Constraint Propagation Analysis and Adjusted Systems

Gen Yoneda
Department of Mathematical Science, Waseda Univ., Japan
yoneda@waseda.jp
Hisa-aki Shinkai
Computational Sci. Div., RIKEN (The Institute of Physical and Chemical Research), Japan
hshinkai@riken.jp

Outline

- Constraint propagation analysis gives us an index of stability
- We try to unify previous treatments (such as hyperbolic and BSSN) by adjusting eqM.

Refs:

review article	gr-qc/0209111 (Nova Science Publ.)
for Ashtekar form.	PRD 60 (1999) 101502, CQG 17 (2000) 4799, CQG 18 (2001) 441
for ADM form.	PRD 63 (2001) 124019, CQG 19 (2002) 1027
for BSSN form.	PRD 66 (2002) 124003
general	CQG 20 (2003) L31

This PDF is available at
www.f.waseda.jp/yoneda/mex.pdf

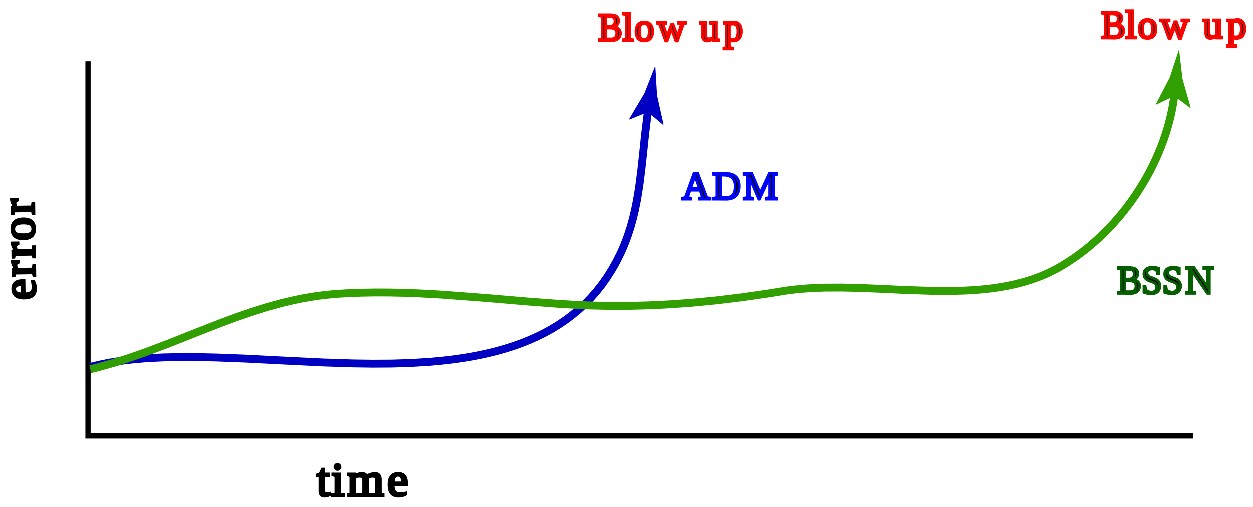
Plan of talks

1. Background of the problem
   - ADM,BSSN
   - hyperbolic
   - asymptotically constrained
2. Constraint propagation analysis
   - For time evolution systems with constraints in general
   - Constraint Amplification Factors (CAF) as an index to express stability (partially new)
   - examples (partially new)
3. adjusted systems
   - For time evolution systems with constraints in general
   - Example:Maxwell equations
   - Example:Einstein eqautions(ADM) (partially new)
   - Example:Einstein eqautions(BSSN) (partially new)
4. Numerical Demonstrations (new) $\longrightarrow$ Hisaaki
5. Discussion (new) $\longrightarrow$ Hisaaki

1 background of the problem

Best formulation of the Einstein eqs. for long-term stable & accurate simulation?

Many (too many) trials and errors, not yet a definit recipe.



Mathematically equivalent formulations, but differ in its stability!

strategy 0: Arnowitt-Deser-Misner formulation

strategy 1: Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

strategy 2: Apply a formulation which reveals a hyperbolicity explicitly

strategy 3: Formulate a system which is "asymptotically constrained" against a violation of constraints

By adding constraints in RHS, we can kill error-growing modes
$\Rightarrow$ How can we understand the features systematically?

strategy 1 Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

– define new variables (ϕ, $\tilde{\gamma}_{ij}$,K,$\tilde{A}_{ij}$,$\tilde{\Gamma}^i$), instead of the ADM's (γ_{ij},K_{ij}) where

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},$$

use momentum constraint in Γ^i-eq., and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

– The set of evolution equations become

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

$$\begin{aligned}
R_{ij} &= \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^\phi_{ij} \\
& R^\phi_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi) \\
& \tilde{R}_{ij} = -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}lm\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}
\end{aligned}$$

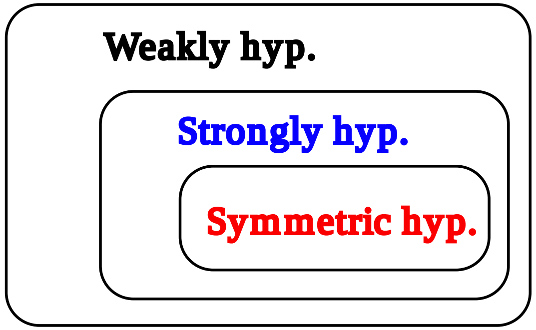
– No explicit explanations why this formulation works better.
AEI group (2000): the replacement by momentum constraint is essential.

strategy 2 <u>Apply a formulation which reveals a hyperbolicity explicitly.</u>

For a first order partial differential equations on a vector u,

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{characteristic part}} + \underbrace{B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{lower order part}}$$

	if the eigenvalues of A are
weakly hyperbolic	all real.
strongly hyperbolic	all real and $\exists$ a complete set of eigenvectors.
symmetric hyperbolic	if A is real and symmetric (Hermitian).



Expectations

– Wellposed behaviour

symmetric hyperbolic system $\Longrightarrow$ WELL-POSED , $||u(t)|| \leq e^{\kappa t}||u(0)||$

– Better boundary treatments $\Longleftarrow$ $\exists$ characteristic field.

– known numerical techniques in Newtonian hydrodynamics.

strategy2 Apply a formulation which reveals a hyperbolicity explicitly (cont.)

Are hyperbolic formluations actulay helpful in numerical simulations?

Unfortunately, we do not have conclusive answer to it yet.

Theoretical issues

- Well-posedness of the non-linear hyperbolic formulations is obtained only locally in time domain.
- Energy inequality indicates boundedness of norm which does not forbid divergence
- The discussion of hyperbolcity only uses characteristic part of evolution equations, and ignore the non-characteristic part.

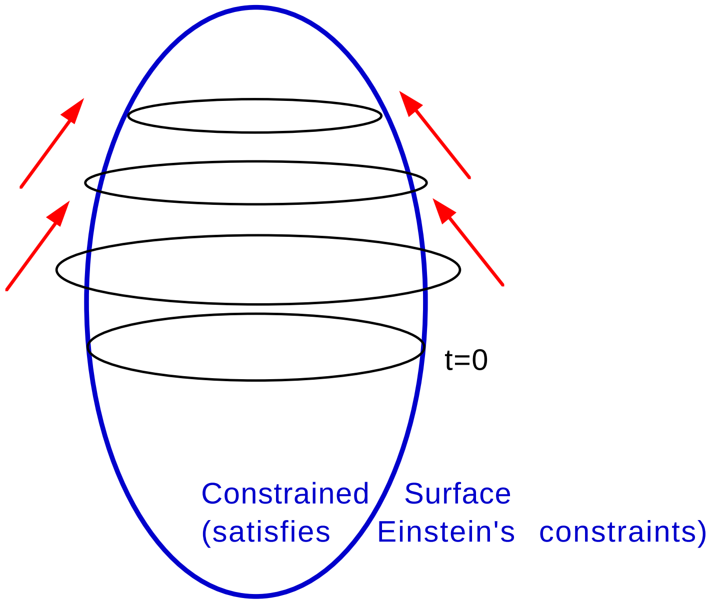
Numerical issues

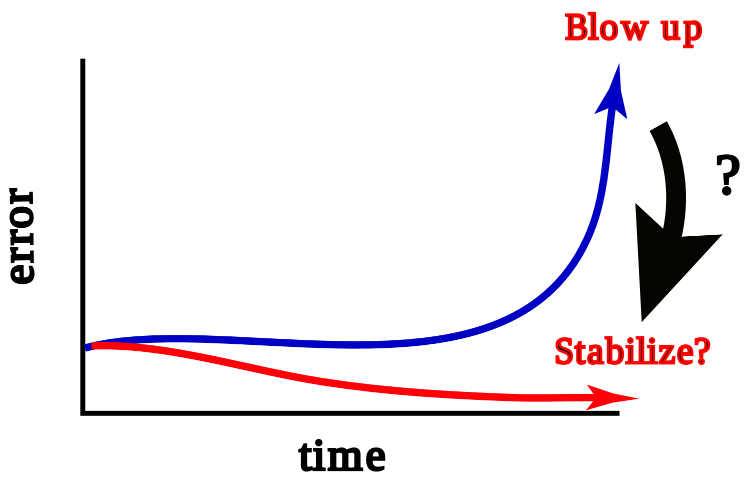
- Earlier numerical comparisons reported the advantages of hyperbolic formulations, but they were against to the standard ADM formulation. [Cornell-Illinois, NCSA, ...]
- If the gauge functions are evolved with hyperbolic equations, then their finite propagation speeds may cause a pathological shock formation [Alcubierre].
- Some group [HS-GY, Hern] reported no drastic numerical differences between three hyperbolic levels, while other group [Calabresse] reported that strongly hyperbolic is good and weakly hyperbolic is bad. Of course, these statements only cast on a particular formulations and models to apply.

Proposed symmetric hyperbolic systems were not always the best one for numerics.

strategy 3 Formulate a system which is "asymptotically constrained" against a violation of constraints

"Asymptotically Constrained System"– Constraint Surface as an Attractor





method 1: λ-system (Brodbeck et al, 2000)

- Add aritificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may prodict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. $\Rightarrow$ for the ADM/BSSN formulation, too!!

Idea of λ-system

Brodbeck, Frittelli, Hübner and Reula, JMP40(99)909

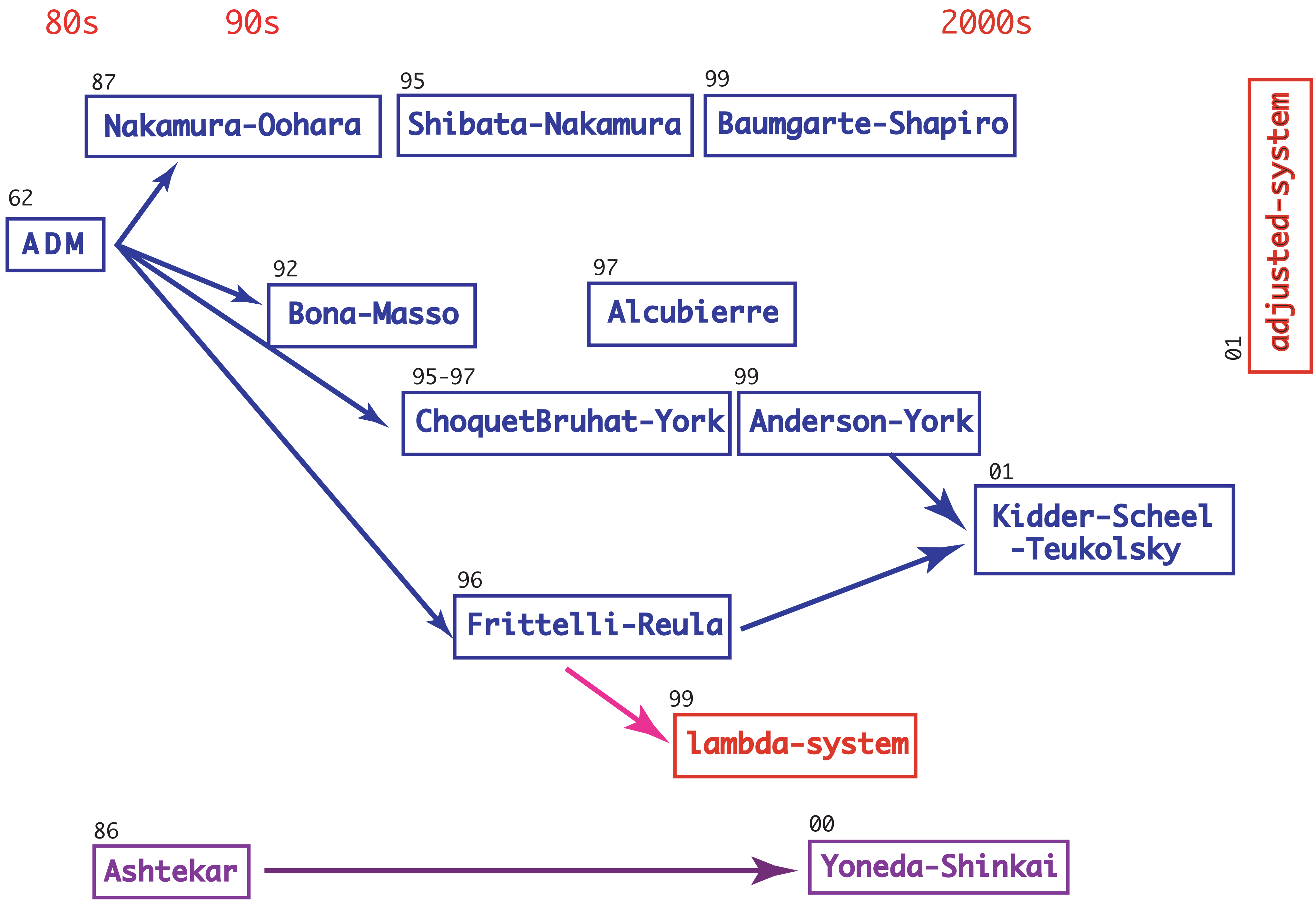
We expect a system that is robust for controlling the violation of constraints

Recipe

1. Prepare a symmetric hyperbolic evolution system $\partial_t u = J \partial_i u + K$
2. Introduce λ as an indicator of violation of constraint which obeys dissipative eqs. of motion $\partial_t \lambda = \alpha C - \beta \lambda$ $(\alpha \neq 0, \beta > 0)$
3. Take a set of (u, λ) as dynamical variables $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$
4. Modify evolution eqs so as to form a symmetric hyperbolic system $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$

Remarks

- BFHR used a sym. hyp. formulation by Frittelli-Reula [PRL76(96)4667]
- The version for the Ashtekar formulation by HS-Yoneda [PRD60(99)101502] for controlling the constraints or reality conditions or both.
- Succeeded in evolution of GW in planar spacetime using Ashtekar vars. [CQG18(2001)441]
- Do the recovered solutions represent true evolution? by Siebel-Hübner [PRD64(2001)024021]



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lambda-system
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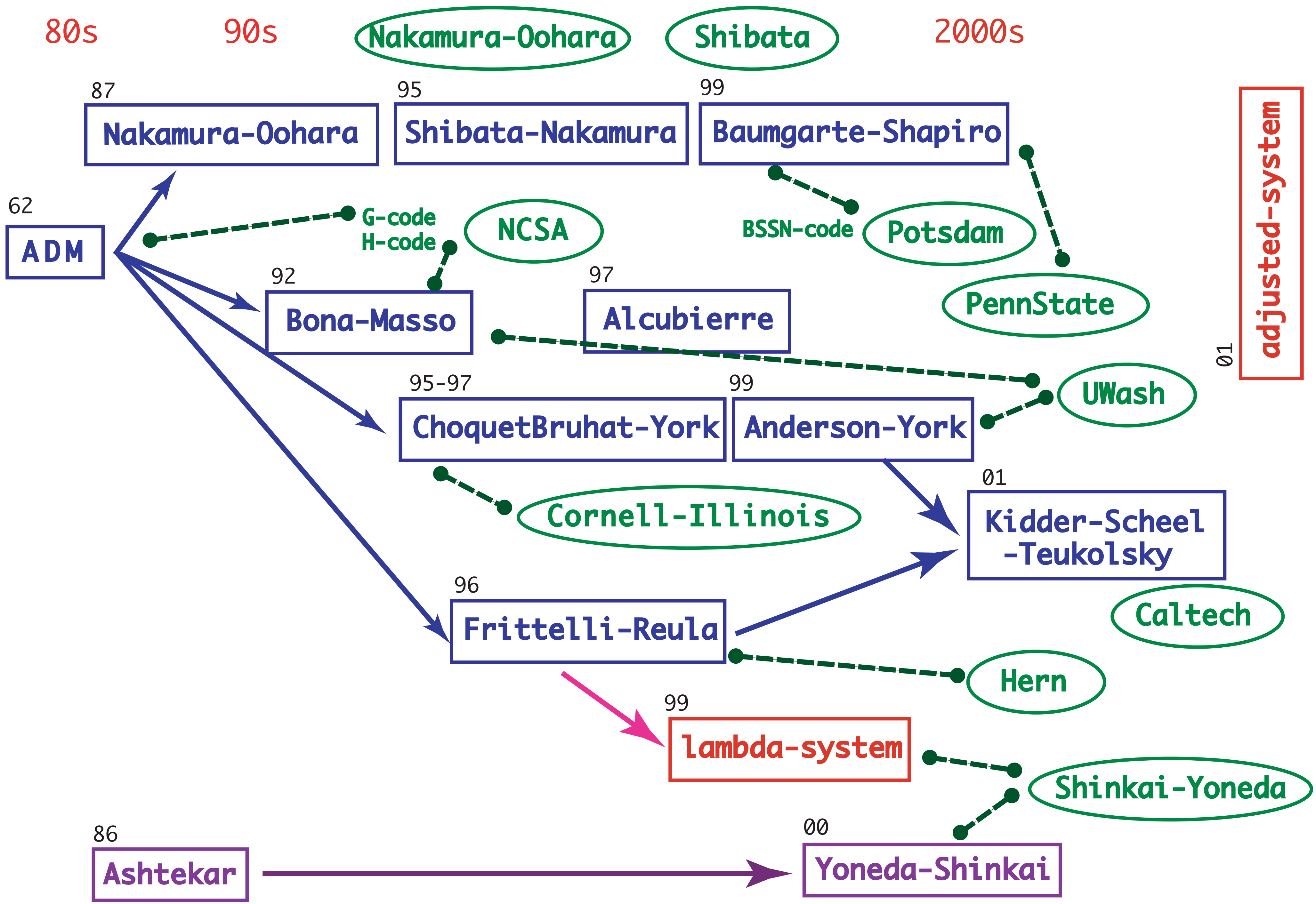
Yoneda-Shinkai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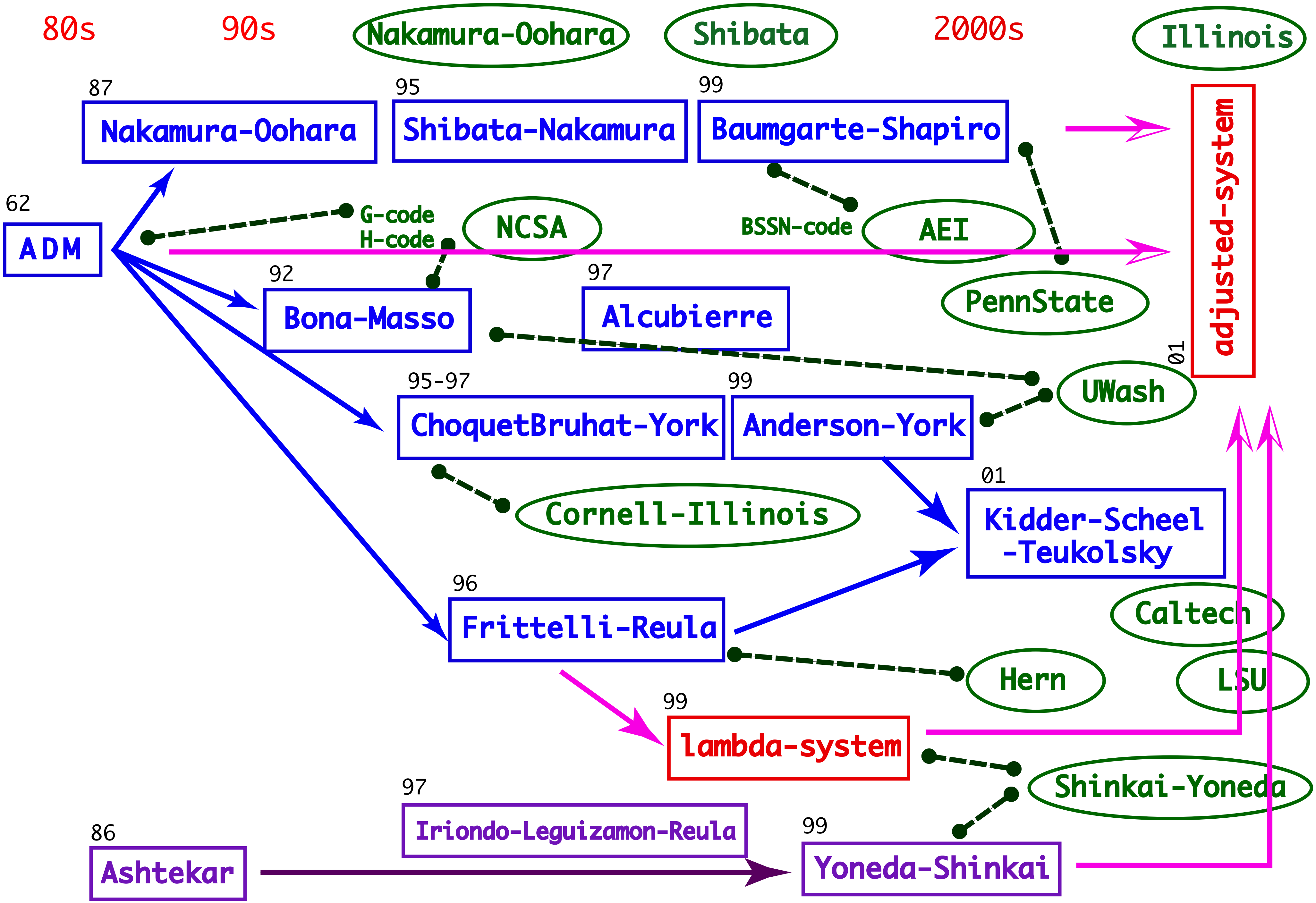

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section 1 Background of the problem

section 2 Constraint propagation analysis

- For time evolution systems with constraints in general
- Constraint Amplification Factors (CAF) as an index to express stability (partially new)
- example1:Maxwell
  example2:ADM
  example3:BSSN

section 3 Adjusted systems
section 4 Numerical experiment
section 5 Discussion

For time evolution systems with constraints in general

$$
\begin{aligned}
\partial_t u^a &= f(u^a, \partial u^a, \partial\partial u^a) \quad \text{evolution equations} \\
C^\alpha &= C^\alpha(u^a, \partial u^a, \partial\partial u^a) \approx 0 \quad \text{constraints}
\end{aligned}
$$

If constraints are first class, constraint propagation takes this form

$$
\partial_t C^\alpha = A_0 C^\alpha + A_1 \partial C^\alpha + A_2 \partial\partial C^\alpha + \cdots \quad \textbf{constraint propagation}
$$

Analytically, constraints are satisfied during evolution. But numerically, does not. By Fourier transformation, we rewrite constraint propagation with each modes, which is ODE.

$$
\begin{aligned}
\partial_t \hat{C}^\alpha &= A_0 \hat{C}^\alpha + A_1 (i\vec{k}) \hat{C}^\alpha + A_2 (i\vec{k})(i\vec{k}) \hat{C}^\alpha + \cdots \\
&= \underbrace{\left(A_0 + A_1(i\vec{k}) + A_2(i\vec{k})(i\vec{k}) + \cdots\right)}_{\text{M}} \hat{C}^\alpha \quad \text{constraint propagation 2}
\end{aligned}
$$

constraint propagation matrix

we substitute background metric into $M \to M_{bg}$

$$
\text{CAF} := \text{Eigenvalues } M_{bg} \quad \textbf{Constraint Amplification Factors}
$$

By evaluating CAFs before simulations, we can predict constraint violation in numerical evolution.

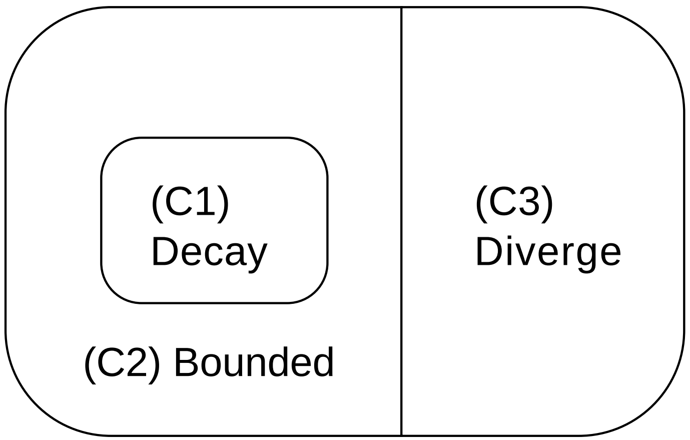
A Classification of Constraint Propagations

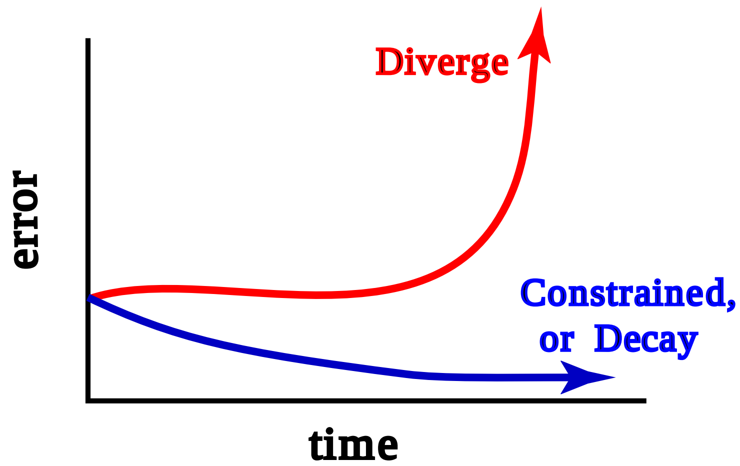
(C1) **Asymptotically constrained :**
Violation of constraints decays (converges to zero).

(C2) **Asymptotically bounded :**
Violation of constraints is bounded at a certain value.

(C3) **Diverge :**
At least one constraint will diverge.

Note that (C1) $\subset$ (C2).





A Classification of Constraint Propagations (cont.)

$\partial_t C = \lambda C \Rightarrow C = C(0) \exp(\lambda t)$

(C1) Asymptotically **constrained** : (Violation of constraints converges to zero.)
≈ all the real part of CAFs are **negative**

(C2) Asymptotically **bounded** : (Violation of constraints is bounded at a certain value.)
≈ all the real part of CAFs are **non-positive**

(C3) **Diverge**: (At least one constraint will diverge.)
≈ there exists CAF with **positive** real part

A Classification of Constraint Propagations (cont.)

$\partial_t C = MC$, CAF = Eigenvalues(M)

(C1) Asymptotically **constrained** : (Violation of constraints decays.)
$\Leftrightarrow$ all the real part of CAFs are **negative**

(C2) Asymptotically **bounded** : (Violation of constraints is bounded at a certain value.)
$\Leftrightarrow$ all the real part of CAFs are **non-positive**
and **Jordan** matrices for eigenvalues with zero real part are **diagonal**
$\Leftarrow$ all the real part CAFs are non-positive and M is **diagonalizable**

(C3) **Diverge**: (At least one constraint will diverge.)
$\Leftrightarrow$ there exists CAF with **positive** real part
or there exists non diagonal **Jordan** matrix for eigenvalues with zero real part

A Classification of Constraint Propagations (cont.)

$\partial_t C = MC$, CAF = Eigenvalues(M)

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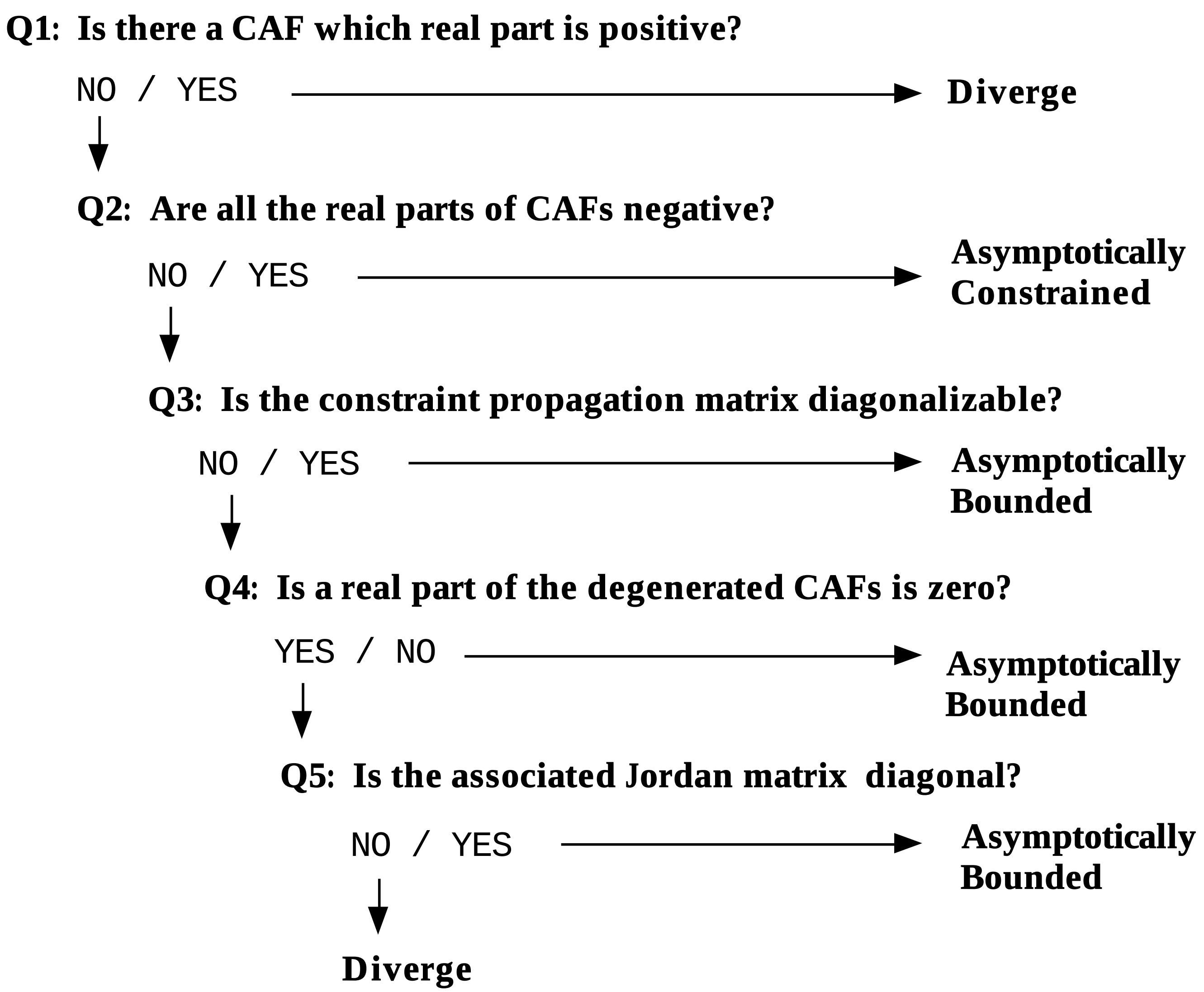
(C3) **Diverge**: (At least one constraint will diverge.)
$\Leftrightarrow$ there exists CAF with **positive** real part
or there exists non diagonal **Jordan** matrix for eigenvalues with zero real part

Each eigenvalue evaluation.
Real part: Nagative is better than zero and positive is worst.
Imaginary part: non-zero is better than zero for avoiding degeneracy.

A flowchart to classify the fate of constraint propagation.



Example1: Maxwell equation

$$
\begin{aligned}
\partial_t E^i &= -c\epsilon^i{}_j{}^l\, \partial_l B^j,\ \partial_t B^i = c\epsilon^i{}_j{}^l\, \partial_l E^j \\
C_E &:= \partial_i E^i \approx 0,,\ C_B := \partial_i B^i \approx 0, \\
\partial_t C_E &= 0,\ \partial_t C_B = 0 \\
\text{CAF} &= (0,0) \quad (\text{asymptotically bounded})
\end{aligned}
$$

Example 2: ADM equation

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, \\
\mathcal{H} &:= R^{(3)} + K^2 - K_{ij} K^{ij}, \\
\mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K, \\
\partial_t \mathcal{H} &= \beta^j (\partial_j \mathcal{H}) - 2\alpha \gamma^{ji} (\partial_i \mathcal{M}_j) + 2\alpha K \mathcal{H} + \alpha (\partial_l \gamma_{mn})(2\gamma^{ml}\gamma^{nj} - \gamma^{mn}\gamma^{lj}) \mathcal{M}_j - 4\gamma^{im}(\partial_m \alpha)\mathcal{M}_i, \\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) + \beta^j(\partial_j \mathcal{M}_i) + \alpha K \mathcal{M}_i - (\partial_i \alpha)\mathcal{H} - \beta^k \gamma^{jm}(\partial_i \gamma_{mk})\mathcal{M}_j + (\partial_i \beta_m)\gamma^{mj}\mathcal{M}_j. \\
\text{CAF} &= (0, 0, \pm\sqrt{-k^2}) \quad (\text{in Minkowskii background}) \quad (\text{asymptotically bounded})
\end{aligned}
$$

Example 3: BSSN

$$
\begin{aligned}
\partial_t^B \varphi &= -(1/6)\alpha K + (1/6)\beta^i(\partial_i\varphi) + (\partial_i\beta^i), \\
\partial_t^B \tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}(\partial_j\beta^k) + \tilde{\gamma}_{jk}(\partial_i\beta^k) - (2/3)\tilde{\gamma}_{ij}(\partial_k\beta^k) + \beta^k(\partial_k\tilde{\gamma}_{ij}), \\
\partial_t^B K &= -D^iD_i\alpha + \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 + \beta^i(\partial_i K), \\
\partial_t^B \tilde{A}_{ij} &= -e^{-4\varphi}(D_iD_j\alpha)^{TF} + e^{-4\varphi}\alpha(R_{ij}^{BSSN})^{TF} + \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{ik}\tilde{A}^k{}_j + (\partial_i\beta^k)\tilde{A}_{kj} + (\partial_j\beta^k)\tilde{A}_{ki} \\
&\quad -(2/3)(\partial_k\beta^k)\tilde{A}_{ij} + \beta^k(\partial_k\tilde{A}_{ij}), \\
\partial_t^B \tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} + 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - (2/3)\tilde{\gamma}^{ij}(\partial_j K) + 6\tilde{A}^{ij}(\partial_j\varphi)) - \partial_j(\beta^k(\partial_k\tilde{\gamma}^{ij}) - \tilde{\gamma}^{kj}(\partial_k\beta^i) \\
&\quad -\tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)). \\
\mathcal{H}^{BSSN} &= R^{BSSN} + K^2 - K_{ij}K^{ij}, \\
\mathcal{M}_i^{BSSN} &= \nabla_j K^j{}_i - \nabla_i K \\
\mathcal{G}^i &= \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk} \\
\mathcal{A} &= \tilde{A}_{ij}\tilde{\gamma}^{ij} \\
\mathcal{S} &= \tilde{\gamma} - 1
\end{aligned}
$$

constraint propagation is given in next page

CAF $= (0\,(\times 3), \pm\sqrt{-k^2}\,$(3 pairs)$)$ (in Minkowskii background) (asymptotically bounded)

A Full set of BSSN constraint propagation eqs.

$$\partial_t^{BS}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_i\\ \mathcal{G}^i\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12} & A_{13} & A_{14} & A_{15}\\ -(1/3)(\partial_i\alpha)+(1/6)\partial_i & \alpha K & A_{23} & 0 & A_{25}\\ 0 & \alpha\tilde\gamma^{ij} & 0 & A_{34} & A_{35}\\ 0 & 0 & 0 & \beta^k(\partial_k\mathcal{S}) & -2\alpha\tilde\gamma\\ 0 & 0 & 0 & 0 & \alpha K+\beta^k\partial_k\end{pmatrix}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_j\\ \mathcal{G}^j\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix}$$

$$\begin{aligned}
A_{11} &= +(2/3)\alpha K+(2/3)\alpha\mathcal{A}+\beta^k\partial_k\\
A_{12} &= -4e^{-4\varphi}\alpha(\partial_k\varphi)\tilde\gamma^{kj}-2e^{-4\varphi}(\partial_k\alpha)\tilde\gamma^{jk}\\
A_{13} &= -2\alpha e^{-4\varphi}\tilde A^k{}_j\partial_k-\alpha e^{-4\varphi}(\partial_j\tilde A_{kl})\tilde\gamma^{kl}-e^{-4\varphi}(\partial_j\alpha)\mathcal{A}-e^{-4\varphi}\beta^k\partial_k\partial_j-(1/2)e^{-4\varphi}\beta^k\tilde\gamma^{-1}(\partial_j\mathcal{S})\partial_k\\
&\quad +(1/6)e^{-4\varphi}\tilde\gamma^{-1}(\partial_j\beta^k)(\partial_k\mathcal{S})-(2/3)e^{-4\varphi}(\partial_k\beta^k)\partial_j\\
A_{14} &= 2\alpha e^{-4\varphi}\tilde\gamma^{-1}\tilde\gamma^{lk}(\partial_l\varphi)\mathcal{A}\partial_k+(1/2)\alpha e^{-4\varphi}\tilde\gamma^{-1}(\partial_l\mathcal{A})\tilde\gamma^{lk}\partial_k+(1/2)e^{-4\varphi}\tilde\gamma^{-1}(\partial_l\alpha)\tilde\gamma^{lk}\mathcal{A}\partial_k+(1/2)e^{-4\varphi}\tilde\gamma^{-1}\beta^m\tilde\gamma^{lk}\partial_m\partial_l\partial_k\\
&\quad -(5/4)e^{-4\varphi}\tilde\gamma^{-2}\beta^m\tilde\gamma^{lk}(\partial_m\mathcal{S})\partial_l\partial_k+e^{-4\varphi}\tilde\gamma^{-1}\beta^m(\partial_m\tilde\gamma^{lk})\partial_l\partial_k+(1/2)e^{-4\varphi}\tilde\gamma^{-1}\beta^i(\partial_j\partial_i\tilde\gamma^{jk})\partial_k\\
&\quad +(3/4)e^{-4\varphi}\tilde\gamma^{-3}\beta^i\tilde\gamma^{jk}(\partial_i\mathcal{S})(\partial_j\mathcal{S})\partial_k-(3/4)e^{-4\varphi}\tilde\gamma^{-2}\beta^i(\partial_i\tilde\gamma^{jk})(\partial_j\mathcal{S})\partial_k+(1/3)e^{-4\varphi}\tilde\gamma^{-1}\tilde\gamma^{pj}(\partial_j\beta^k)\partial_p\partial_k\\
&\quad -(5/12)e^{-4\varphi}\tilde\gamma^{-2}\tilde\gamma^{jk}(\partial_k\beta^i)(\partial_i\mathcal{S})\partial_j+(1/3)e^{-4\varphi}\tilde\gamma^{-1}(\partial_k\tilde\gamma^{ij})(\partial_j\beta^k)\partial_i-(1/6)e^{-4\varphi}\tilde\gamma^{-1}\tilde\gamma^{mk}(\partial_k\partial_l\beta^l)\partial_m\\
A_{15} &= (4/9)\alpha K\mathcal{A}-(8/9)\alpha K^2+(4/3)\alpha e^{-4\varphi}(\partial_i\partial_j\varphi)\tilde\gamma^{ij}+(8/3)\alpha e^{-4\varphi}(\partial_k\varphi)(\partial_l\tilde\gamma^{lk})+\alpha e^{-4\varphi}(\partial_j\tilde\gamma^{jk})\partial_k\\
&\quad +8\alpha e^{-4\varphi}\tilde\gamma^{jk}(\partial_j\varphi)\partial_k+\alpha e^{-4\varphi}\tilde\gamma^{jk}\partial_j\partial_k+8e^{-4\varphi}(\partial_l\alpha)(\partial_k\varphi)\tilde\gamma^{lk}+e^{-4\varphi}(\partial_l\alpha)(\partial_k\tilde\gamma^{lk})+2e^{-4\varphi}(\partial_l\alpha)\tilde\gamma^{lk}\partial_k\\
&\quad +e^{-4\varphi}\tilde\gamma^{lk}(\partial_l\partial_k\alpha)\\
A_{23} &= \alpha e^{-4\varphi}\tilde\gamma^{km}(\partial_k\varphi)(\partial_j\tilde\gamma_{mi})-(1/2)\alpha e^{-4\varphi}\tilde\Gamma^m_{kl}\tilde\gamma^{kl}(\partial_j\tilde\gamma_{mi})\\
&\quad +(1/2)\alpha e^{-4\varphi}\tilde\gamma^{mk}(\partial_k\partial_j\tilde\gamma_{mi})+(1/2)\alpha e^{-4\varphi}\tilde\gamma^{-2}(\partial_i\mathcal{S})(\partial_j\mathcal{S})-(1/4)\alpha e^{-4\varphi}(\partial_i\tilde\gamma_{kl})(\partial_j\tilde\gamma^{kl})+\alpha e^{-4\varphi}\tilde\gamma^{km}(\partial_k\varphi)\tilde\gamma_{ji}\partial_m\\
&\quad +\alpha e^{-4\varphi}(\partial_j\varphi)\partial_i-(1/2)\alpha e^{-4\varphi}\tilde\Gamma^m_{kl}\tilde\gamma^{kl}\tilde\gamma_{ji}\partial_m+\alpha e^{-4\varphi}\tilde\gamma^{mk}\tilde\Gamma_{ijk}\partial_m+(1/2)\alpha e^{-4\varphi}\tilde\gamma^{lk}\tilde\gamma_{ji}\partial_k\partial_l\\
&\quad +(1/2)e^{-4\varphi}\tilde\gamma^{mk}(\partial_j\tilde\gamma_{im})(\partial_k\alpha)+(1/2)e^{-4\varphi}(\partial_j\alpha)\partial_i+(1/2)e^{-4\varphi}\tilde\gamma^{mk}\tilde\gamma_{ji}(\partial_k\alpha)\partial_m\\
A_{25} &= -\tilde A^k{}_i(\partial_k\alpha)+(1/9)(\partial_i\alpha)K+(4/9)\alpha(\partial_iK)+(1/9)\alpha K\partial_i-\alpha\tilde A^k{}_i\partial_k\\
A_{34} &= -(1/2)\beta^k\tilde\gamma^{il}\tilde\gamma^{-2}(\partial_l\mathcal{S})\partial_k-(1/2)(\partial_l\beta^i)\tilde\gamma^{lk}\tilde\gamma^{-1}\partial_k+(1/3)(\partial_l\beta^l)\tilde\gamma^{ik}\tilde\gamma^{-1}\partial_k-(1/2)\beta^l\tilde\gamma^{in}(\partial_l\tilde\gamma_{mn})\tilde\gamma^{mk}\tilde\gamma^{-1}\partial_k\\
&\quad +(1/2)\beta^k\tilde\gamma^{il}\tilde\gamma^{-1}\partial_l\partial_k\\
A_{35} &= -(\partial_k\alpha)\tilde\gamma^{ik}+4\alpha\tilde\gamma^{ik}(\partial_k\varphi)-\alpha\tilde\gamma^{ik}\partial_k
\end{aligned}$$

section 1 Background of the problem
section 2 Constraint propagation analysis

section 3 Adjusted systems

- For time evolution systems with constraints in general
- Example:Maxwell equations
- Example:Einstein eqautions(Ashtekar)
- Example:Einstein eqautions(ADM) (partially new)
- Example:Einstein eqautions(BSSN) (partially new)

section 4 Numerical experiment
section 5 Discussion

Idea of adjusted system

$$
\begin{aligned}
& \text{Add constraint terms to evolution equations (adjust)} \\
\partial_t u^a &= f(u^a, \partial u^a, \partial\partial u^a) + F(C^\alpha, \partial C^\alpha, \partial\partial C^\alpha) \\
& \text{constraint propagation changes depending on them, too} \\
\partial_t C^\alpha &= A_0 C^\alpha + A_1 \partial C^\alpha + A_2 \partial\partial C^\alpha + \cdots + B_0 C^\alpha + B_1 \partial C^\alpha + B_2 \partial\partial C^\alpha + \cdots \\
& \text{CAF changes depending on them, too}
\end{aligned}
$$

We should adjust so that CAFs improve.

Advantage of adjusted system

1. Available even if the base system is not a symmetric hyperbolic.
2. Keep the number of the variables same with the original system.
3. Unified understanding for formulation problem is possible using the notions of adjustment and CAF

Example: the Maxwell equations

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Maxwell evolution equations.

$$
\begin{aligned}
\partial_t E_i &= c\epsilon_i{}^{jk}\partial_j B_k + P_i\, C_E + Q_i\, C_B, \\
\partial_t B_i &= -c\epsilon_i{}^{jk}\partial_j E_k + R_i\, C_E + S_i\, C_B, \\
& C_E = \partial_i E^i \approx 0, \quad C_B = \partial_i B^i \approx 0,
\end{aligned}
\qquad
\begin{cases}
\text{sym. hyp} & \Leftrightarrow \ P_i = Q_i = R_i = S_i = 0, \\
\text{strongly hyp} & \Leftrightarrow \ (P_i - S_i)^2 + 4R_iQ_i > 0, \\
\text{weakly hyp} & \Leftrightarrow \ (P_i - S_i)^2 + 4R_iQ_i \geq 0
\end{cases}
$$

Constraint propagation equations

$$
\begin{aligned}
\partial_t C_E &= (\partial_i P^i)C_E + P^i(\partial_i C_E) + (\partial_i Q^i)C_B + Q^i(\partial_i C_B), \\
\partial_t C_B &= (\partial_i R^i)C_E + R^i(\partial_i C_E) + (\partial_i S^i)C_B + S^i(\partial_i C_B),
\end{aligned}
$$

$$
\begin{cases}
\text{sym. hyp} & \Leftrightarrow \ Q_i = R_i, \\
\text{strongly hyp} & \Leftrightarrow \ (P_i - S_i)^2 + 4R_iQ_i > 0, \\
\text{weakly hyp} & \Leftrightarrow \ (P_i - S_i)^2 + 4R_iQ_i \geq 0
\end{cases}
$$

CAFs?

$$
\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} = \begin{pmatrix} \partial_i P^i + P^i k_i & \partial_i Q^i + Q^i k_i \\ \partial_i R^i + R^i k_i & \partial_i S^i + S^i k_i \end{pmatrix} \partial_l \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} \approx \begin{pmatrix} P^i k_i & Q^i k_i \\ R^i k_i & S^i k_i \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} =: T \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix}
$$

$$
\Rightarrow \ \text{CAFs} = (P^i k_i + S^i k_i \pm \sqrt{(P^i k_i + S^i k_i)^2 + 4(Q^i k_i R^j k_j - P^i k_i S^j k_j)})/2
$$

Therefore CAFs become negative-real when

$$
P^i k_i + S^i k_i < 0, \qquad \text{and} \qquad Q^i k_i R^j k_j - P^i k_i S^j k_j < 0
$$

Example: Maxwell adjusted (cont.)

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$$\begin{aligned} \partial_t E^i &= -c\epsilon^i{}_j{}^l\,\partial_l B^j + \kappa\partial^i C_E,\ \partial_t B^i = c\epsilon^i{}_j{}^l\,\partial_l E^j + \kappa\partial^i C_B \\ C_E &:= \partial_i E^i \approx 0,\ ,\ C_B := \partial_i B^i \approx 0, \\ \partial_t C_E &= \kappa\partial_i\partial^i C_E,\ \partial_t C_B = \kappa\partial_i\partial^i C_B \\ \text{CAF} &= (-\kappa, -\kappa) \end{aligned}$$

original Maxwell($\kappa = 0$), CAF$= (0, 0)$, averaged VN factor for FTCS=1.013

adjusted Maxwell($\kappa = 0.1$), CAF$= (-0.1, -0.1)$, averaged VN factor for FTCS=0.8017

adjusted Maxwell($\kappa = -0.1$), CAF$= (+0.1, +0.1)$, averaged VN factor for FTCS=1.2246

Adjustments also reduce the von Neumann factors.
In other words, adjustment is just like adding viscosity terms to evolution equations.

Example: the Ashtekar equations

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Adjusted dynamical equations:

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c \underbrace{+X^i_a \mathcal{C}_H + Y^{ij}_a \mathcal{C}_{Mj} + P^{ib}_a \mathcal{C}_{Gb}}_{adjust}$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i \underbrace{+Q^a_i \mathcal{C}_H + R^{aj}_i \mathcal{C}_{Mj} + Z^{ab}_i \mathcal{C}_{Gb}}_{adjust}$$

Adjusted and linearized:

$$X = Y = Z = 0,\ P^{ia}_b = \kappa_1(iN^i\delta^a_b),\ Q^a_i = \kappa_2(e^{-2}\underset{\sim}{N}\tilde{E}^a_i),\ R^{aj}{}_i = \kappa_3(-ie^{-2}\underset{\sim}{N}\epsilon^{ac}{}_d \tilde{E}^d_i \tilde{E}^j_c)$$

Fourier transform and extract 0th order of the characteristic matrix:

$$\partial_t \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mi} \\ \hat{\mathcal{C}}_{Ga} \end{pmatrix} = \begin{pmatrix} 0 & i(1+2\kappa_3)k_j & 0 \\ i(1-2\kappa_2)k_i & \kappa_3 \epsilon^{kj}{}_i k_k & 0 \\ 0 & 2\kappa_3 \delta^j_a & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mj} \\ \hat{\mathcal{C}}_{Gb} \end{pmatrix}$$

Eigenvalues:

$$\left(0, 0, 0, \pm\kappa_3\sqrt{-kx^2 - ky^2 - kz^2}, \pm\sqrt{(-1+2\kappa_2)(1+2\kappa_3)(kx^2+ky^2+kz^2)}\right)$$

In order to obtain non-positive real eigenvalues:

$$(-1+2\kappa_2)(1+2\kappa_3) < 0$$

Adjusted ADM systems

We adjust the standard ADM system using constraints as:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \tag{1}$$

$$+P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \tag{2}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, \tag{3}$$

$$+R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \tag{4}$$

with constraint equations

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij}K^{ij}, \tag{5}$$

$$\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K. \tag{6}$$

We can write the adjusted constraint propagation equations as

$$\partial_t \mathcal{H} = \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], \tag{7}$$

$$\partial_t \mathcal{M}_i = \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. \tag{8}$$

Standard ADM v.s. Original ADM in Minkowskii background

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} \\
&\quad + \beta^k \nabla_k K_{ij} + \kappa_F \alpha \gamma_{ij} \mathcal{H} \\
&\quad \text{standard} : \kappa_F = 0, \\
&\quad \text{original} : \kappa_F = -1/4 \\
\text{CAFs} &= (0, 0, \pm\sqrt{-k^2(1 + 4\kappa_F)}) \\
&\quad \text{standard} : (0, 0, \Im, \Im) \text{ (asymptotically bounded)} \\
&\quad \text{original} : (0, 0, 0, 0) \text{ (diverge)}
\end{aligned}
$$

Standard ADM is better than original ADM.

Example 1: standard ADM vs original ADM (in Schwarzschild coordinate)

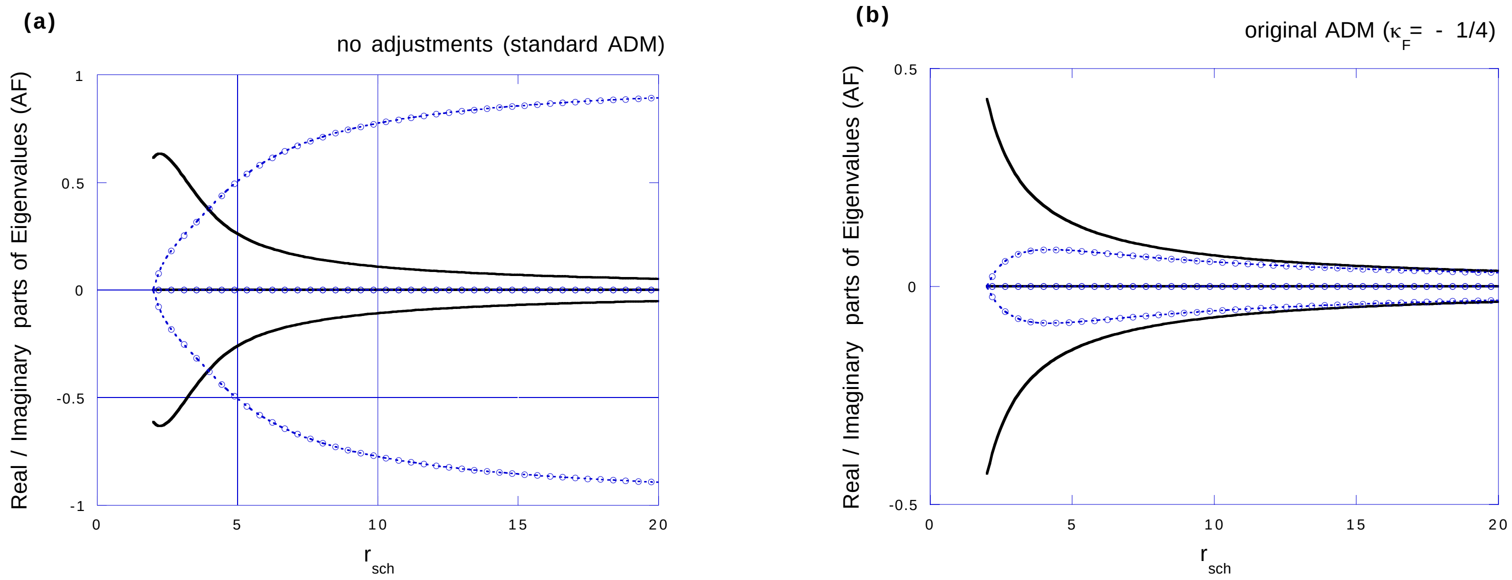

Figure 1: Amplification factors (AFs, eigenvalues of homogenized constraint propagation equations) are shown for the standard Schwarzschild coordinate, with (a) no adjustments, i.e., standard ADM, (b) original ADM ($\kappa_F = -1/4$). The solid lines and the dotted lines with circles are real parts and imaginary parts, respectively. They are four lines each, but actually the two eigenvalues are zero for all cases. Plotting range is $2 < r \leq 20$ using Schwarzschild radial coordinate. We set $k = 1, l = 2$, and $m = 2$ throughout the article.

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + \kappa_F \alpha \gamma_{ij} \mathcal{H},
\end{aligned}
$$

Example 2: Detweiler-type adjusted (in Schwarzschild coord.)

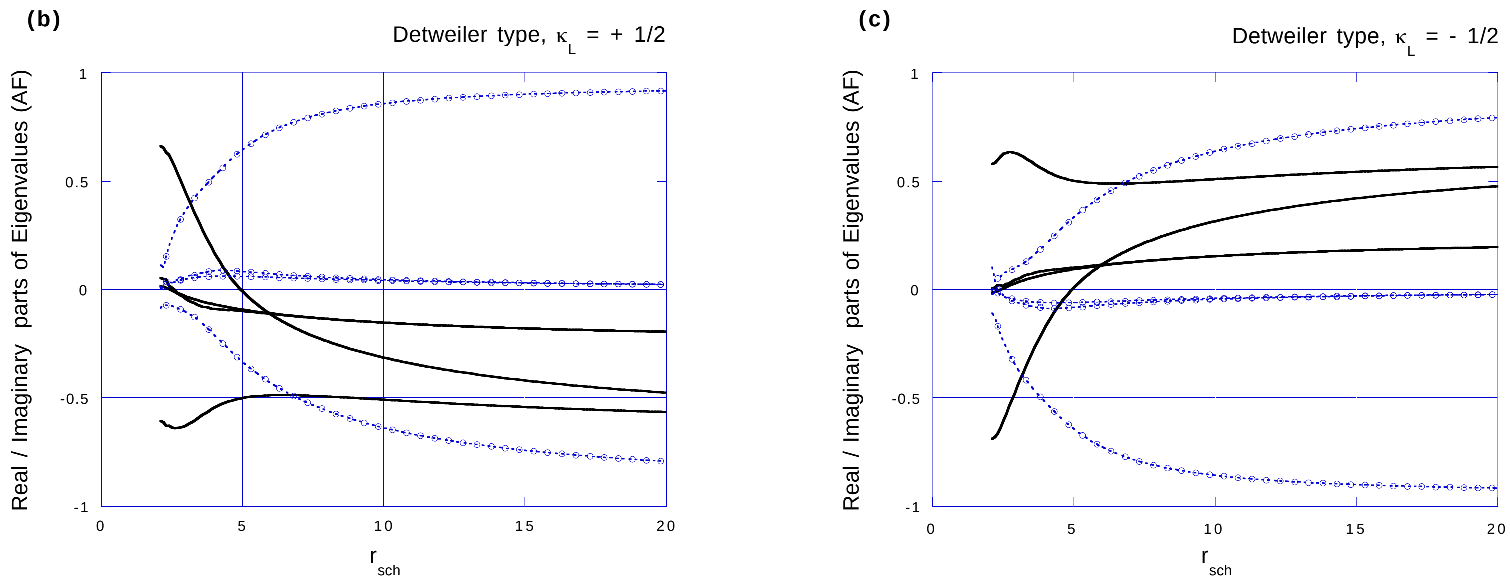

Figure 2: Amplification factors of the standard Schwarzschild coordinate, with Detweiler type adjustments. Multipliers used in the plot are (b) $\kappa_L = +1/2$, and (c) $\kappa_L = -1/2$.

$$\partial_t \gamma_{ij} = \text{(original terms)} + P_{ij}\mathcal{H},$$

$$\partial_t K_{ij} = \text{(original terms)} + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l),$$

where $P_{ij} = -\kappa_L \alpha^3 \gamma_{ij}, \quad R_{ij} = \kappa_L \alpha^3 (K_{ij} - (1/3)K\gamma_{ij}),$

$$S^k{}_{ij} = \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], \quad s^{kl}{}_{ij} = \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}],$$

Example 4: Detweiler-type adjusted (in iEF/PG coord.)

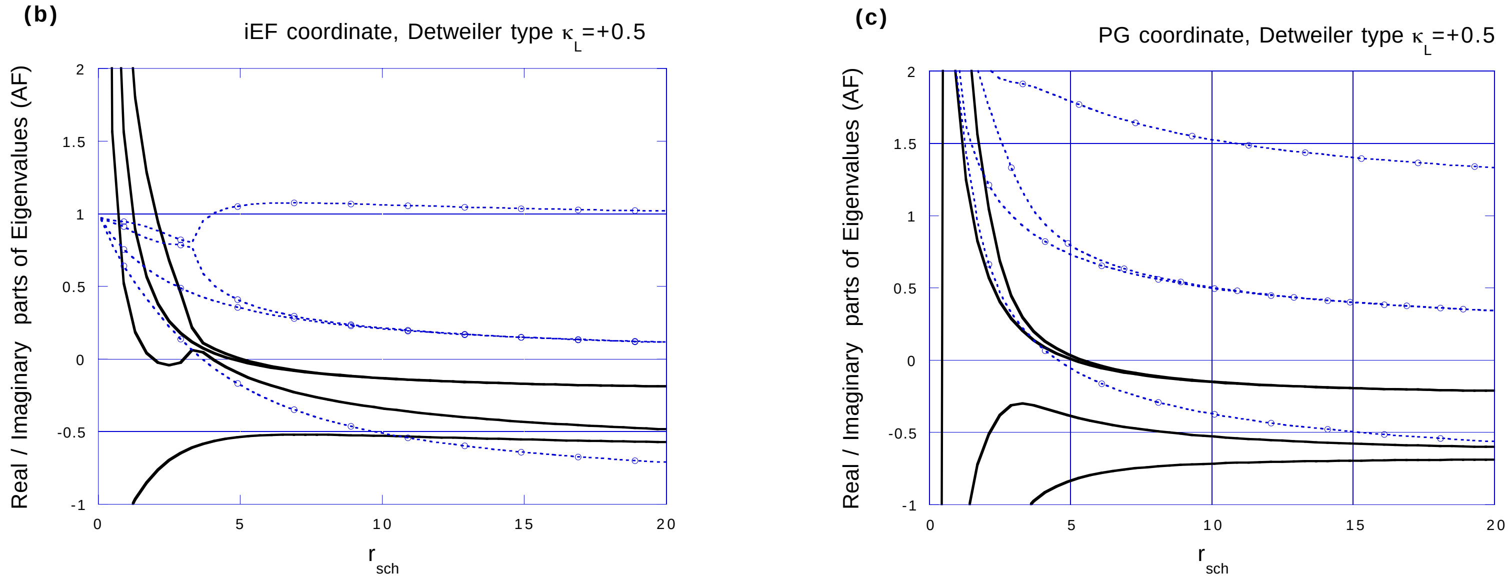

Figure 4: Similar comparison for Detweiler adjustments. $\kappa_L = +1/2$ for all plots.

Table 3. List of adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers in terms of (13) and (14). The column '1st?' and 'TRS' are the same as in table 1. The effects to amplification factors (when $\kappa > 0$) are commented for each coordinate system and for real/imaginary parts of AFs, respectively. The 'N/A' means that there is no effect due to the coordinate properties; 'not apparent' means the adjustment does not change the AFs effectively according to our conjecture; 'enl./red./min.' means enlarge/reduce/minimize, and 'Pos./Neg.' means positive/negative, respectively. These judgements are made at the $r \sim O(10M)$ region on their $t = 0$ slice.

					Schwarzschild/isotropic coordinates			iEF/PG coordinates	
No	No in table 1		Adjustment	1st?	TRS	Real	Imaginary	Real	Imaginary
0	0	–	no adjustments	yes	–	–	–	–	–
P-1	2-P	P_{ij}	$-\kappa_L \alpha^3 \gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-2	3	P_{ij}	$-\kappa_L \alpha \gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-3	–	P_{ij}	$P_{rr} = -\kappa$ or $P_{rr} = -\kappa\alpha$	no	no	slightly enl.Neg.	not apparent	slightly enl.Neg.	not apparent
P-4	–	P_{ij}	$-\kappa \gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-5	–	P_{ij}	$-\kappa \gamma_{rr}$	no	no	red. Pos./enl.Neg.	not apparent	red.Pos./enl.Neg.	not apparent
Q-1	–	$Q^k{}_{ij}$	$\kappa \alpha \beta^k \gamma_{ij}$	no	no	N/A	N/A	$\kappa \sim 1.35$ min. vals.	not apparent
Q-2	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
Q-3	–	$Q^k{}_{ij}$	$Q^r{}_{ij} = \kappa \gamma_{ij}$ or $Q^r{}_{ij} = \kappa \alpha \gamma_{ij}$	no	yes	red. abs vals.	not apparent	enl.Neg.	enl. vals.
Q-4	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa \gamma_{rr}$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
R-1	1	R_{ij}	$\kappa_F \alpha \gamma_{ij}$	yes	yes	$\kappa_F = -1/4$ min. abs vals.		$\kappa_F = -1/4$ min. vals.	
R-2	4	R_{ij}	$R_{rr} = -\kappa_\mu \alpha$ or $R_{rr} = -\kappa_\mu$	yes	no	not apparent	not apparent	red.Pos./enl.Neg.	enl. vals.
R-3	–	R_{ij}	$R_{rr} = -\kappa \gamma_{rr}$	yes	no	enl. vals.	not apparent	red.Pos./enl.Neg.	enl. vals.
S-1	2-S	$S^k{}_{ij}$	$\kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}]$	yes	no	not apparent	not apparent	not apparent	not apparent
S-2	–	$S^k{}_{ij}$	$\kappa \alpha \gamma^{lk} (\partial_l \gamma_{ij})$	yes	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
p-1	–	$p^k{}_{ij}$	$p^r{}_{ij} = -\kappa \alpha \gamma_{ij}$	no	no	red. Pos.	red. vals.	red. Pos.	enl. vals.
p-2	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa \alpha$	no	no	red. Pos.	red. vals.	red.Pos/enl.Neg.	enl. vals.
p-3	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa \alpha \gamma_{rr}$	no	no	makes 2 Neg.	enl. vals.	red. Pos. vals.	red. vals.
q-1	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{ij} = \kappa \alpha \gamma_{ij}$	no	no	$\kappa = 1/2$ min. vals.	red. vals.	not apparent	enl. vals.
q-2	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{rr} = -\kappa \alpha \gamma_{rr}$	no	yes	red. abs vals.	not apparent	not apparent	not apparent
r-1	–	$r^k{}_{ij}$	$r^r{}_{ij} = \kappa \alpha \gamma_{ij}$	no	yes	not apparent	not apparent	not apparent	enl. vals.
r-2	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa \alpha$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
r-3	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa \alpha \gamma_{rr}$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
s-1	2-s	$s^{kl}{}_{ij}$	$\kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}]$	no	no	makes 4 Neg.	not apparent	makes 4 Neg.	not apparent
s-2	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{ij} = -\kappa \alpha \gamma_{ij}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.
s-3	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{rr} = -\kappa \alpha \gamma_{rr}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.

Constraints in BSSN system

The normal Hamiltonian and momentum constraints

$$\mathcal{H}^{BSSN} = R^{BSSN} + K^2 - K_{ij}K^{ij}, \tag{1}$$

$$\mathcal{M}_i^{BSSN} = \mathcal{M}_i^{ADM}, \tag{2}$$

Additionally, we regard the following three as the constraints:

$$\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \tag{3}$$

$$\mathcal{A} = \tilde{A}_{ij}\tilde{\gamma}^{ij}, \tag{4}$$

$$\mathcal{S} = \tilde{\gamma} - 1, \tag{5}$$

Adjustments in evolution equations

$$\partial_t^B \varphi = \partial_t^A \varphi + (1/6)\alpha\mathcal{A} - (1/12)\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\beta^j, \tag{6}$$

$$\partial_t^B \tilde{\gamma}_{ij} = \partial_t^A \tilde{\gamma}_{ij} - (2/3)\alpha\tilde{\gamma}_{ij}\mathcal{A} + (1/3)\tilde{\gamma}^{-1}(\partial_k\mathcal{S})\beta^k\tilde{\gamma}_{ij}, \tag{7}$$

$$\partial_t^B K = \partial_t^A K - (2/3)\alpha K\mathcal{A} - \alpha\mathcal{H}^{BSSN} + \alpha e^{-4\varphi}(\tilde{D}_j\mathcal{G}^j), \tag{8}$$

$$\begin{aligned}\partial_t^B \tilde{A}_{ij} = {} & \partial_t^A \tilde{A}_{ij} + ((1/3)\alpha\tilde{\gamma}_{ij}K - (2/3)\alpha\tilde{A}_{ij})\mathcal{A} + ((1/2)\alpha e^{-4\varphi}(\partial_k\tilde{\gamma}_{ij}) - (1/6)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}))\mathcal{G}^k \\ & + \alpha e^{-4\varphi}\tilde{\gamma}_{k(i}(\partial_{j)}\mathcal{G}^k) - (1/3)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}(\partial_k\mathcal{G}^k)\end{aligned} \tag{9}$$

$$\begin{aligned}\partial_t^B \tilde{\Gamma}^i = {} & \partial_t^A \tilde{\Gamma}^i - ((2/3)(\partial_j\alpha)\tilde{\gamma}^{ji} + (2/3)\alpha(\partial_j\tilde{\gamma}^{ji}) + (1/3)\alpha\tilde{\gamma}^{ji}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) - 4\alpha\tilde{\gamma}^{ij}(\partial_j\varphi))\mathcal{A} - (2/3)\alpha\tilde{\gamma}^{ji}(\partial_j\mathcal{A}) \\ & + 2\alpha\tilde{\gamma}^{ij}\mathcal{M}_j - (1/2)(\partial_k\beta^i)\tilde{\gamma}^{kj}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (1/6)(\partial_j\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}) + (1/3)(\partial_k\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) \\ & + (5/6)\beta^k\tilde{\gamma}^{-2}\tilde{\gamma}^{ij}(\partial_k\mathcal{S})(\partial_j\mathcal{S}) + (1/2)\beta^k\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\mathcal{S}) + (1/3)\beta^k\tilde{\gamma}^{-1}(\partial_j\tilde{\gamma}^{ji})(\partial_k\mathcal{S}).\end{aligned} \tag{10}$$

Effect of adjustments

No.		Constraints (number of components) $\mathcal{H}$ (1)	$\mathcal{M}_i$ (3)	$\mathcal{G}^i$ (3)	$\mathcal{A}$ (1)	$\mathcal{S}$ (1)	diag?	Constr. Amp. Factors in Minkowskii background
0.	standard ADM	use	use	-	-	-	yes	$(0, 0, \Im, \Im)$
1.	BSSN no adjustment	use	use	use	use	use	yes	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
2.	the BSSN	use+adj	use+adj	use+adj	use+adj	use+adj	no	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
3.	no $\mathcal{S}$ adjustment	use+adj	use+adj	use+adj	use+adj	use	no	no difference in flat background
4.	no $\mathcal{A}$ adjustment	use+adj	use+adj	use+adj	use	use+adj	no	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
5.	no $\mathcal{G}^i$ adjustment	use+adj	use+adj	use	use+adj	use+adj	no	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
6.	no $\mathcal{M}_i$ adjustment	use+adj	use	use+adj	use+adj	use+adj	no	$(0, 0, 0, 0, 0, 0, 0, \Re, \Re)$ Growing modes
7.	no $\mathcal{H}$ adjustment	use	use+adj	use+adj	use+adj	use+adj	no	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
8.	ignore $\mathcal{G}^i$, $\mathcal{A}$, $\mathcal{S}$	use+adj	use+adj	-	-	-	no	$(0, 0, 0, 0)$
9.	ignore $\mathcal{G}^i$, $\mathcal{A}$	use+adj	use+adj	use+adj	-	-	yes	$(0, \Im, \Im, \Im, \Im, \Im, \Im)$
10.	ignore $\mathcal{G}^i$	use+adj	use+adj	-	use+adj	use+adj	no	$(0, 0, 0, 0, 0, 0)$
11.	ignore $\mathcal{A}$	use+adj	use+adj	use+adj	-	use+adj	yes	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
12.	ignore $\mathcal{S}$	use+adj	use+adj	use+adj	use+adj	-	yes	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$

New Proposals :: Improved (adjusted) BSSN systems

TRS breaking adjustments

In order to break time reversal symmetry (TRS) of the evolution eqs, to adjust $\partial_t\phi$,$\partial_t\tilde{\gamma}_{ij}$,$\partial_t\tilde{\Gamma}^i$ using $\mathcal{S}, \mathcal{G}^i$, or to adjust $\partial_t K, \partial_t \tilde{A}_{ij}$ using $\tilde{\mathcal{A}}$.

$$\begin{aligned}
\partial_t\phi &= \partial_t^{BS}\phi + \kappa_{\phi\mathcal{H}}\alpha\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\alpha\tilde{D}_k\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}\alpha\mathcal{S} + \kappa_{\phi\mathcal{S}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{S} \\
\partial_t\tilde{\gamma}_{ij} &= \partial_t^{BS}\tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}\mathcal{H}}\alpha\tilde{\gamma}_{ij}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{G}2}\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{S}1}\alpha\tilde{\gamma}_{ij}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{S} \\
\partial_t K &= \partial_t^{BS}K + \kappa_{K\mathcal{M}}\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k) + \kappa_{K\tilde{\mathcal{A}}1}\alpha\tilde{\mathcal{A}} + \kappa_{K\tilde{\mathcal{A}}2}\alpha\tilde{D}^j\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{A}_{ij} &= \partial_t^{BS}\tilde{A}_{ij} + \kappa_{A\mathcal{M}1}\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k) + \kappa_{A\mathcal{M}2}\alpha(\tilde{D}_{(i}\mathcal{M}_{j)}) + \kappa_{A\tilde{\mathcal{A}}1}\alpha\tilde{\gamma}_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{\mathcal{A}}2}\alpha\tilde{D}_i\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{\Gamma}^i &= \partial_t^{BS}\tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}\mathcal{H}}\alpha\tilde{D}^i\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}\alpha\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\alpha\tilde{D}^i\mathcal{H}^{BS}
\end{aligned}$$

or in the flat background

$$\begin{aligned}
\partial_t^{ADJ(1)}\phi &= +\kappa_{\phi\mathcal{H}}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\partial_k{}^{(1)}\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}{}^{(1)}\mathcal{S} + \kappa_{\phi\mathcal{S}2}\partial_j\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}\tilde{\gamma}_{ij} &= +\kappa_{\tilde{\gamma}\mathcal{H}}\delta_{ij}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{G}^k + (1/2)\kappa_{\tilde{\gamma}\mathcal{G}2}(\partial_j{}^{(1)}\mathcal{G}^i + \partial_i{}^{(1)}\mathcal{G}^j) + \kappa_{\tilde{\gamma}\mathcal{S}1}\delta_{ij}{}^{(1)}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\partial_i\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}K &= +\kappa_{K\mathcal{M}}\partial_j{}^{(1)}\mathcal{M}_j + \kappa_{K\tilde{\mathcal{A}}1}{}^{(1)}\tilde{\mathcal{A}} + \kappa_{K\tilde{\mathcal{A}}2}\partial_j\partial_j{}^{(1)}\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{A}_{ij} &= +\kappa_{A\mathcal{M}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{M}_k + (1/2)\kappa_{A\mathcal{M}2}(\partial_i\mathcal{M}_j + \partial_j\mathcal{M}_i) + \kappa_{A\tilde{\mathcal{A}}1}\delta_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{\mathcal{A}}2}\partial_i\partial_j\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{\Gamma}^i &= +\kappa_{\tilde{\Gamma}\mathcal{H}}\partial_i{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\partial_j\partial_j{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\partial_i\partial_j{}^{(1)}\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\partial_i{}^{(1)}\mathcal{S}
\end{aligned}$$

Constraint Amplification Factors with each adjustment

adjustment		CAFs	diag?	effect of the adjustment	
$\partial_t \phi$	$\kappa_{\phi\mathcal{H}}\, \alpha \mathcal{H}$	$(0, 0, \pm\sqrt{-k^2}(*3), 8\kappa_{\phi\mathcal{H}} k^2)$	no	$\kappa_{\phi\mathcal{H}} < 0$ makes 1 Neg.	
$\partial_t \phi$	$\kappa_{\phi\mathcal{G}}\, \alpha \tilde{D}_k \mathcal{G}^k$	$(0, 0, \pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\phi\mathcal{G}} < 0$ makes 2 Neg. 1 Pos.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{SD}\, \alpha \tilde{\gamma}_{ij} \mathcal{H}$	$(0, 0, \pm\sqrt{-k^2}(*3), (3/2)\kappa_{SD} k^2)$	yes	$\kappa_{SD} < 0$ makes 1 Neg.	Case (B)
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}1}\, \alpha \tilde{\gamma}_{ij} \tilde{D}_k \mathcal{G}^k$	$(0, 0, \pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}1} > 0$ makes 1 Neg.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}2}\, \alpha \tilde{\gamma}_{k(i} \tilde{D}_{j)} \mathcal{G}^k$	(0,0, $(1/4)k^2\kappa_{\tilde{\gamma}\mathcal{G}2} \pm \sqrt{k^2(-1 + k^2\kappa_{\tilde{\gamma}\mathcal{G}2}/16)}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}2} < 0$ makes 6 Neg. 1 Pos.	Case (E1)
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}1}\, \alpha \tilde{\gamma}_{ij} \mathcal{S}$	$(0, 0, \pm\sqrt{-k^2}(*3), 3\kappa_{\tilde{\gamma}\mathcal{S}1})$	no	$\kappa_{\tilde{\gamma}\mathcal{S}1} < 0$ makes 1 Neg.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}2}\, \alpha \tilde{D}_i \tilde{D}_j \mathcal{S}$	$(0, 0, \pm\sqrt{-k^2}(*3), -\kappa_{\tilde{\gamma}\mathcal{S}2} k^2)$	no	$\kappa_{\tilde{\gamma}\mathcal{S}2} > 0$ makes 1 Neg.	
$\partial_t K$	$\kappa_{K\mathcal{M}}\, \alpha \tilde{\gamma}^{jk} (\tilde{D}_j \mathcal{M}_k)$	$(0, 0, 0, \pm\sqrt{-k^2}(*2),$ $(1/3)\kappa_{K\mathcal{M}} k^2 \pm (1/3)\sqrt{k^2(-9 + k^2\kappa_{K\mathcal{M}}^2)})$	no	$\kappa_{K\mathcal{M}} < 0$ makes 2 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{M}1}\, \alpha \tilde{\gamma}_{ij} (\tilde{D}^k \mathcal{M}_k)$	$(0, 0, \pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{M}1} k^2)$	yes	$\kappa_{A\mathcal{M}1} > 0$ makes 1 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{M}2}\, \alpha (\tilde{D}_{(i} \mathcal{M}_{j)})$	(0,0, $-k^2\kappa_{A\mathcal{M}2}/4 \pm \sqrt{k^2(-1 + k^2\kappa_{A\mathcal{M}2}/16)}(*2)$, long expressions)	yes	$\kappa_{A\mathcal{M}2} > 0$ makes 7 Neg	Case (D)
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{A}1}\, \alpha \tilde{\gamma}_{ij} \mathcal{A}$	$(0, 0, \pm\sqrt{-k^2}(*3), 3\kappa_{A\mathcal{A}1})$	yes	$\kappa_{A\mathcal{A}1} < 0$ makes 1 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{A}2}\, \alpha \tilde{D}_i \tilde{D}_j \mathcal{A}$	$(0, 0, \pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2} k^2)$	yes	$\kappa_{A\mathcal{A}2} > 0$ makes 1 Neg.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{H}}\, \alpha \tilde{D}^i \mathcal{H}$	$(0, 0, \pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2} k^2)$	no	$\kappa_{\tilde{\Gamma}\mathcal{H}} > 0$ makes 1 Neg.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}1}\, \alpha \mathcal{G}^i$	$(0, 0, (1/2)\kappa_{\tilde{\Gamma}\mathcal{G}1} \pm \sqrt{-k^2 + \kappa_{\tilde{\Gamma}\mathcal{G}1}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}1} < 0$ makes 6 Neg. 1 Pos.	Case (E2)
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}2}\, \alpha \tilde{D}^j \tilde{D}_j \mathcal{G}^i$	$(0, 0, -(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}2} \pm \sqrt{-k^2 + \kappa_{\tilde{\Gamma}\mathcal{G}2}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}2} > 0$ makes 2 Neg. 1 Pos.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}3}\, \alpha \tilde{D}^i \tilde{D}_j \mathcal{G}^j$	$(0, 0, -(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}3} \pm \sqrt{-k^2 + \kappa_{\tilde{\Gamma}\mathcal{G}3}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}3} > 0$ makes 2 Neg. 1 Pos.	

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Summary (1)

[Keyword 1] Adjusted Systems

Adjusting the EoM with constraints is common to all previous approaches.

[Keyword 2] Constraint Propagation Analysis

By evaluating the propagation eqs of constraints, predict the suitable adjustments to the EoM.

(Step 1) Fourier mode expression of all terms of constraint propagation eqs.

(Step 2) Eigenvalues and Diagonalizability of constraint propagation matrix.

Eigenvalues = Constraint Amplification Factors

[Keyword 3] Adjusted ADM systems

We show the standard ADM has constraint violating mode.

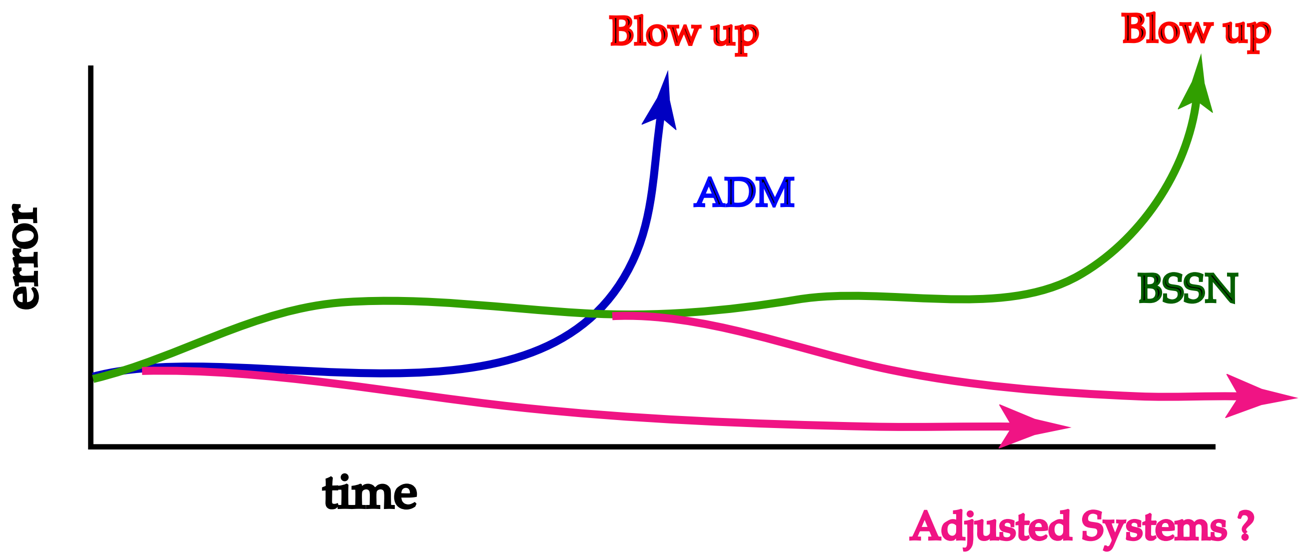
We predict several adjustments, which may give better stability.

[Keyword 3] Adjusted BSSN systems

We show the advantage of BSSN is the adjustment using M.

We predict several adjustments, which may give better stability.

4. Numerical Demonstrations



Comparisons of Adjusted ADM systems (Teukolsky wave)
3-dim, harmonic slice, periodic BC **HS original Cactus/GR code**

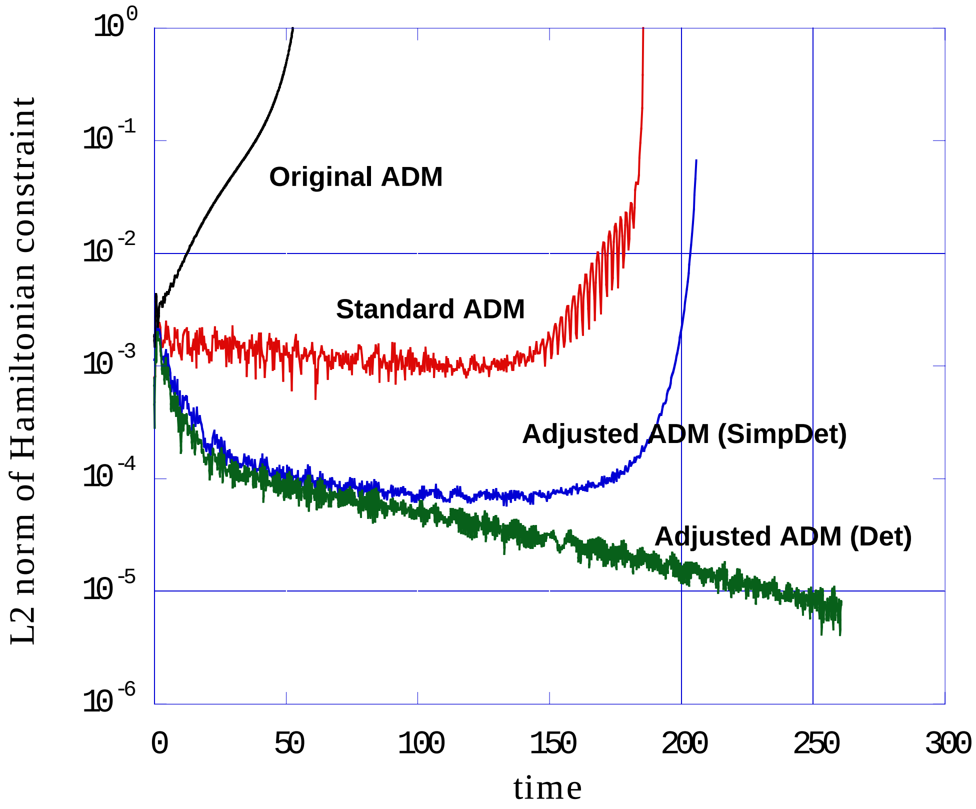


Figure 1: Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR/evolveADMeq code was used. Grid $= 24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method.

Comparisons of Adjusted ADM systems (Teukolsky wave) :: Detweiler type
3-dim, harmonic slice, periodic BC HS original Cactus/GR code

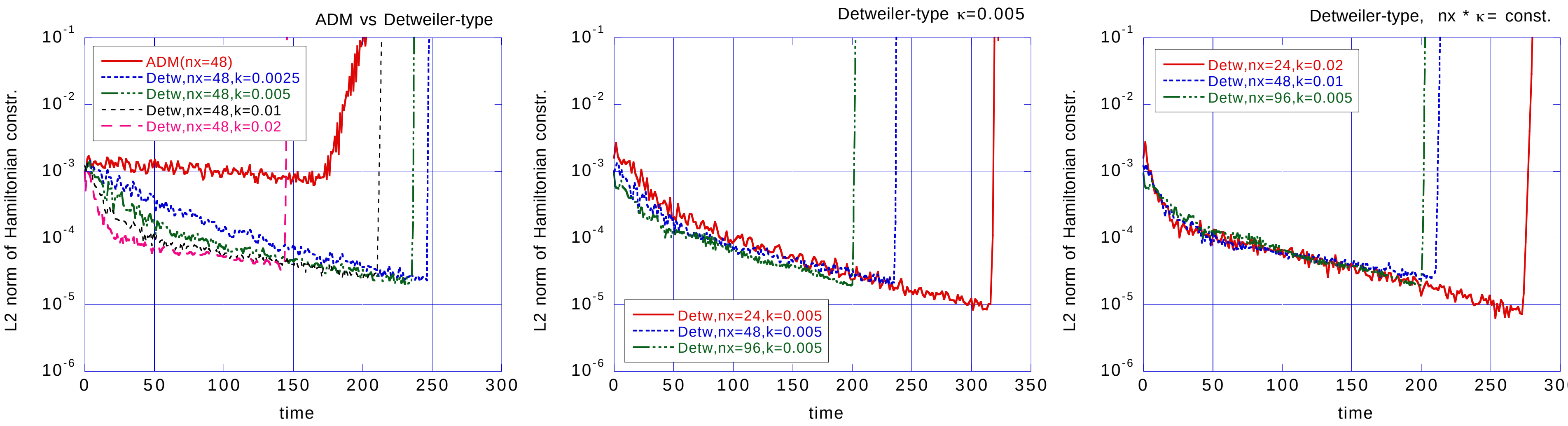


Figure 2: Violation of Hamiltonian constraints versus time: Adjusted ADM (Detweiler-type) system is applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR/evolveADMeq code was used. $(x, y, z) = [-3, 3]$, iterative Crank-Nicholson method.

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij}\,\mathcal{H} \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij})\,\mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}]\,\mathcal{M}_k \\
&\quad + \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}]\,(\nabla_k \mathcal{M}_l)
\end{aligned}
$$

Comparisons of Adjusted ADM systems (Teukolsky wave) :: Simplified-Detweiler type
3-dim, harmonic slice, periodic BC HS original Cactus/GR code

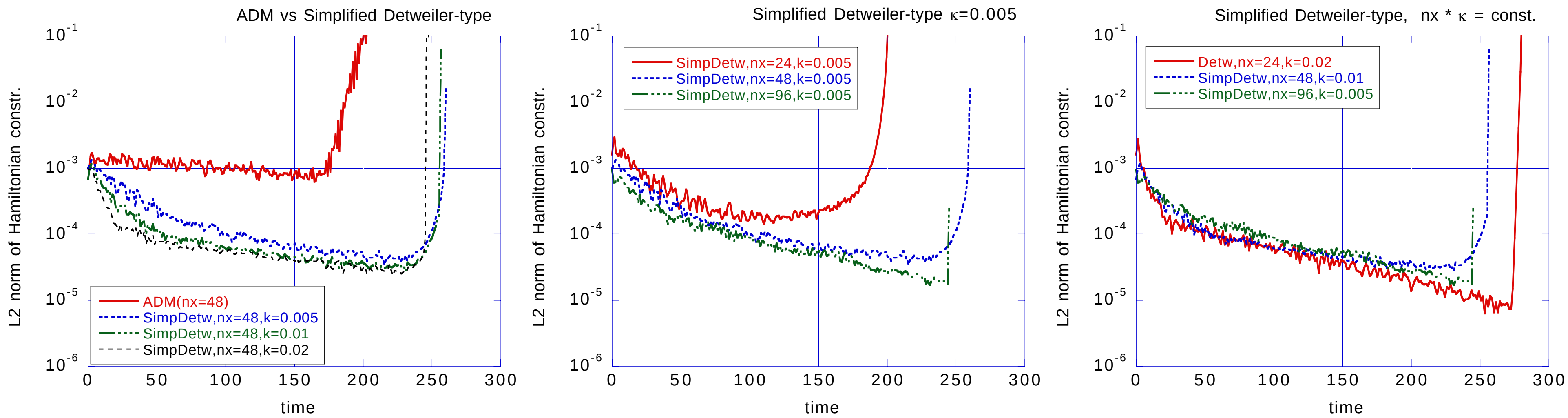


Figure 3: Violation of Hamiltonian constraints versus time: Adjusted ADM (Simplified Detweiler-type) system is applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR/evolveADMeq code was used. $(x, y, z) = [-3, 3]$, iterative Crank-Nicholson method.

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha \gamma_{ij} \mathcal{H}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$

Comparisons of Adjusted ADM systems (Teukolsky wave)

3-dim, harmonic slice, periodic BC

HS original Cactus/GR code

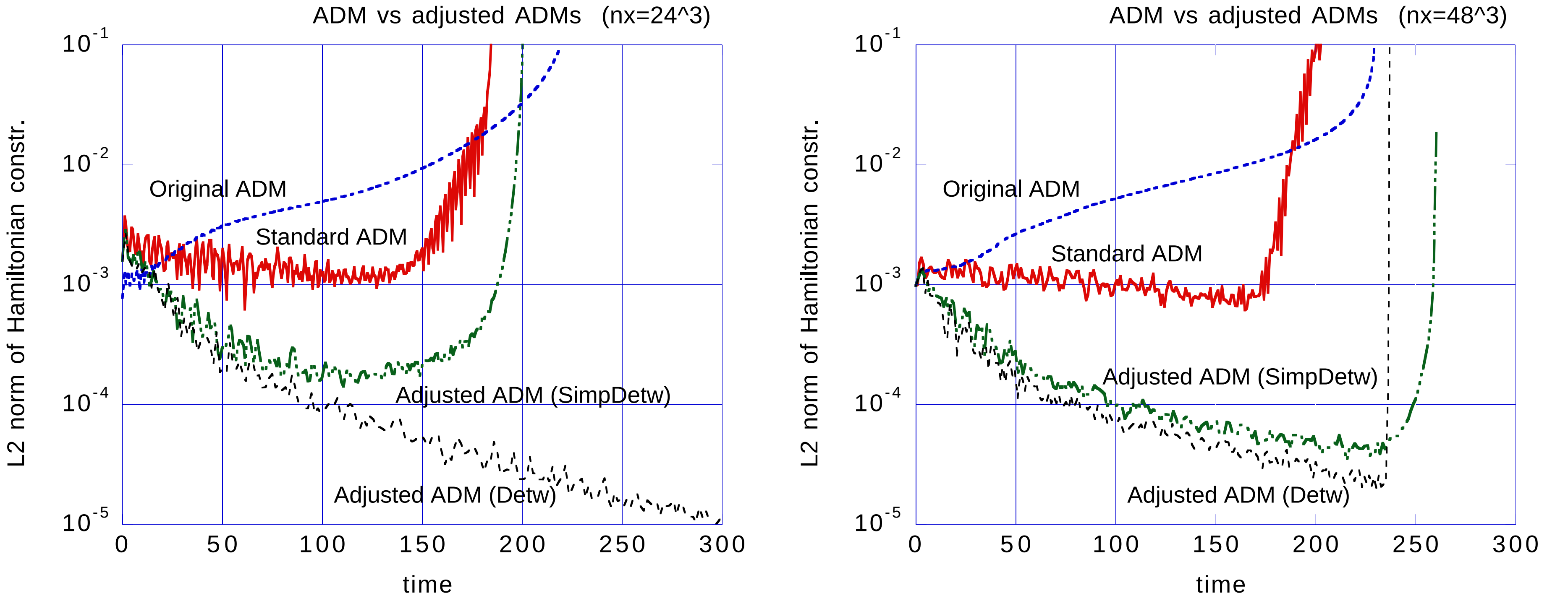


Figure 1: Violation of Hamiltonian constraints versus time: Adjusted ADM systems are applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR/evolveADMeq code was used. Grid $= 24^3$ and 48^3, with $(x, y, z) = [-3, 3]$, iterative Crank-Nicholson method. Adjusted parameters $\kappa = 0.005$.

Table 3. List of adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers in terms of (13) and (14). The column '1st?' and 'TRS' are the same as in table 1. The effects to amplification factors (when $\kappa > 0$) are commented for each coordinate system and for real/imaginary parts of AFs, respectively. The 'N/A' means that there is no effect due to the coordinate properties; 'not apparent' means the adjustment does not change the AFs effectively according to our conjecture; 'enl./red./min.' means enlarge/reduce/minimize, and 'Pos./Neg.' means positive/negative, respectively. These judgements are made at the $r \sim O(10M)$ region on their $t = 0$ slice.

					Schwarzschild/isotropic coordinates			iEF/PG coordinates	
No	No in table 1		Adjustment	1st?	TRS	Real	Imaginary	Real	Imaginary
0	0	–	no adjustments	yes	–	–	–	–	–
P-1	2-P	P_{ij}	$-\kappa_L\alpha^3\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-2	3	P_{ij}	$-\kappa_L\alpha\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-3	–	P_{ij}	$P_{rr} = -\kappa$ or $P_{rr} = -\kappa\alpha$	no	no	slightly enl.Neg.	not apparent	slightly enl.Neg.	not apparent
P-4	–	P_{ij}	$-\kappa\gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-5	–	P_{ij}	$-\kappa\gamma_{rr}$	no	no	red. Pos./enl.Neg.	not apparent	red.Pos./enl.Neg.	not apparent
Q-1	–	$Q^k{}_{ij}$	$\kappa\alpha\beta^k\gamma_{ij}$	no	no	N/A	N/A	$\kappa \sim 1.35$ min. vals.	not apparent
Q-2	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
Q-3	–	$Q^k{}_{ij}$	$Q^r{}_{ij} = \kappa\gamma_{ij}$ or $Q^r{}_{ij} = \kappa\alpha\gamma_{ij}$	no	yes	red. abs vals.	not apparent	enl.Neg.	enl. vals.
Q-4	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa\gamma_{rr}$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
R-1	1	R_{ij}	$\kappa_F\alpha\gamma_{ij}$	yes	yes	$\kappa_F = -1/4$ min. abs vals.		$\kappa_F = -1/4$ min. vals.	
R-2	4	R_{ij}	$R_{rr} = -\kappa_\mu\alpha$ or $R_{rr} = -\kappa_\mu$	yes	no	not apparent	not apparent	red.Pos./enl.Neg.	enl. vals.
R-3	–	R_{ij}	$R_{rr} = -\kappa\gamma_{rr}$	yes	no	enl. vals.	not apparent	red.Pos./enl.Neg.	enl. vals.
S-1	2-S	$S^k{}_{ij}$	$\kappa_L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}]$	yes	no	not apparent	not apparent	not apparent	not apparent
S-2	–	$S^k{}_{ij}$	$\kappa\alpha\gamma^{lk}(\partial_l\gamma_{ij})$	yes	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
p-1	–	$p^k{}_{ij}$	$p^r{}_{ij} = -\kappa\alpha\gamma_{ij}$	no	no	red. Pos.	red. vals.	red. Pos.	enl. vals.
p-2	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa\alpha$	no	no	red. Pos.	red. vals.	red.Pos/enl.Neg.	enl. vals.
p-3	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	enl. vals.	red. Pos. vals.	red. vals.
q-1	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{ij} = \kappa\alpha\gamma_{ij}$	no	no	$\kappa = 1/2$ min. vals.	red. vals.	not apparent	enl. vals.
q-2	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	not apparent	not apparent	not apparent
r-1	–	$r^k{}_{ij}$	$r^r{}_{ij} = \kappa\alpha\gamma_{ij}$	no	yes	not apparent	not apparent	not apparent	enl. vals.
r-2	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa\alpha$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
r-3	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
s-1	2-s	$s^{kl}{}_{ij}$	$\kappa_L\alpha^3[\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}]$	no	no	makes 4 Neg.	not apparent	makes 4 Neg.	not apparent
s-2	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{ij} = -\kappa\alpha\gamma_{ij}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.
s-3	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.

HS-Yoneda, CQG 19 (2002) 1027

Comparisons of Adjusted BSSN systems (linear wave)

Mexico NR 2002 Workshop participants

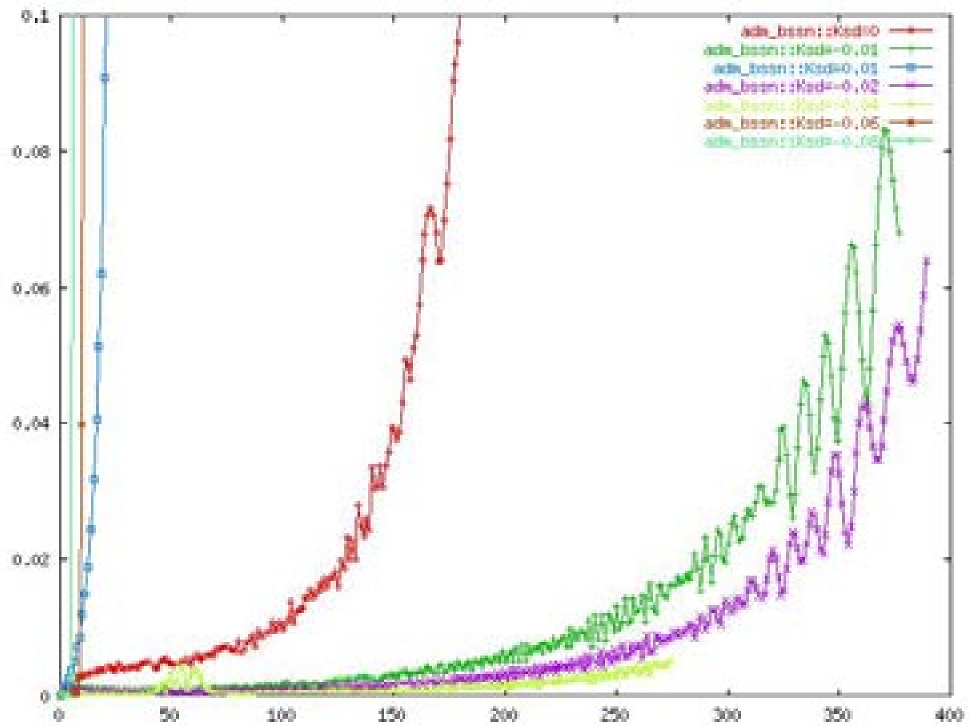
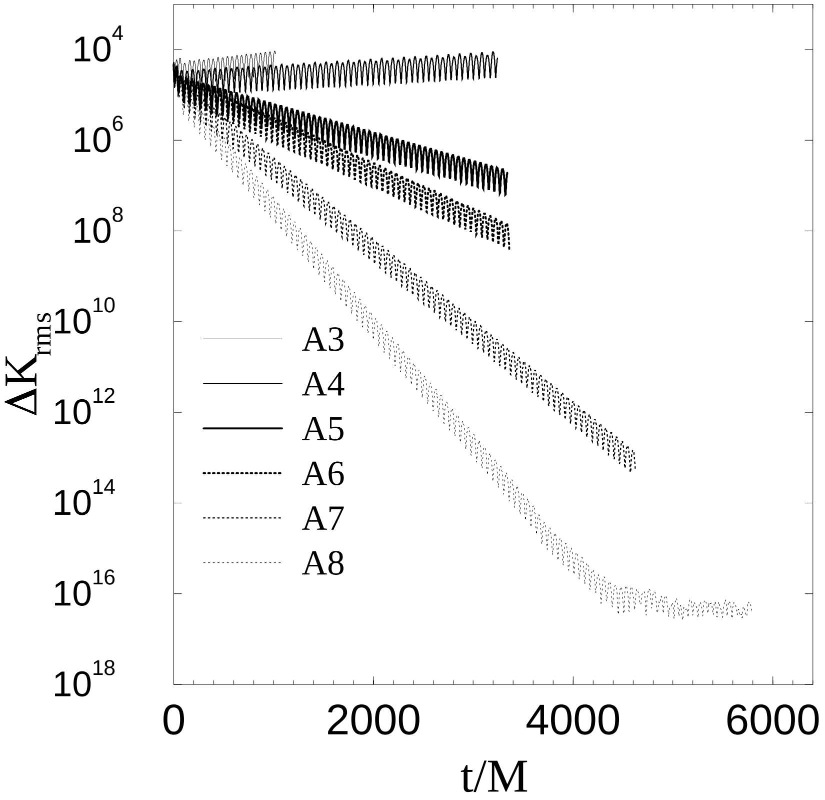
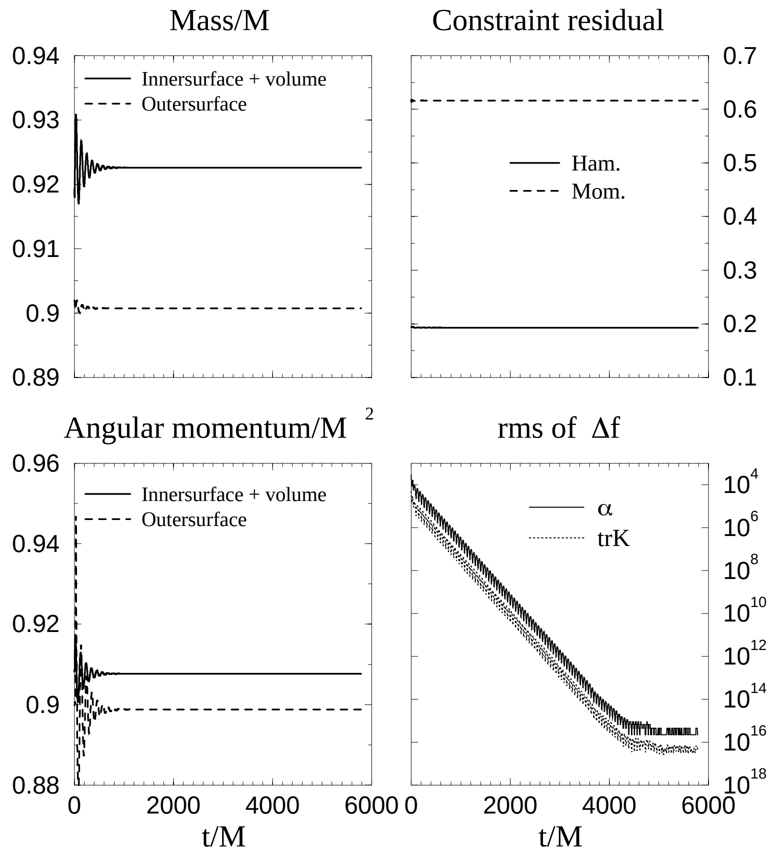


Figure 2: Violation of Hamiltonian constraints versus time: Adjusted BSSN systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/AEIThorns/BSSN code was used. Grid $= 24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method. Courtesy of N. Dorband and D. Pollney (AEI).

An Evolution of Adjusted BSSN Formulation

by Yo-Baumgarte-Shapiro, PRD 66 (2002) 084026





Kerr-Schild BH (0.9 J/M), excision with cube, $1 + \log$-lapse, Γ-driver shift.

$$\partial_t \tilde{\Gamma}^i = (\cdots) + \frac{2}{3}\tilde{\Gamma}^i \beta^i{}_{,j} - (\chi + \frac{2}{3})\mathcal{G}^i \beta^j{}_{,j} \qquad \chi = 2/3 \;\; \textbf{for (A4)-(A8)}$$

$$\partial_t \tilde{\gamma}_{ij} = (\cdots) - \kappa \alpha \tilde{\gamma}_{ij} \mathcal{H} \qquad \kappa = 0.1 \sim 0.2 \;\; \textbf{for (A5), (A6) and (A8)}$$

Constraint Amplification Factors with each adjustment

adjustment		CAFs	diag?	effect of the adjustment	
$\partial_t \phi$	$\kappa_{\phi\mathcal{H}}\, \alpha \mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), 8\kappa_{\phi\mathcal{H}}k^2)$	no	$\kappa_{\phi\mathcal{H}} < 0$ makes 1 Neg.	
$\partial_t \phi$	$\kappa_{\phi\mathcal{G}}\, \alpha \tilde{D}_k \mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\phi\mathcal{G}} < 0$ makes 2 Neg. 1 Pos.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{SD}\, \alpha \tilde{\gamma}_{ij} \mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), (3/2)\kappa_{SD}k^2)$	yes	$\kappa_{SD} < 0$ makes 1 Neg.	Case (B)
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}1}\, \alpha \tilde{\gamma}_{ij} \tilde{D}_k \mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}1} > 0$ makes 1 Neg.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}2}\, \alpha \tilde{\gamma}_{k(i} \tilde{D}_{j)} \mathcal{G}^k$	(0,0, $(1/4)k^2\kappa_{\tilde{\gamma}\mathcal{G}2} \pm \sqrt{k^2(-1+k^2\kappa_{\tilde{\gamma}\mathcal{G}2}/16)}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}2} < 0$ makes 6 Neg. 1 Pos.	Case (E1)
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}1}\, \alpha \tilde{\gamma}_{ij} \mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{\tilde{\gamma}\mathcal{S}1})$	no	$\kappa_{\tilde{\gamma}\mathcal{S}1} < 0$ makes 1 Neg.	
$\partial_t \tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}2}\, \alpha \tilde{D}_i \tilde{D}_j \mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{\tilde{\gamma}\mathcal{S}2}k^2)$	no	$\kappa_{\tilde{\gamma}\mathcal{S}2} > 0$ makes 1 Neg.	
$\partial_t K$	$\kappa_{K\mathcal{M}}\, \alpha \tilde{\gamma}^{jk}(\tilde{D}_j \mathcal{M}_k)$	$(0,0,0,\pm\sqrt{-k^2}(*2), (1/3)\kappa_{K\mathcal{M}}k^2 \pm (1/3)\sqrt{k^2(-9+k^2\kappa_{K\mathcal{M}}^2)})$	no	$\kappa_{K\mathcal{M}} < 0$ makes 2 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{M}1}\, \alpha \tilde{\gamma}_{ij}(\tilde{D}^k \mathcal{M}_k)$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{M}1}k^2)$	yes	$\kappa_{A\mathcal{M}1} > 0$ makes 1 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{M}2}\, \alpha(\tilde{D}_{(i}\mathcal{M}_{j)})$	(0,0, $-k^2\kappa_{A\mathcal{M}2}/4 \pm \sqrt{k^2(-1+k^2\kappa_{A\mathcal{M}2}/16)}(*2)$, long expressions)	yes	$\kappa_{A\mathcal{M}2} > 0$ makes 7 Neg	Case (D)
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{A}1}\, \alpha \tilde{\gamma}_{ij} \mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{A\mathcal{A}1})$	yes	$\kappa_{A\mathcal{A}1} < 0$ makes 1 Neg.	
$\partial_t \tilde{A}_{ij}$	$\kappa_{A\mathcal{A}2}\, \alpha \tilde{D}_i \tilde{D}_j \mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2}k^2)$	yes	$\kappa_{A\mathcal{A}2} > 0$ makes 1 Neg.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{H}}\, \alpha \tilde{D}^i \mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2}k^2)$	no	$\kappa_{\tilde{\Gamma}\mathcal{H}} > 0$ makes 1 Neg.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}1}\, \alpha \mathcal{G}^i$	$(0,0,(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}1} \pm \sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}1}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}1} < 0$ makes 6 Neg. 1 Pos.	Case (E2)
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}2}\, \alpha \tilde{D}^j \tilde{D}_j \mathcal{G}^i$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}2} \pm \sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}2}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}2} > 0$ makes 2 Neg. 1 Pos.	
$\partial_t \tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}3}\, \alpha \tilde{D}^i \tilde{D}_j \mathcal{G}^j$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}3} \pm \sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}3}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}3} > 0$ makes 2 Neg. 1 Pos.	

Yoneda-HS, PRD66 (2002) 124003

5 Discussion

5.1 Application 1 : Constraint Propagation in $N+1$ dim. space-time

HS-Yoneda, submitted to GRG (2003)

Dynamical equation has N-dependency

Only the matter term in $\partial_t K_{ij}$ has N-dependency.

$$
\begin{aligned}
0 \approx \mathcal{C}_H &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu n^\nu = \frac{1}{2}({}^{(N)}R + K^2 - K^{ij}K_{ij}) - 8\pi\rho_H - \Lambda, \\
0 \approx \mathcal{C}_{Mi} &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu \perp^\nu_i = D_j K^j_i - D_i K - 8\pi J_i, \\
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_j \beta_i + D_i \beta_j, \\
\partial_t K_{ij} &= \alpha {}^{(N)}R_{ij} + \alpha K K_{ij} - 2\alpha K^\ell{}_j K_{i\ell} - D_i D_j \alpha \\
&\quad + \beta^k (D_k K_{ij}) + (D_j \beta^k) K_{ik} + (D_i \beta^k) K_{kj} - 8\pi\alpha \left(S_{ij} - \frac{1}{N-1} \gamma_{ij} T \right) - \frac{2\alpha}{N-1} \gamma_{ij} \Lambda,
\end{aligned}
$$

Constraint Propagations remain the same

From the Bianchi identity, $\nabla^\nu \mathcal{S}_{\mu\nu} = 0$ with $\mathcal{S}_{\mu\nu} = X n_\mu n_\nu + Y_\mu n_\nu + Y_\nu n_\mu + Z_{\mu\nu}$, we get

$$
\begin{aligned}
0 = n^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= -Z_{\mu\nu}(\nabla^\mu n^\nu) - \nabla^\mu Y_\mu + Y_\nu n^\mu \nabla_\mu n^\nu - 2 Y_\mu n_\nu (\nabla^\nu n^\mu) - X(\nabla^\mu n_\mu) - n_\mu (\nabla^\mu X), \\
0 = h_i{}^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= \nabla^\mu Z_{i\mu} + Y_i(\nabla^\mu n_\mu) + Y_\mu (\nabla^\mu n_i) + X(\nabla^\mu n_i) n_\mu + n_\mu (\nabla^\mu Y_i).
\end{aligned}
$$

- $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (T_{\mu\nu}, \rho_H, J_i, S_{ij})$ with $\nabla^\mu T_{\mu\nu} = 0 \Rightarrow$ matter eq.
- $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (G_{\mu\nu} - 8\pi T_{\mu\nu}, \mathcal{C}_H, \mathcal{C}_{Mi}, \kappa\gamma_{ij}\mathcal{C}_H)$ with $\nabla^\mu (G_{\mu\nu} - 8\pi T_{\mu\nu}) = 0 \Rightarrow$ CP eq.

5.2 Application 2 : Constraint Propagation of Maxwell field in Curved space

HS-Yoneda, in preparation

Towards a robust GR-MHD system:

- **Maxwell eqs in curved space-time**

$$\begin{aligned}\partial_t E^i &= \epsilon^{ijk} D_j(\alpha B_k) - 4\pi\alpha J^i + \alpha K E^i + \pounds_\beta E^i \\ \partial_t B^i &= -\epsilon^{ijk} D_j(\alpha E_k) + \alpha K B^i + \pounds_\beta B^i \\ \mathcal{C}_E &:= D_i E^i - 4\pi\rho_e \\ \mathcal{C}_B &:= D_i B^i\end{aligned}$$

- **CP of Maxwell system in curved space-time**

$$\begin{aligned}\partial_t C_E &= \alpha K C_E + \beta^j D_j C_E \\ \partial_t C_B &= \alpha K C_B + \beta^j D_j C_B\end{aligned}$$

- **CP of ADM+Maxwell**

$$\partial_t \begin{pmatrix} \mathcal{C}_E \\ \mathcal{C}_B \\ \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \begin{pmatrix} \mathcal{C}_E \\ \mathcal{C}_B \\ \mathcal{H} \\ \mathcal{M}_i \end{pmatrix}$$

- **CP of ADM+Maxwell+Hydro**
  in progress.

5.3 Future : Construct a robust adjusted system

HS-Yoneda, in preparation

(1) dynamic & automatic determination of κ under a suitable principle.

e.g.) Efforts in Multi-body Constrained Dynamics simulations

$$\frac{\partial}{\partial t} p_i = F_i + \lambda_a \frac{\partial C^a}{\partial x^i}, \qquad \text{with} \quad C^a(x_i, t) \approx 0$$

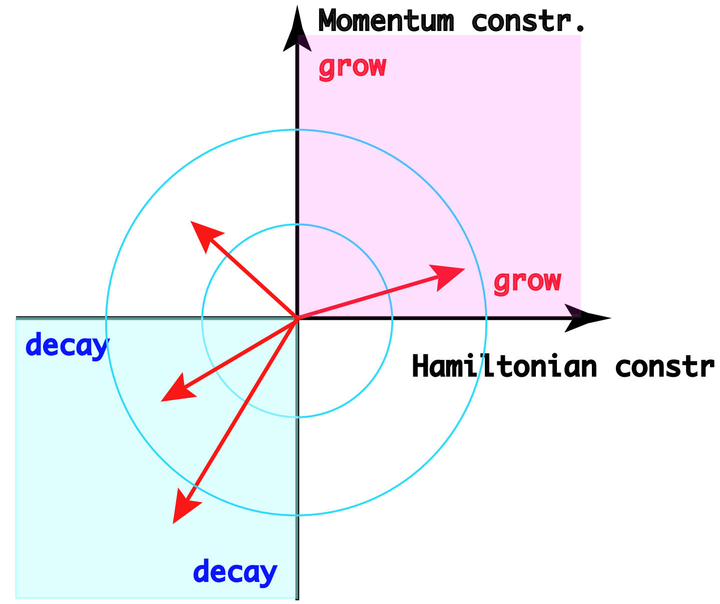
- **J. Baumgarte (1972, Comp. Methods in Appl. Mech. Eng.)**
  Replace a holonomic constraint $\partial_t^2 C = 0$ as $\partial_t^2 C + \alpha \partial_t C + \beta^2 C = 0$.
- **Park-Chiou (1988, J. Guidance), "penalty method"**
  Derive "stabilization eq." for Lagrange multiplier $\lambda(t)$.
- **Nagata (2002, Multibody Dyn.)**
  Introduce a scaled norm, $J = C^T S C$, apply $\partial_t J + w^2 J = 0$, and adjust $\lambda(t)$.

e.g.) Efforts in Molecular Dynamics simulations

- **Constant pressure ······ potential piston!**
- **Constant temperature ······ potential thermostat!! (Nosé, 1991, PTP)**

(2) target to control each constraint violation by adjusting multipliers.

CP-eigenvectors indicate directions of constraint grow/decay, if CP-matrix is diagonalizable.



(3) clarify the reasons of non-linear violation in the last stage of current test evolutions.

(4) Alternative new ideas?

– control theories, optimization methods (convex functional theories), mathematical programming methods, or ....

(5) Numerical comparisons of formulations, links to other systems, ...

– "Comparisons of Formulations" (Mexico NR workshop, 2002-2003-2004-)

– with MHD people, mini-symposium at The 5th International Congress on Industrial and Applied Mathematics (Sydney, July 2003).

Appendix : By the way, ….

From Jan. 2004, Hisaaki will change his primary job to
the coordinator of Kyoto prize committee at the Inamori foundation.

http://www.inamori-f.or.jp/

Advanced Technology / Basic Science / Arts and Philosophy

awarded with Medal and 50 Million yen (500 Mil Mex Pesos)

Eugene N Parker, George M Whitesides, Donald E Knuth, Claude E Shannon, Jan H Oort, Avram N Chomsky, John Cage, Maurice Bejart, Akira Kurosawa, ...

See you in Kyoto!

shinkai@einstein1905.info

Summary

Towards a stable and accurate formulation for numerical relativity

We tried to understand the background in an unified way.

- Our proposal = "Evaluate eigenvalues of constraint propagation eqns"
  We give satisfactory conditions for stable evolutions.
  Fourier-mode analysis allows us to discuss lower-order terms.

- Our Observation = "Stability will change by adding constraints in RHS"
  Named "Adjusted System".
  Theoretical supports are given by Constraint Propagation Analysis.
  - Maxwell system
  - Ashtekar system
  - ADM system .... (also explain effective parameter ranges of ADM-Detweiler)
  - BSSN system

When re-formulating the system, evaluation of CAFs may be an alternative guideline to hyperbolization.