Controlling Constraint Violation using Adjusted ADM Systems

Hisa-aki Shinkai (Osaka Institute of Technology, Japan)
Gen Yoneda (Waseda University, Japan)

Summary & Outlook

- Formulation problem of Numerical Relativity
- Longer Evolutions by Adjusted ADM Systems (ADM+Lagrange multipliers)
- Teukolsky wave, 3+1propagataion
  ⇒ Standard ADM x 1.5~4 life-time

- Trying to keep the Error at small value is better than to force the Error to zero.

  ⇒ Next Step: Develop Auto-Control system of Lagrange multipliers
  ⇒ Similar results must be hold either at Adjusted BSSN Systems.

Formulation Problem?

Numerical Relativity
= Necessary for unveiling the strong nature of gravity

- GWs from NS-NS, BH-BH, NS-BH coalescences
- Relativistic phenomena like Cosmology, AGN, ...
- Mathematical feedbacks to Singularity, Exact Solutions, ...
- Laboratory of Gravitational theories; Higher dimensional models, ...

Current Standard Formulation

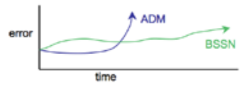
- BH-BH, NS-BH simulations
  (2005-- Pretorius, UTB, NASA, PSU, LSU, ...)

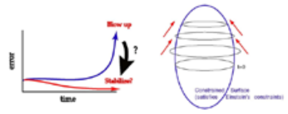
- BSSN formulation
  lapse function: 1+log slicing
  shift vector: Gamma-freezing driver
  initial data: puncture initial data

This combination works, anyway.

Why? Alternatives?

blow-up





For a review, please take a look
Shinkai & Yoneda, gr-qc/0209111

Adjusted Systems

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) + F(C^a, \partial_b C^a, \cdots)$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$ to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \cdots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) + G(C^a, \partial_b C^a, \cdots)$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = A(\hat{C}^a)\, \hat{C}^k$

「+0」 appropriate adjustments ⇒ Better Stability

ADM vs BSSN | Adjusted ADM | Adjusted BSSN

Guidelines for Better Formulation

Eigenvalue-analysis of Constraint Propagation eqs.

Conjecture on Constraint Amplification Factors (CAFs):

$$\partial_t \begin{pmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{pmatrix} = \begin{pmatrix} \text{Constraint} \\ \text{Propagation} \\ \text{Matrix} \end{pmatrix} \begin{pmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{pmatrix},$$

Eigenvalues = CAFs

We see more stable evolution, if CAFs have

(A) negative real-part (the constraints are forced to be diminished), or

(B) non-zero imaginary-part (the constraints are propagating away).

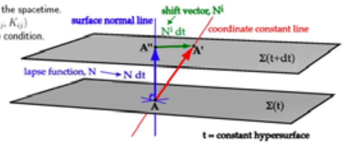
Adjusted ADM | ADM vs BSSN | Adjusted BSSN

- Standard ADM has constraint violating mode!
- Better ADM must be available ☜ ☜ This Work
- Better BSSN must be available

The Standard ADM Formulation

(Arnowitt-Deser-Misner, 1962; York 1978)

3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.



	Maxwell eqs.	ADM Einstein eq.
constraints	$\text{div}\, \mathbf{E} = 4\pi\rho$ $\text{div}\, \mathbf{B} = 0$	${}^{(3)}R + (\text{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \text{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} = \text{rot}\, \mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -\text{rot}\, \mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \text{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \text{tr}S)\}$

BSSN Formulation

(Nakamura et al, 1987; Shibata-Nakamura 1995, Baumgarte-Shapiro 1999)

Adjusted ADM formulation (1)

We adjust the standard ADM system using constraints as:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \quad (1)$$
$$+ P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \quad (2)$$
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k)K_{kj} + (\nabla_j \beta^k)K_{ki} + \beta^k \nabla_k K_{ij} \quad (3)$$
$$+ R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \quad (4)$$

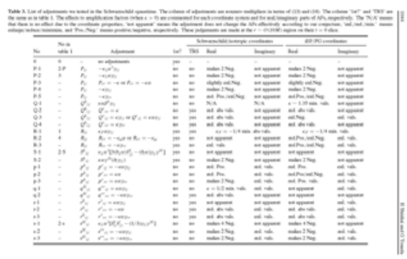
with constraint equations

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij}K^{ij}, \quad (5)$$
$$\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K. \quad (6)$$

We can write the adjusted constraint propagation equations as

$$\partial_t \mathcal{H} = (\text{original terms}) + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], \quad (7)$$
$$\partial_t \mathcal{M}_i = (\text{original terms}) + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. \quad (8)$$

Adjusted ADM formulation (2)



HS, Yoneda, CQG 19 (2002) 1027

Numerical Tests (method)

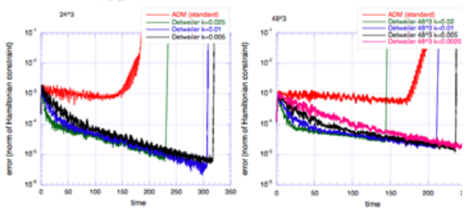
- Cactus-based original "GR" code
  http://www.cactuscode.org/
  [CactusBase+CactusPUGH+GR]
- 3+1dim, linear wave evolution (Teukolsky wave)
- harmonic slice
- periodic boundary, [-3,+3]
- iterative Crank-Nicholson method
- 12^3, 24^3, 48^3, 96^3

Towards standard testbeds for numerical relativity
Mexico Numerical Relativity Workshop 2002 Participants
CQG 21 (2004) 589-613

Numerical Tests (Detweiler-type)

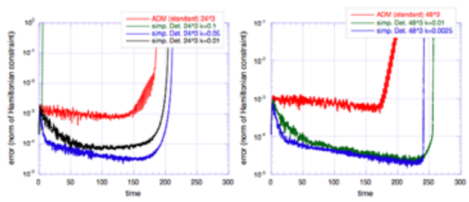
PRD35(1987)1095

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij}\mathcal{H}$$
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k)K_{kj} + (\nabla_j \beta^k)K_{ki} + \beta^k \nabla_k K_{ij}$$
$$+ \kappa_L \alpha^3 (K_{ij} - (1/3)K\gamma_{ij})\mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}]\mathcal{M}_k$$
$$+ \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}](\nabla_k \mathcal{M}_l)$$



Numerical Tests (Simplified Detweiler)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha \gamma_{ij}\mathcal{H}$$
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k)K_{kj} + (\nabla_j \beta^k)K_{ki} + \beta^k \nabla_k K_{ij}$$



Numerical Tests (Detweiler, k-adjust)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij}\mathcal{H}$$
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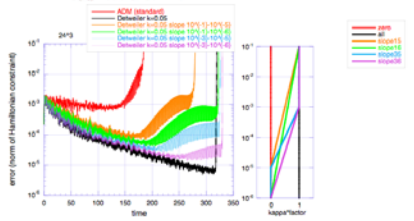

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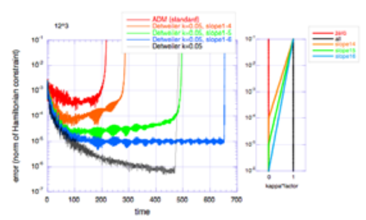

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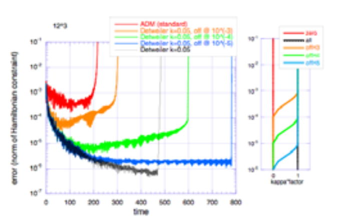

shinkai@is.oit.ac.jp
http://www.is.oit.ac.jp/~shinkai/

@ GRG18, Sydney, Australia, 2007/7/8-14