Toward the dynamics in Gauss-Bonnet gravity

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- We present the (N+1)-dimensional Gauss-Bonnet gravity equations in the ADM-like space-time decomposed form which shall be used for future numerical simulations.
- We also prepared the set of equations for obtaining the initial data using the standard conformal approach to solve the constraint equations.
- The set of dynamical equations seem to require an iterative scheme.
- Coding is in progress.

Einstein-Gauss-Bonnet action

- $(N+1)$-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$

$$S=\int_{\mathcal{M}} d^{N+1}X\sqrt{-g}\left[\frac{1}{2\kappa^2}(\mathcal{R}-2\Lambda+\alpha_{GB}\mathcal{L}_{GB})+\mathcal{L}_{\rm matter}\right] \tag{1}$$

$$\mathcal{L}_{GB}=\mathcal{R}^2-4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

- The action gives the gravitational equation

$$\mathcal{G}_{\mu\nu}+\alpha_{GB}\mathcal{H}_{\mu\nu}=\kappa^2\mathcal{T}_{\mu\nu} \tag{2}$$

where

$$\begin{aligned}
\mathcal{G}_{\mu\nu} &= \mathcal{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\mathcal{R}+\Lambda g_{\mu\nu},\\
\mathcal{H}_{\mu\nu} &= 2\left[\mathcal{R}\mathcal{R}_{\mu\nu}-2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu}-2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta}+\mathcal{R}_{\mu}{}^{\alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}\right]-\frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB},\\
\mathcal{T}_{\mu\nu} &= -2\frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}}+g_{\mu\nu}\mathcal{L}_{\rm matter}.
\end{aligned}$$

Projections to Hypersurface Σ_N (spacelike or timelike) (1)

- the projection operator,

$$\perp_{\mu\nu}=g_{\mu\nu}-\varepsilon n_\mu n_\nu,\quad n_\mu n^\mu=\varepsilon$$

where n_μ is the unit-normal vector to Σ with n_μ is timelike (if $\varepsilon=-1$) or spaceli
Σ is spacelike (timelike) if n_μ is timelike (spacelike).

- The projections of the gravitational equation:

$$\begin{aligned}
(\mathcal{G}_{\mu\nu}+\alpha_{GB}\mathcal{H}_{\mu\nu})n^\mu n^\nu &= \kappa^2 T_{\mu\nu}n^\mu n^\nu =: \kappa^2\rho_H,\\
(\mathcal{G}_{\mu\nu}+\alpha_{GB}\mathcal{H}_{\mu\nu})n^\mu \perp^\nu_\rho &= \kappa^2 T_{\mu\nu}n^\mu \perp^\nu_\rho =: -\kappa^2 J_\rho,\\
(\mathcal{G}_{\mu\nu}+\alpha_{GB}\mathcal{H}_{\mu\nu})\perp^\mu_\rho\perp^\nu_\sigma &= \kappa^2 T_{\mu\nu}\perp^\mu_\rho\perp^\nu_\sigma =: \kappa^2 S_{\rho\sigma},
\end{aligned}$$

where we defined

$$T_{\mu\nu}=\rho_H n_\mu n_\nu+J_\mu n_\nu+J_\nu n_\mu+S_{\mu\nu},\quad T=-\rho_H+S^\ell{}_\ell$$

- Introduce the extrinsic curvature K_{ij}

$$K_{ij}:=-\frac{1}{2}\pounds_n h_{ij}=-\perp^\alpha_i\perp^\beta_j\nabla_\alpha n_\beta,$$

where $\pounds_n$ denotes the Lie derivative in the n-direction and ∇ and D_i is the co
entiation with respect to $g_{\mu\nu}$ and γ_{ij}, respectively.

Projections to Hypersurface Σ_N (spacelike or timelike) (2)

- Projection of the $(N+1)$-dimensional Riemann tensor onto Σ_N

Gauss eq. $\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha_i\perp^\beta_j\perp^\gamma_k\perp^\delta_l=R_{ijkl}-\varepsilon K_{ik}K_{jl}+\varepsilon K_{il}K_{jk},$ (8)

Codacci eq. $\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha_i\perp^\beta_j\perp^\gamma_k n^\delta=-2D_{[i}K_{j]k},$ (9)

$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha_i\perp^\gamma_k n^\beta n^\delta=\pounds_n K_{ik}+K_{i\ell}K^\ell{}_k,$ (10)

- Curvature relations

$$\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}-\varepsilon(K_{\mu\rho}K_{\nu\sigma}-K_{\mu\sigma}K_{\nu\rho}-n_\mu D_\rho K_{\nu\sigma}+n_\mu D_\sigma K_{\rho\nu}+n_\nu D_\rho K_{\sigma\mu}-n_\nu D_\sigma K_{\rho\mu}\\
&\quad -n_\rho D_\mu K_{\nu\sigma}+n_\rho D_\nu K_{\mu\sigma}+n_\sigma D_\mu K_{\nu\rho}-n_\sigma D_\nu K_{\mu\rho})\\
&\quad +n_\mu n_\rho K_{\nu\alpha}K^\alpha{}_\sigma-n_\mu n_\sigma K_{\nu\alpha}K^\alpha{}_\rho-n_\nu n_\rho K_{\mu\alpha}K^\alpha{}_\sigma+n_\nu n_\sigma K_{\mu\alpha}K^\alpha{}_\rho\\
&\quad +n_\mu n_\rho\pounds_n K_{\nu\sigma}-n_\mu n_\sigma\pounds_n K_{\nu\rho}-n_\nu n_\rho\pounds_n K_{\mu\sigma}+n_\nu n_\sigma\pounds_n K_{\mu\rho} \qquad (11)\\
\mathcal{R}_{\mu\nu} &= R_{\mu\nu}-\varepsilon[KK_{\mu\nu}-2K_{\mu\alpha}K^\alpha{}_\nu+n_\mu(D_\alpha K^\alpha{}_\nu-D_\nu K)+n_\nu(D_\alpha K^\alpha{}_\mu-D_\mu K)]\\
&\quad +n_\mu n_\nu K_{\alpha\beta}K^{\alpha\beta}+\varepsilon\pounds_n K_{\mu\nu}+n_\mu n_\nu\gamma^{\alpha\beta}\pounds_n K_{\alpha\beta}, \qquad (12)\\
\mathcal{R} &= R-\varepsilon(K^2-3K_{\alpha\beta}K^{\alpha\beta}-2\gamma^{\alpha\beta}\pounds_n K_{\alpha\beta}). \qquad (13)
\end{aligned}$$

$N+1$ Einstein-Gauss-Bonnet equations

Substituting (11)-(13) into (3) or (4)-(6), we find:

(a) dynamical equations for γ_{ij}:

$$\begin{aligned}
&M_{ij}-\frac{1}{2}M\gamma_{ij}-\varepsilon(-K_{ia}K^a{}_j+\gamma_{ij}K_{ab}K^{ab}-\pounds_n K_{ij}+\gamma_{ij}\gamma^{ab}\pounds_n K_{ab})\\
&+2\alpha_{GB}[H_{ij}+\varepsilon(M\pounds_n K_{ij}-2M_i{}^a\pounds_n K_{aj}-2M_j{}^a\pounds_n K_{ai}-W_{ij}{}^{ab}\pounds_n K_{ab})]=\kappa^2\mathcal{T}_{\mu\nu}\gamma^\mu{}_i\gamma^\nu{}_j
\end{aligned}$$

(b) Hamiltonian constraint equation:

$$M+\alpha_{GB}(M^2-4M_{ab}M^{ab}+M_{abcd}M^{abcd})=-2\varepsilon\kappa^2\mathcal{T}_{\mu\nu}n^\mu n^\nu$$

(c) momentum constraint equation:

$$N_i+2\alpha_{GB}(MN_i-2M_i{}^aN_a+2M^{ab}N_{iab}-M_i{}^{cab}N_{abc})=-\kappa^2\mathcal{T}_{\mu\nu}n^\mu\gamma^\nu{}_i$$

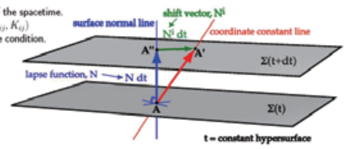
$$\begin{aligned}
M_{ijkl} &= R_{ijkl}-\varepsilon(K_{ik}K_{jl}-K_{il}K_{jk})\\
M_{ij} &= \gamma^{ab}M_{aibj}=R_{ij}-\varepsilon(KK_{ij}-K_{ia}K^a{}_j)\\
M &= \gamma^{ab}M_{ab}=R-\varepsilon(K^2-K_{ab}K^{ab})\\
N_{ijk} &= D_iK_{jk}-D_jK_{ik}\\
N_i &= \gamma^{ab}N_{aib}=D_aK_i^a-D_iK\\
W_{ij}{}^{kl} &= M\gamma_{ij}\gamma^{kl}-2M_{ij}\gamma^{kl}-2\gamma_{ij}M^{kl}+2M_{iajb}\gamma^{ak}\gamma^{bl}
\end{aligned}$$

$$\begin{aligned}
H_{ij} = {}& MM_{ij}-2(M_{ia}M^a{}_j+M^{ab}M_{iajb})+M_{iabc}M_j{}^{abc}\\
&-2\varepsilon[-K_{ab}K^{ab}M_{ij}-\tfrac{1}{2}MK_{ia}K^a{}_j+K_{ab}K^b{}_jM^a{}_i+K_{ia}K^b{}_jM^a{}_b+K^{a}{}_{c}K^{bc}M_{iajb}\\
&+N_iN_j-N^a(N_{aij}+N_{aji})-\tfrac{1}{2}N_{abi}N^{ab}{}_j-N_{iab}N_j{}^{ab}]\\
&-\tfrac{1}{4}\gamma_{ij}[M^2-4M_{ab}M^{ab}+M_{abcd}M^{abcd}]\\
&-\varepsilon\gamma_{ij}[K_{ab}K^{ab}M-2M_{ab}K^{ac}K_c{}^b-2N_aN^a+N_{abc}N^{abc}]
\end{aligned}$$

The Standard ADM Formulation

(Arnowitt-Deser-Misner, 1962; York 1978)

3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.



	Maxwell eqs.	ADM Einstein eq.
constraints	$\mathrm{div}\,\mathbf{E}=4\pi\rho$ $\mathrm{div}\,\mathbf{B}=0$	${}^{(3)}R+(\mathrm{tr}K)^2-K_{ij}K^{ij}=2\kappa\rho_H+2\Lambda$ $D_jK^j{}_i-D_i\mathrm{tr}K=\kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t\mathbf{E}=\mathrm{rot}\,\mathbf{B}-\frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t\mathbf{B}=-\mathrm{rot}\,\mathbf{E}$	$\partial_t\gamma_{ij}=-2NK_{ij}+D_jN_i+D_iN_j,$ $\partial_tK_{ij}=N({}^{(3)}R_{ij}+\mathrm{tr}KK_{ij})-2NK_{il}K^l{}_j-D_iD_jN$ $+(D_jN^m)K_{mi}+(D_iN^m)K_{mj}+N^mD_mK_{ij}-N\gamma_{ij}\Lambda$ $-\kappa\alpha\{S_{ij}+\frac{1}{2}\gamma_{ij}(\rho_H-\mathrm{tr}S)\}$

Conformal Approach to solve constraints : Initial Data construction

Conformal approach by York and ÓMurchadha (1974)

- Conformal transformation

solution $\gamma_{ij}=\psi^4\hat{\gamma}_{ij}$ trial metric

this gives $${}^{(N)}R=\frac{{}^{(N)}\hat{R}}{\psi^4}-\frac{4(N-1)g^{ac}\nabla_a\nabla_c\psi}{\psi^5}-\frac{4(N-1)(N-3)g^{ac}(\nabla_a\psi)(\nabla_c\psi)}{\psi^6}$$

- decompose the extrinsic curvature K_{ij} as $K_{ij}\equiv A_{ij}+\frac{1}{N}\gamma_{ij}K$, and assume

$$K\equiv\hat{K},\quad A^{ij}\equiv\psi^{-10}\hat{A}^{ij},\quad A_{ij}=\psi^{-2}\hat{A}_{ij}.$$

- decompose A_{ij} as $\hat{A}^{ij}=\hat{A}^{ij}_{TT}+(\hat{\mathbb{I}}W)^{ij}$, where $\hat{D}_j\hat{A}^{ij}_{TT}=0$.
The latter part can be expressed as $(\hat{\mathbb{I}}W)^{ij}=\hat{D}^iW^j+\hat{D}^jW^i-\frac{2}{N}\hat{\gamma}^{ij}\hat{D}_kW^k$
- If the matter term exists, assume

$$\rho=\psi^{-n}\hat{\rho},\qquad J^i=\psi^{-10}\hat{J}^i$$

(A) Hamiltonian constraint

$$\begin{aligned}
&4(N-1)\hat{\Delta}\psi+4(N-1)(N-3)(\nabla_a\psi)(\nabla^a\psi)\psi^{-1}\\
&={}^{(N)}\hat{R}\psi-(\hat{A}_{ij}\hat{A}^{ij})\psi^{-7}+\left[\frac{N-1}{N}(\mathrm{tr}K)^2-2\Lambda\right]\psi^5-16\pi G\hat{\rho}\psi^{5-n}\\
&\quad+\alpha_{GB}\left(M^2-4M_{ij}M^{ij}+M_{ijkl}M^{ijkl}\right)\psi^5
\end{aligned}$$

(B) momentum constraint

$$\begin{aligned}
&\hat{\Delta}W^i+\frac{N-2}{N}\hat{D}^i\hat{D}_kW^k+\hat{R}^i{}_kW^k\\
&=\frac{N-1}{N}\psi^6\hat{D}^iK+8\pi G\hat{J}^i-2\alpha_{GB}\left(MN^i-2M^{ik}N_k+2M^{kl}N^i_{kl}-M^{ijk}N_{jkl}\right)
\end{aligned}$$

Procedures to construct the initial hypersurface data $(\gamma_{ij}, K_{ij}, \rho, J^i)$

1. Give the initial assumption (trial values) for $\hat{\gamma}_{ij}$, tr$K$, $\hat{A}^{TT}_{ij}$ and $\hat{\rho}$, $\hat{J}$.
2. Solve (A) and (B) for ψ and W^i.
3. inverse conformal transformations,

$$\gamma_{ij}=\psi^4\hat{\gamma}_{ij},\qquad K_{ij}=\psi^{-2}[\hat{A}^{TT}_{ij}+(\hat{\mathbb{I}}W)_{ij}]+\frac{1}{N}\psi^4\hat{\gamma}_{ij}\mathrm{tr}K,$$
$$\rho=\psi^{-n}\hat{\rho},\qquad J^i=\psi^{-10}\hat{J}^i$$

$N+1$ Einstein-Gauss-Bonnet evolution equations

$$\begin{aligned}
&M_{ij}-\frac{1}{2}M\gamma_{ij}-\varepsilon(-K_{ia}K^a{}_j+\gamma_{ij}K_{ab}K^{ab}-\pounds_nK_{ij}+\gamma_{ij}\gamma^{ab}\pounds_nK_{ab})\\
&+2\alpha_{GB}[H_{ij}+\varepsilon(M\pounds_nK_{ij}-2M_i{}^a\pounds_nK_{aj}-2M_j{}^a\pounds_nK_{ai}-W_{ij}{}^{ab}\pounds_nK_{ab})]=\kappa^2\mathcal{T}_{\mu\nu}\gamma^\mu{}_i\gamma^\nu{}_j
\end{aligned}$$

- $\pounds_nK_{\mu\nu}$ terms appear only in the linear form, due to the quasi-linear property of the Gauss-Bonnet gravity.
- Iterative scheme is necessary.

$$\begin{pmatrix}\pounds_n\gamma_{11}\\ \pounds_n\gamma_{12}\\ \pounds_n\gamma_{13}\\ \vdots\\ \pounds_nK_{11}\\ \pounds_nK_{12}\\ \pounds_nK_{13}\\ \vdots\end{pmatrix}=\begin{pmatrix}O & O\\ O & \text{Mixing}\end{pmatrix}\begin{pmatrix}\pounds_n\gamma_{11}\\ \pounds_n\gamma_{12}\\ \pounds_n\gamma_{13}\\ \vdots\\ \pounds_nK_{11}\\ \pounds_nK_{12}\\ \pounds_nK_{13}\\ \vdots\end{pmatrix}+\begin{pmatrix}K_{11}\\ K_{12}\\ K_{13}\\ \vdots\\ \text{Source}\end{pmatrix}$$

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@ JGRG17, Nagoya, 2007/12/3-7