$

with constraint equations

$$
\begin{aligned}
\mathcal{H} &:= R^{(3)} + K^2 - K_{ij} K^{ij}, && (5)\\
\mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K. && (6)
\end{aligned}
$$

We can write the adjusted constraint propagation equations as

$$
\begin{aligned}
\partial_t \mathcal{H} &= \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], && (7)\\
\partial_t \mathcal{M}_i &= \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. && (8)
\end{aligned}
$$

Original ADM vs Standard ADM

Original ADM (ADM, 1962) the pair of (h_{ij}, π^{ij}).

$$\begin{aligned}\mathcal{L} &= \sqrt{-g}R = \sqrt{h}N[{}^{(3)}R - K^2 + K_{ij}K^{ij}], \qquad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}), \\ \mathcal{H} &= \pi^{ij}\dot{h}_{ij} - \mathcal{L}\end{aligned}$$

$$\begin{cases} \partial_t h_{ij} = \dfrac{\delta \mathcal{H}}{\delta \pi^{ij}} = 2\dfrac{N}{\sqrt{h}}(\pi_{ij} - \dfrac{1}{2}h_{ij}\pi) + 2D_{(i}N_{j)}, \\ \partial_t \pi^{ij} = -\dfrac{\delta \mathcal{H}}{\delta h_{ij}} = -\sqrt{h}N({}^{(3)}R^{ij} - \dfrac{1}{2}{}^{(3)}Rh^{ij}) + \dfrac{1}{2}\dfrac{N}{\sqrt{h}}h^{ij}(\pi_{mn}\pi^{mn} - \dfrac{1}{2}\pi^2) - 2\dfrac{N}{\sqrt{h}}(\pi^{in}\pi_n{}^j - \dfrac{1}{2}\pi\pi^{ij}) \\ \qquad +\sqrt{h}(D^iD^jN - h^{ij}D^mD_mN) + \sqrt{h}D_m(h^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)} \end{cases}$$

Standard ADM (York, 1979) the pair of (h_{ij}, K_{ij}).

$$\begin{cases} \partial_t h_{ij} = -2NK_{ij} + D_jN_i + D_iN_j, \\ \partial_t K_{ij} = N({}^{(3)}R_{ij} + KK_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN + (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} \end{cases}$$

In this converting process, $\mathcal{H}$ was used.
That is, the standard ADM is already adjusted.

Constraint propagation of ADM systems

(1) Original ADM vs Standard ADM

With the adjustment $\underline{\ R_{ij} = \kappa_1 \alpha \gamma_{ij}\ \text{ and other multiplier zero,}}$ where $\kappa_1 = \begin{cases} 0 & \text{the standard ADM} \\ -1/4 & \text{the original ADM} \end{cases}$

- The constraint propagation eqs keep the first-order form (cf Frittelli, PRD55(97)5992):

$$\partial_t \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} \simeq \begin{pmatrix} \beta^l & -2\alpha\gamma^{jl} \\ -(1/2)\alpha\delta^l_i + R^l{}_i - \delta^l_i R & \beta^l \delta^j_i \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_j \end{pmatrix}. \tag{1}$$

The eigenvalues of the characteristic matrix:

$$\lambda^l = (\beta^l, \beta^l, \beta^l \pm \sqrt{\alpha^2 \gamma^{ll}(1 + 4\kappa_1)})$$

The hyperbolicity of (1): $\begin{cases} \text{symmetric hyperbolic} & \text{when } \kappa_1 = 3/2 \\ \text{strongly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) > 0 \\ \text{weakly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) \geq 0 \end{cases}$

- On the Minkowskii background metric, the linear order terms of the Fourier-transformed constraint propagation equations gives the eigenvalues

$$\Lambda^l = (0, 0, \pm\sqrt{-k^2(1 + 4\kappa_1)}).$$

That is, $\begin{cases} \text{(two 0s, two pure imaginary)} & \text{for the standard ADM} \\ \text{(four 0s)} & \text{for the original ADM} \end{cases}$ BETTER STABILITY

Example 1: standard ADM vs original ADM (in Schwarzschild coordinate)

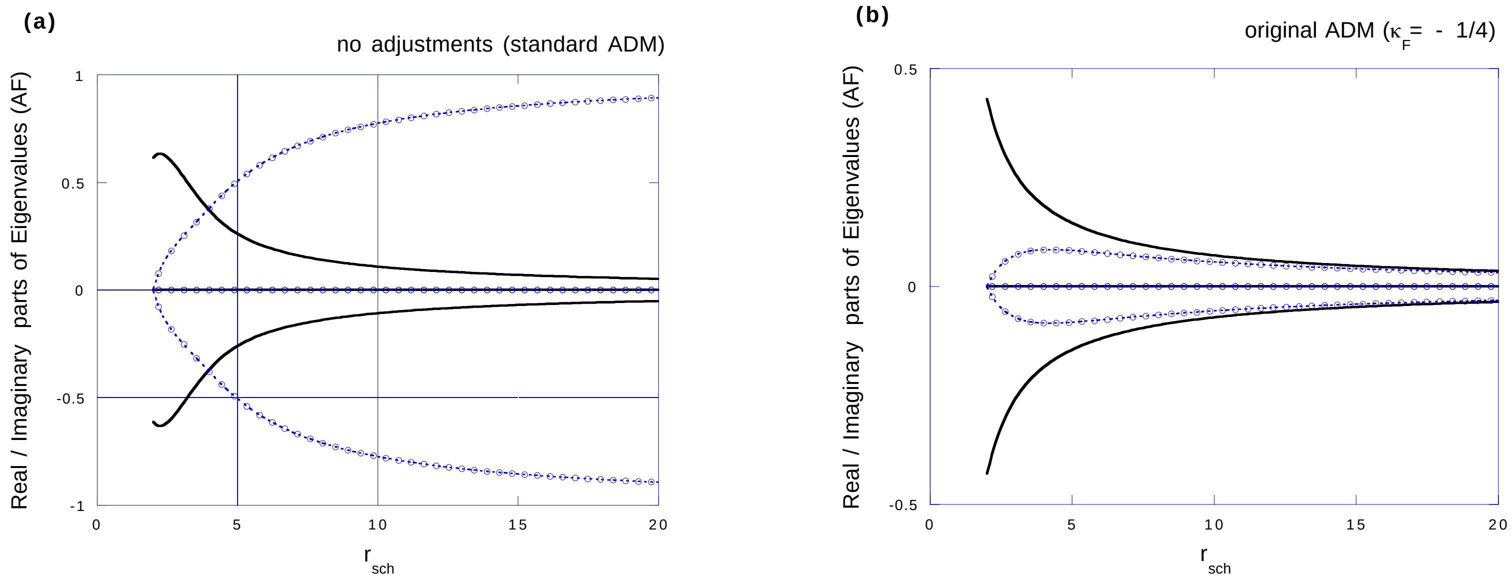


Figure 1: Amplification factors (AFs, eigenvalues of homogenized constraint propagation equations) are shown for the standard Schwarzschild coordinate, with (a) no adjustments, i.e., standard ADM, (b) original ADM ($\kappa_F = -1/4$). The solid lines and the dotted lines with circles are real parts and imaginary parts, respectively. They are four lines each, but actually the two eigenvalues are zero for all cases. Plotting range is $2 < r \leq 20$ using Schwarzschild radial coordinate. We set $k = 1, l = 2$, and $m = 2$ throughout the article.

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + \kappa_F \alpha \gamma_{ij} \mathcal{H},
\end{aligned}
$$

Constraint propagation of ADM systems

(2) Detweiler's system

Detweiler's modification to ADM [PRD35(87)1095] can be realized in our notation as:

$$\begin{aligned}
P_{ij} &= -L\alpha^3\gamma_{ij}, \\
R_{ij} &= L\alpha^3(K_{ij} - (1/3)K\gamma_{ij}), \\
S^k_{ij} &= L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}], \\
s^{kl}_{ij} &= L\alpha^3[2\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}],
\end{aligned}$$

and else zero, where L is a constant.

- This adjustment does not make constraint propagation equation in the first order form, so that we can not discuss the hyperbolicity nor the characteristic speed of the constraints.
- For the Minkowskii background spacetime, the adjusted constraint propagation equations with above choice of multiplier become

$$\begin{aligned}
\partial_t\mathcal{H} &= -2(\partial_j\mathcal{M}_j) + 4L(\partial_j\partial_j\mathcal{H}), \\
\partial_t\mathcal{M}_i &= -(1/2)(\partial_i\mathcal{H}) + (L/2)(\partial_k\partial_k\mathcal{M}_i) + (L/6)(\partial_i\partial_k\mathcal{M}_k).
\end{aligned}$$

Constraint Amp. Factors (the eigenvalues of their Fourier expression) are

$$\Lambda^l = (-(L/2)k^2(\text{multiplicity 2}), -(7L/3)k^2 \pm (1/3)\sqrt{k^2(-9+25L^2k^2)}.)$$

This indicates negative real eigenvalues if we chose small positive L.

Detweiler's criteria vs Our criteria

- Detweiler calculated the L2 norm of the constraints, C_α, over the 3-hypersurface and imposed its negative definiteness of its evolution,

$$\text{Detweiler's criteria} \;\Leftrightarrow\; \partial_t \int \sum_\alpha C_\alpha^2 \, dV < 0,$$

This is rewritten by supposing the constraint propagation to be $\partial_t \hat{C}_\alpha = A_\alpha{}^\beta \hat{C}_\beta$ in the Fourier components,

$$\Leftrightarrow \quad \partial_t \int \sum_\alpha \hat{C}_\alpha \bar{\hat{C}}_\alpha \, dV = \int \sum_\alpha A_\alpha{}^\beta \hat{C}_\beta \bar{\hat{C}}_\alpha + \hat{C}_\alpha \bar{A}_\alpha{}^\beta \bar{\hat{C}}_\beta \, dV < 0, \; \forall \text{ non zero } \hat{C}_\alpha$$

$$\Leftrightarrow \quad \underline{\text{eigenvalues of } (A + A^\dagger) \text{ are all negative for } \forall k}.$$

- Our criteria is that the eigenvalues of A are all negative. Therefore,

$$\boxed{\text{Our criteria} \ni \text{Detweiler's criteria}}$$

- We remark that Detweiler's truncations on higher order terms in C-norm corresponds our perturbative analysis, both based on the idea that the deviations from constraint surface (the errors expressed non-zero constraint value) are initially small.

Example 2: Detweiler-type adjusted (in Schwarzschild coord.)

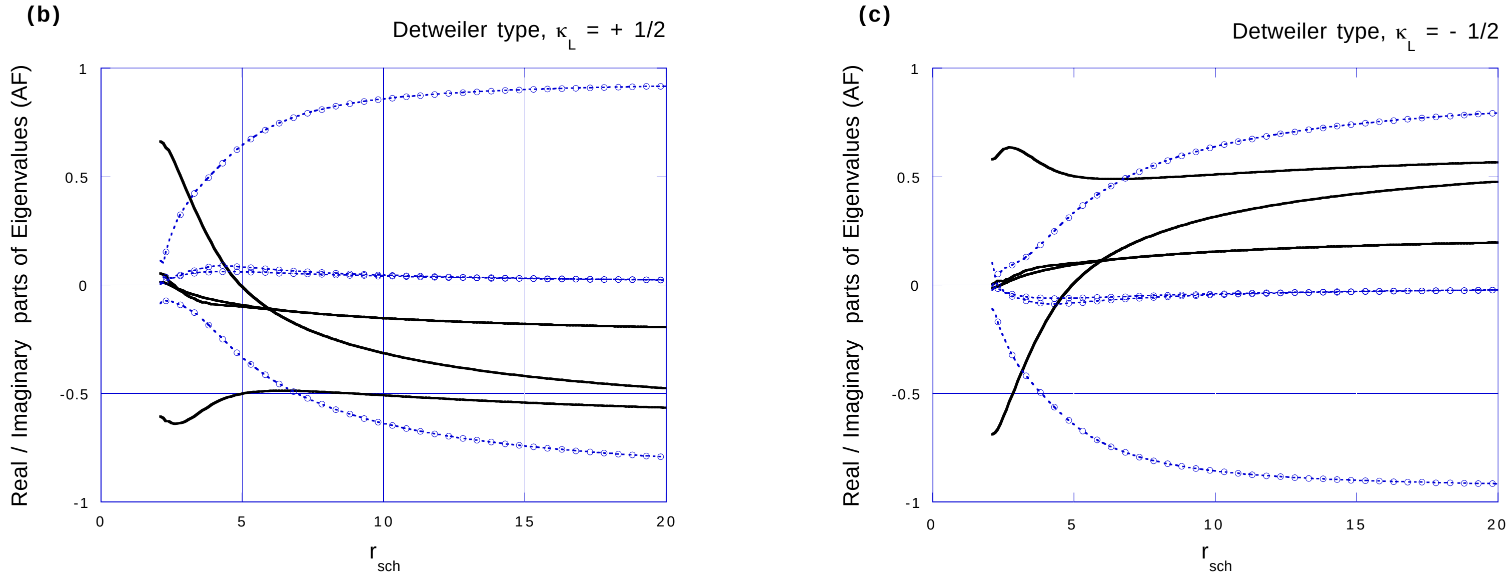
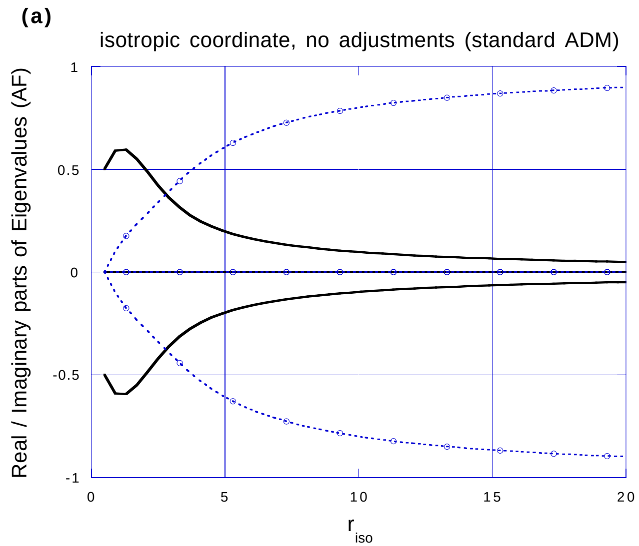


Figure 2: Amplification factors of the standard Schwarzschild coordinate, with Detweiler type adjustments. Multipliers used in the plot are (b) $\kappa_L = +1/2$, and (c) $\kappa_L = -1/2$.

$$
\begin{aligned}
\partial_t \gamma_{ij} &= \text{(original terms)} + P_{ij}\mathcal{H}, \\
\partial_t K_{ij} &= \text{(original terms)} + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \\
& \text{where} \quad P_{ij} = -\kappa_L \alpha^3 \gamma_{ij}, \quad R_{ij} = \kappa_L \alpha^3 (K_{ij} - (1/3)K\gamma_{ij}), \\
& \qquad\quad S^k{}_{ij} = \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], \quad s^{kl}{}_{ij} = \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}],
\end{aligned}
$$

Example 3: standard ADM (in isotropic/iEF coord.)



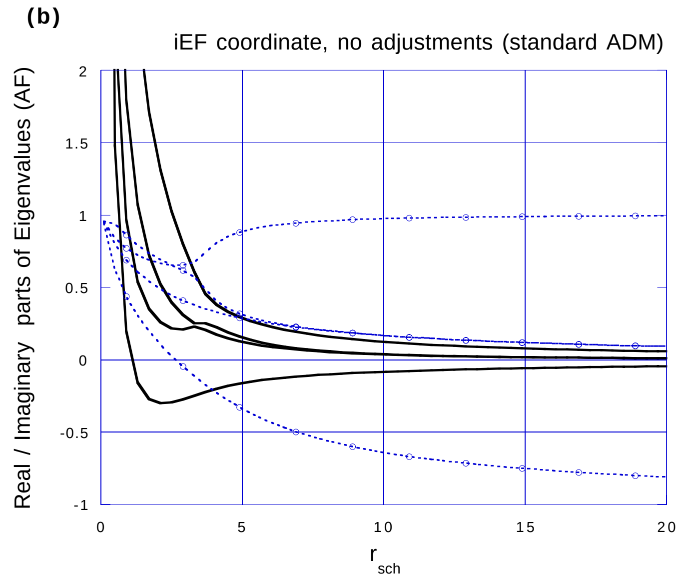


Figure 3: Comparison of amplification factors between different coordinate expressions for the standard ADM formulation (i.e. no adjustments). Fig. (a) is for the isotropic coordinate (1), and the plotting range is $1/2 \leq r_{iso}$. Fig. (b) is for the iEF coordinate (1) and we plot lines on the $t = 0$ slice for each expression. The solid four lines and the dotted four lines with circles are real parts and imaginary parts, respectively.

Example 4: Detweiler-type adjusted (in iEF/PG coord.)

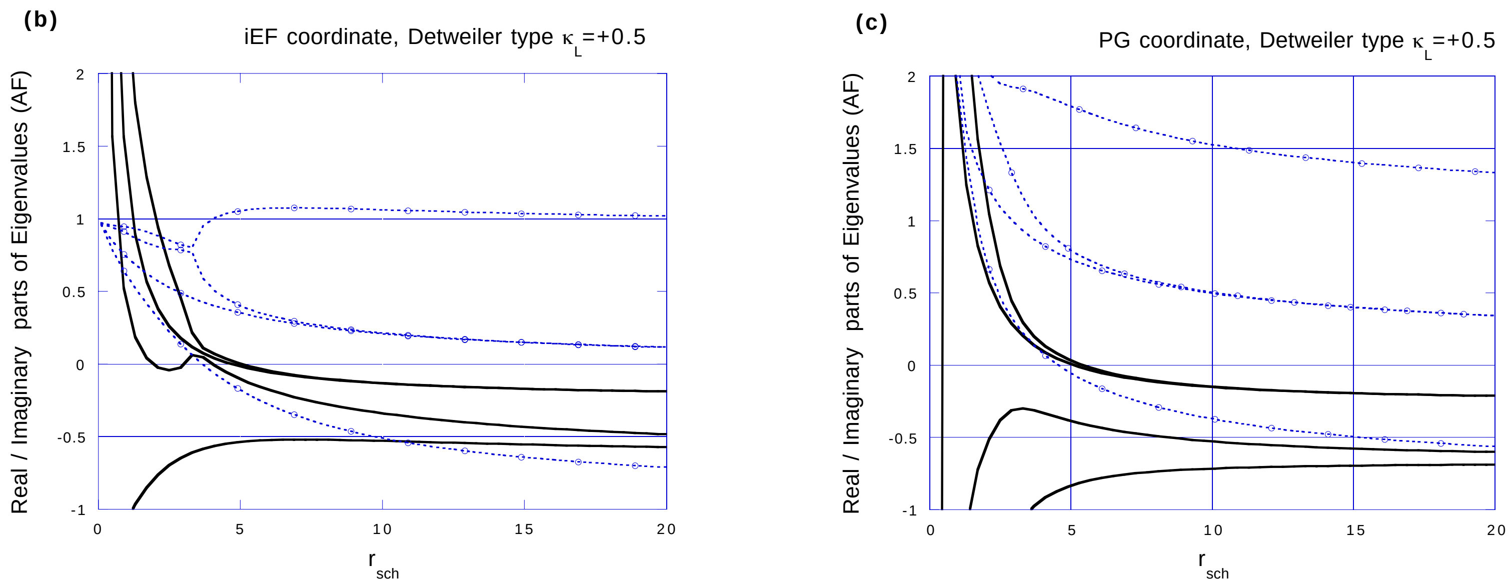


Figure 4: Similar comparison for Detweiler adjustments. $\kappa_L = +1/2$ for all plots.

Table 3. List of adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers in terms of (13) and (14). The column '1st?' and 'TRS' are the same as in table 1. The effects to amplification factors (when $\kappa > 0$) are commented for each coordinate system and for real/imaginary parts of AFs, respectively. The 'N/A' means that there is no effect due to the coordinate properties; 'not apparent' means the adjustment does not change the AFs effectively according to our conjecture; 'enl./red./min.' means enlarge/reduce/minimize, and 'Pos./Neg.' means positive/negative, respectively. These judgements are made at the $r \sim O(10M)$ region on their $t = 0$ slice.

					Schwarzschild/isotropic coordinates			iEF/PG coordinates	
No	No in table 1		Adjustment	1st?	TRS	Real	Imaginary	Real	Imaginary
0	0	–	no adjustments	yes	–	–	–	–	–
P-1	2-P	P_{ij}	$-\kappa_L\alpha^3\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-2	3	P_{ij}	$-\kappa_L\alpha\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-3	–	P_{ij}	$P_{rr} = -\kappa$ or $P_{rr} = -\kappa\alpha$	no	no	slightly enl.Neg.	not apparent	slightly enl.Neg.	not apparent
P-4	–	P_{ij}	$-\kappa\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-5	–	P_{ij}	$-\kappa\gamma_{rr}$	no	no	red. Pos./enl.Neg.	not apparent	red.Pos./enl.Neg.	not apparent
Q-1	–	$Q^k{}_{ij}$	$\kappa\alpha\beta^k\gamma_{ij}$	no	no	N/A	N/A	$\kappa \sim 1.35$ min. vals.	not apparent
Q-2	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
Q-3	–	$Q^k{}_{ij}$	$Q^r{}_{ij} = \kappa\gamma_{ij}$ or $Q^r{}_{ij} = \kappa\alpha\gamma_{ij}$	no	yes	red. abs vals.	not apparent	enl.Neg.	enl. vals.
Q-4	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa\gamma_{rr}$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
R-1	1	R_{ij}	$\kappa_F\alpha\gamma_{ij}$	yes	yes	$\kappa_F = -1/4$ min. abs vals.		$\kappa_F = -1/4$ min. vals.	
R-2	4	R_{ij}	$R_{rr} = -\kappa_\mu\alpha$ or $R_{rr} = -\kappa_\mu$	yes	no	not apparent	not apparent	red.Pos./enl.Neg.	enl. vals.
R-3	–	R_{ij}	$R_{rr} = -\kappa\gamma_{rr}$	yes	no	enl. vals.	not apparent	red.Pos./enl.Neg.	enl. vals.
S-1	2-S	$S^k{}_{ij}$	$\kappa_L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}]$	yes	no	not apparent	not apparent	not apparent	not apparent
S-2	–	$S^k{}_{ij}$	$\kappa\alpha\gamma^{lk}(\partial_l\gamma_{ij})$	yes	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
p-1	–	$p^k{}_{ij}$	$p^r{}_{ij} = -\kappa\alpha\gamma_{ij}$	no	no	red. Pos.	red. vals.	red. Pos.	enl. vals.
p-2	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa\alpha$	no	no	red. Pos.	red. vals.	red.Pos/enl.Neg.	enl. vals.
p-3	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	enl. vals.	red. Pos. vals.	red. vals.
q-1	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{ij} = \kappa\alpha\gamma_{ij}$	no	no	$\kappa = 1/2$ min. vals.	red. vals.	not apparent	enl. vals.
q-2	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	not apparent	not apparent	not apparent
r-1	–	$r^k{}_{ij}$	$r^r{}_{ij} = \kappa\alpha\gamma_{ij}$	no	yes	not apparent	not apparent	not apparent	enl. vals.
r-2	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa\alpha$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
r-3	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
s-1	2-s	$s^{kl}{}_{ij}$	$\kappa_L\alpha^3[\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}]$	no	no	makes 4 Neg.	not apparent	makes 4 Neg.	not apparent
s-2	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{ij} = -\kappa\alpha\gamma_{ij}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.
s-3	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.

HS and Yoneda, Classical Quantum Gravity, 19 (2002) 1027

Comparisons of Adjusted ADM systems (Teukolsky wave)

3-dim, harmonic slice, periodic BC HS original Cactus/GR code

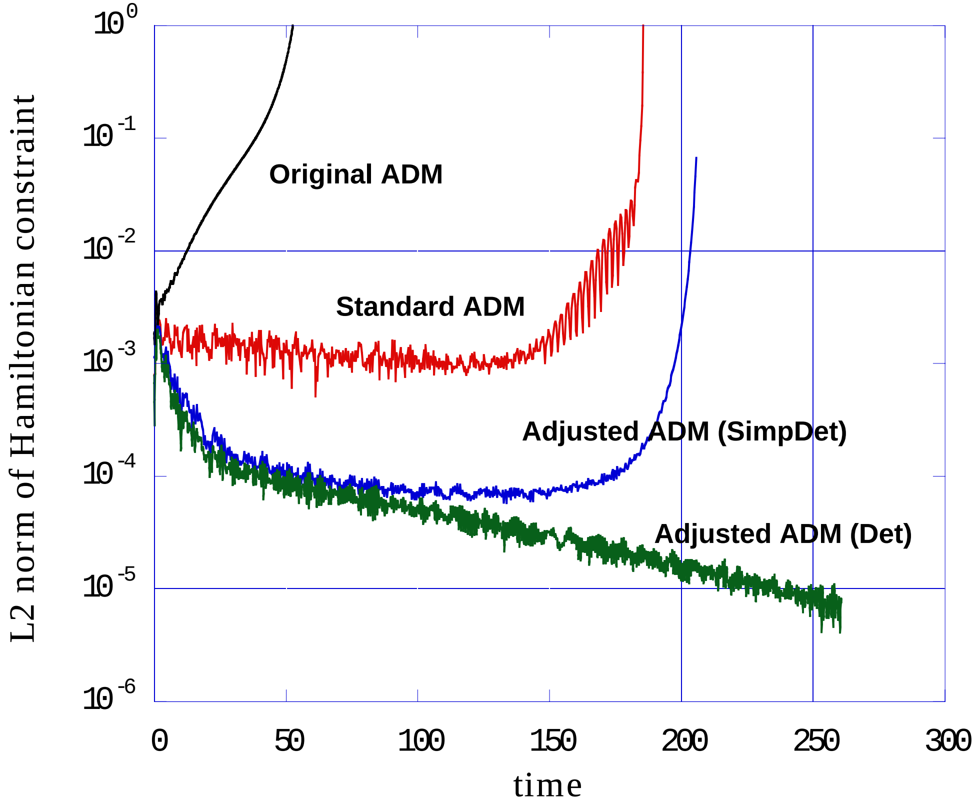


Figure 1: Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR/evolveADMeq code was used. Grid $= 24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method.

"Einstein equations" are time-reversal invariant. So ...

Why all negative amplification factors (AFs) are available?

Explanation by the time-reversal invariance (TRI)

- the adjustment of the system I,

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{K_{ij}}_{(-)} = \kappa_1 \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

preserves TRI. ... so the AFs remain zero (unchange).

- the adjustment by (a part of) Detweiler

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{\gamma_{ij}}_{(+)} = -L \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

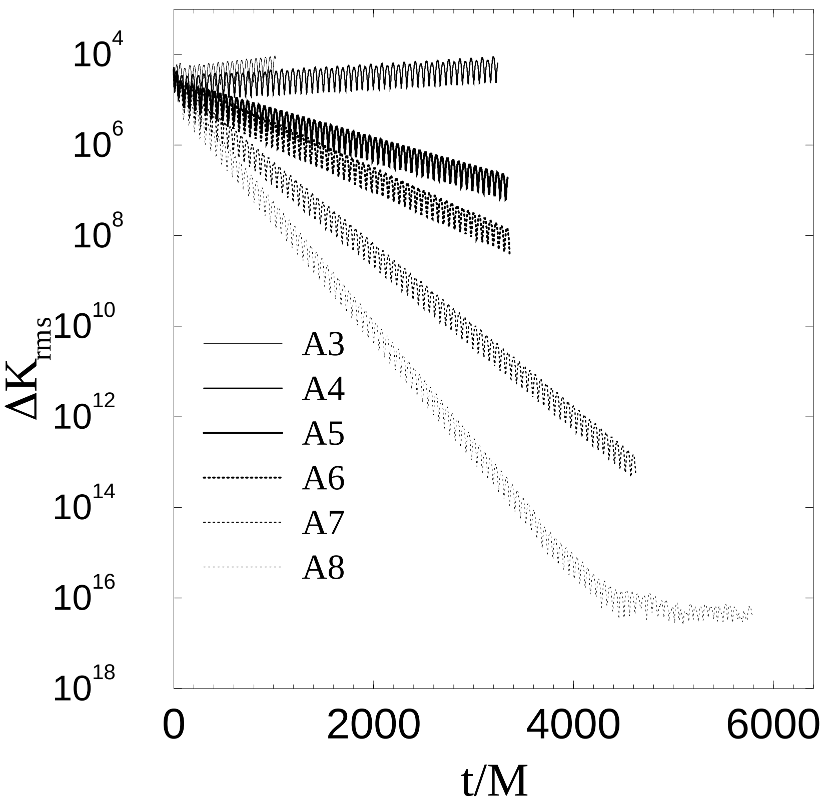
violates TRI. ... so the AFs can become negative.

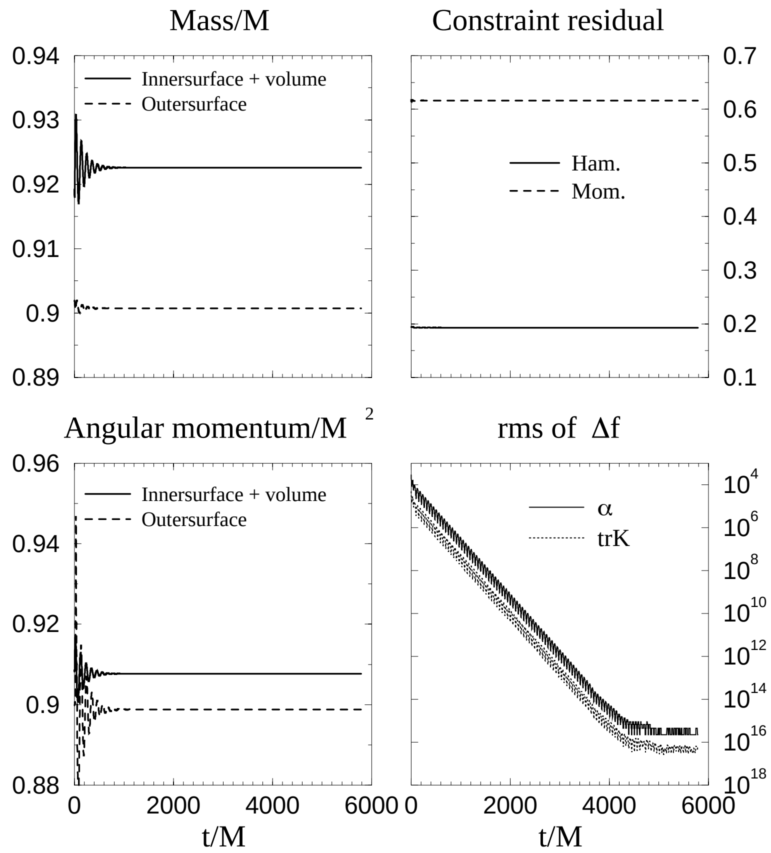
Therefore

$$\boxed{\text{We can break the time-reversal invariant feature of the ``ADM equations''.}}$$

An Evolution of Adjusted BSSN Formulation

by Yo-Baumgarte-Shapiro, PRD 66 (2002) 084026





Kerr-Schild BH (0.9 J/M), excision with cube, $1 + \log$-lapse, Γ-driver shift.

$$\partial_t \tilde{\Gamma}^i = (\cdots) + \frac{2}{3}\tilde{\Gamma}^i \beta^i{}_{,j} - (\chi + \frac{2}{3})\mathcal{G}^i \beta^j{}_{,j} \qquad \chi = 2/3 \ \ \textbf{for (A4)-(A8)}$$

$$\partial_t \tilde{\gamma}_{ij} = (\cdots) - \kappa\alpha\tilde{\gamma}_{ij}\mathcal{H} \qquad \kappa = 0.1 \sim 0.2 \ \ \textbf{for (A5), (A6) and (A8)}$$

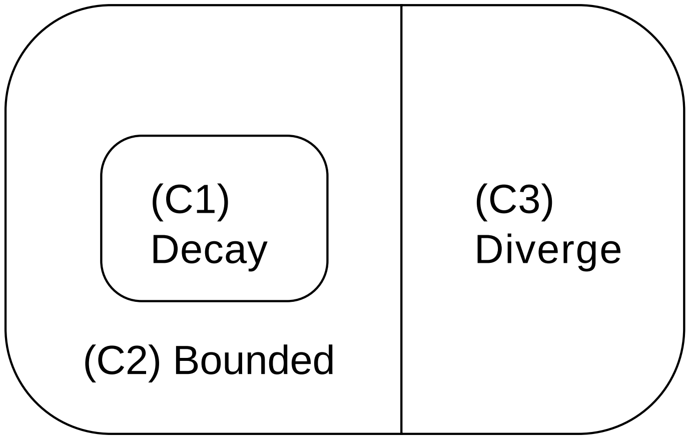
A Classification of Constraint Propagations

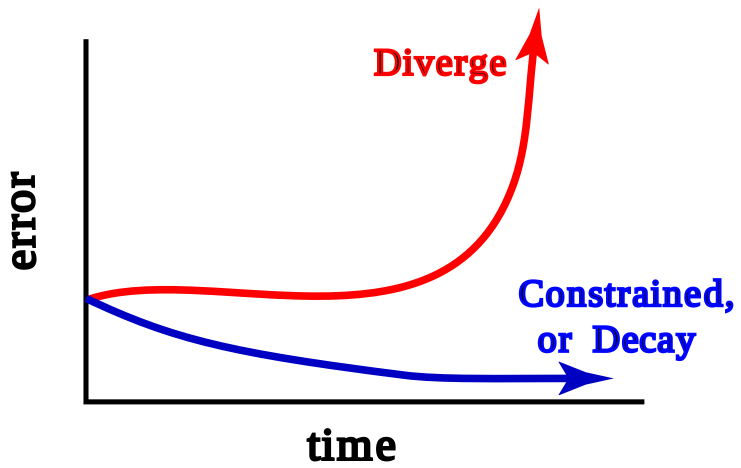
(C1) Asymptotically constrained :
Violation of constraints decays (converges to zero).

(C2) Asymptotically bounded :
Violation of constraints is bounded at a certain value.

(C3) Diverge :
At least one constraint will diverge.

Note that (C1) $\subset$ (C2).





A Classification of Constraint Propagations (cont.)

CQG 20 (2003) L31

(C1) Asymptotically constrained :

Violation of constraints decays (converges to zero).

> $\Leftrightarrow$ **All the real parts of CAFs are negative.**

(C2) Asymptotically bounded :

Violation of constraints is bounded at a certain value.

> $\Leftrightarrow$
>
> **(a) All the real parts of CAFs are not positive, and**
>
> **(b1) the CP matrix $M^\alpha{}_\beta$ is diagonalizable, or**
>
> **(b2) the real part of the degenerated CAFs is not zero.**

(C3) Diverge :

At least one constraint will diverge.

A flowchart to classify the fate of constraint propagation.

Q1: Is there a CAF which real part is positive?

NO / YES → **Diverge**

↓

Q2: Are all the real parts of CAFs negative?

NO / YES → **Asymptotically Constrained**

↓

Q3: Is the constraint propagation matrix diagonalizable?

NO / YES → **Asymptotically Bounded**

↓

Q4: Is a real part of the degenerated CAFs is zero?

YES / NO → **Asymptotically Bounded**

↓

Q5: Is the associated Jordan matrix diagonal?

NO / YES → **Asymptotically Bounded**

↓

Diverge

Towards a stable and accurate formulation for numerical relativity

We tried to understand the background in an unified way.

1. Our Observation = "Stability will change by adding constraints in RHS"
   - $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) \underbrace{+F(C^a, \partial_b C^a, \cdots)}$, named **"Adjusted System"**
2. Our proposal = "Evaluate eigenvalues of constraint propagation eqns"
   - $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) \underbrace{+G(C^a, \partial_b C^a, \cdots)}$
   - Fourier-mode analysis allows us to discuss lower-order terms.

$$\partial_t \begin{pmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{pmatrix} = \begin{pmatrix} \text{Constraint} \\ \text{Propagation} \\ \text{Matrix} \end{pmatrix} \begin{pmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{pmatrix}, \quad \text{Eigenvalues} = \text{CAFs}$$

   - Conjecture: Stable formulation if CAFs have more

   (A) negative real-part (the constraints are forced to be diminished), or

   (B) non-zero imaginary-part (the constraints are propagating away).

> When re-formulating the system, evaluation of CAFs may be an alternative guideline to hyperbolization.

Studies in progress ...(1)...

- **Construct a robust adjusted system**

(1) dynamic & automatic determination of κ under a suitable principle.

e.g.) Efforts in Multi-body Constrained Dynamics simulations

$$\frac{\partial}{\partial t} p_i = F_i + \lambda_a \frac{\partial C^a}{\partial x^i}, \qquad \text{with} \quad C^a(x_i, t) \approx 0$$

- **J. Baumgarte (1972, Comp. Methods in Appl. Mech. Eng.)**
  Replace a holonomic constraint $\partial_t^2 C = 0$ as $\partial_t^2 C + \alpha \partial_t C + \beta^2 C = 0$.
- **Park-Chiou (1988, J. Guidance), "penalty method"**
  Derive "stabilization eq." for Lagrange multiplier $\lambda(t)$.
- **Nagata (2002, Multibody Dyn.)**
  Introduce a scaled norm, $J = C^T SC$, apply $\partial_t J + w^2 J = 0$, and adjust $\lambda(t)$.

e.g.) Efforts in Molecular Dynamics simulations

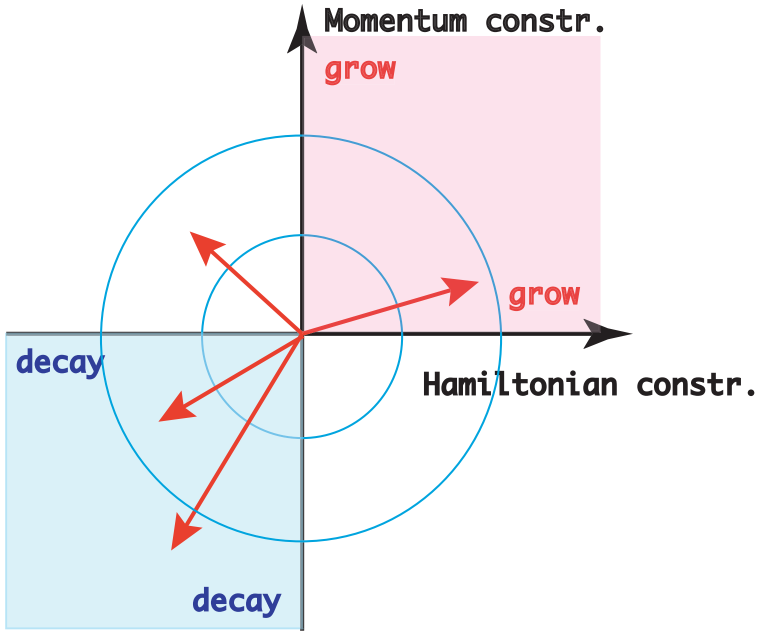
- **Constant pressure $\cdots\cdots$ potential piston!**
- **Constant temperature $\cdots\cdots$ potential thermostat!! (Nosé, 1991, PTP)**

Studies in progress ...(2)... http://atlas.riken.go.jp/~shinkai/

- **Construct a robust adjusted system**

  (2) target to control each constraint violation by adjusting multipliers.

  CP-eigenvectors indicate directions of constraint grow/decay, if CP-matrix is diagonalizable.



  (3) clarify the reasons of non-linear violation in the last stage of current test evolutions.

- **Numerical comparisons of formulations, links to other systems, ...**
  - "Comparisons of Formulations" (Mexico NR workshop, 2002), gr-qc/0305023.
  - with people here in Sydney!!