

# Autoregressive Approach to Extract Ring-down Gravitational Wave of Black-hole Merger



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## Outline & Summary

We apply an autoregressive (AR) model for identifying the ring-down part of gravitational wave of binary black-hole mergers.

This approach enables us to extract signals without templates, and is effective for short-period data. After having experience of parameters using mock data, we apply to extract the ring-down frequency of the remnant black-hole of GW150914, GW170104, and GW170814, of which ring-down waves are expected to be included in LIGO data.

We find that AR analysis extracts ringdown part for GW150914 and GW170814 with consistent mass and spin of the remnant BH which were reported by LIGO/Virgo group. However, we failed for GW170104, which might be due to the small S/N (=13) compared to the others (S/N=23.7 & 18).

## Example with mock data

### Mock data example (1) fitting well

Mock Data (Nakano02 p)

Fitting data with linear func.

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \dots + a_M x_{n-M} + \epsilon$$

$$= \sum_{j=1}^M a_j x_{n-j} + \epsilon$$

a.1 = -2.235e+00  
a.2 = 1.869e+00  
a.3 = -5.545e-01

### Mock data example (2) spectrogram

Mock Data (Nakano02 p)

power spectrum

$$p(f) = \frac{\sigma^2}{\left| 1 - \sum_{j=1}^M a_j e^{-i2\pi j f \Delta t} \right|^2}$$

### Mock data example (3) characteristic eq.

Mock Data (Nakano02 p)

Fitting data with linear func.

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \dots + a_M x_{n-M} + \epsilon$$

$$= \sum_{j=1}^M a_j x_{n-j} + \epsilon$$

a.1 = -2.235e+00  
a.2 = 1.869e+00  
a.3 = -5.545e-01

characteristic eq.

$$f(z) = 1 - \sum_{j=1}^M a_j z^j = 0$$

$|z_k|$  says amplitude,  
 $\arg(z_k)$  says frequency.

x.r	x.i	f_R [Hz]	x	f_I [Hz]
1	0.962	0.566	346.800	8.025e-01
2	0.962	-0.566	-346.800	8.025e-01
3	1.447	0.000	0.000	4.775e-01

### Mock data example (4) identify ring-down freq.

Mock Data (Nakano02 p)

characteristic eq.

$$f(z) = 1 - \sum_{j=1}^M a_j z^j = 0$$

$$x_{n-1} = z x_n$$

$$z = \exp[-2\pi i f \Delta t]$$

$|z_k|$  says amplitude,  
 $\arg(z_k)$  says frequency.

x.r	x.i	f_R [Hz]	x	f_I [Hz]
1	0.962	0.566	346.800	8.025e-01
2	0.962	-0.566	-346.800	8.025e-01
3	1.447	0.000	0.000	4.775e-01

## Method (general)

### Auto-Regressive model (Method, general) I

Fitting data with linear func.

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \dots + a_M x_{n-M} + \epsilon$$

$$= \sum_{j=1}^M a_j x_{n-j} + \epsilon$$

e.g.  $x_n = A e^{-r n \Delta t} \cos(\omega n \Delta t)$   
 $Z_1 = e^{-(r-j\omega)\Delta t} \rightarrow x_n = \frac{A}{2} (Z_1^n + Z_2^n) = (Z_1 + Z_2)x_{n-1} - Z_1 Z_2 x_{n-2}$   
 $Z_2 = e^{-(r+j\omega)\Delta t}$

can be applied also to noisy data by adjusting M

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### Auto-Regressive model (Method, general) II

Fitting data with linear func.

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \dots + a_M x_{n-M} + \epsilon$$

$$= \sum_{j=1}^M a_j x_{n-j} + \epsilon$$

- find  $a_j, M$  (maximum entropy method, final prediction error method)
- re-construct wave signal from fitted function
- apply FFT with arbitrary precision.

characteristic eq.  $f(z) = 1 - \sum_{j=1}^M a_j z^j = 0$   
 $|z_k|$  says amplitude,  
 $\arg(z_k)$  says frequency.

power spectrum  $p(f) = \frac{\sigma^2}{\left| 1 - \sum_{j=1}^M a_j e^{-i2\pi j f \Delta t} \right|^2}$

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### Auto-Regressive model vs Short FFT

segment = 1/128 sec = 32 points

shift = 1/1024 sec = 4 points

The order M can be fixed at 2~8.

Even for short segment, AR model shows precise power-spectrum.

## Application to GW150914, GW170104, GW170814

### Ringdown wave of GW150914

LIGO paper

4096 sampling rate  
100-400 Hz filter  
1 segment = 1/64 sec = 64 points  
1 shift = 1/512 sec = 8 points

AR model Hanford

max M = 3

### Ringdown wave of GW150914

16384 sampling rate

4096 sampling rate  
100-400 Hz filter  
1 segment = 1/64 sec = 64 points  
1 shift = 1/512 sec = 8 points

AR model Livingston

max M = 3

### Ringdown wave of GW150914

t	f real	f imag	
4486	1	0.1642258E+02	0.313508E+03
4486x	2	0.1642258E+02	0.313508E+03
4489	1	0.164314E+02	0.317807E+03
4490	1	0.164402E+02	0.322106E+03
4491	1	0.164490E+02	0.326405E+03
4492	1	0.164578E+02	0.330704E+03
4493	1	0.164666E+02	0.335003E+03
4494	1	0.164754E+02	0.339302E+03
4495	1	0.164842E+02	0.343601E+03
4496	1	0.164930E+02	0.347900E+03
4497	1	0.165018E+02	0.352199E+03
4498	1	0.165106E+02	0.356498E+03
4499	1	0.165194E+02	0.360797E+03
4500	1	0.165282E+02	0.365096E+03
4501	1	0.165370E+02	0.369395E+03
4502	1	0.165458E+02	0.373694E+03
4503	1	0.165546E+02	0.377993E+03
4504	1	0.165634E+02	0.382292E+03
4505	1	0.165722E+02	0.386591E+03
4506	1	0.165810E+02	0.390890E+03
4507	1	0.165898E+02	0.395189E+03
4508	1	0.165986E+02	0.399488E+03
4509	1	0.166074E+02	0.403787E+03
4510	1	0.166162E+02	0.408086E+03
4511	1	0.166250E+02	0.412385E+03
4512	1	0.166338E+02	0.416684E+03
4513	1	0.166426E+02	0.420983E+03
4514	1	0.166514E+02	0.425282E+03
4515	1	0.166602E+02	0.429581E+03
4516	1	0.166690E+02	0.433880E+03
4517	1	0.166778E+02	0.438179E+03
4518	1	0.166866E+02	0.442478E+03
4519	1	0.166954E+02	0.446777E+03
4520	1	0.167042E+02	0.451076E+03
4521	1	0.167130E+02	0.455375E+03
4522	1	0.167218E+02	0.459674E+03
4523	1	0.167306E+02	0.463973E+03
4524	1	0.167394E+02	0.468272E+03
4525	1	0.167482E+02	0.472571E+03
4526	1	0.167570E+02	0.476870E+03
4527	1	0.167658E+02	0.481169E+03
4528	1	0.167746E+02	0.485468E+03
4529	1	0.167834E+02	0.489767E+03
4530	1	0.167922E+02	0.494066E+03
4531	1	0.168010E+02	0.498365E+03
4532	1	0.168098E+02	0.502664E+03
4533	1	0.168186E+02	0.506963E+03
4534	1	0.168274E+02	0.511262E+03
4535	1	0.168362E+02	0.515561E+03
4536	1	0.168450E+02	0.519860E+03
4537	1	0.168538E+02	0.524159E+03
4538	1	0.168626E+02	0.528458E+03
4539	1	0.168714E+02	0.532757E+03
4540	1	0.168802E+02	0.537056E+03
4541	1	0.168890E+02	0.541355E+03
4542	1	0.168978E+02	0.545654E+03
4543	1	0.169066E+02	0.549953E+03
4544	1	0.169154E+02	0.554252E+03
4545	1	0.169242E+02	0.558551E+03
4546	1	0.169330E+02	0.562850E+03
4547	1	0.169418E+02	0.567149E+03
4548	1	0.169506E+02	0.571448E+03
4549	1	0.169594E+02	0.575747E+03
4550	1	0.169682E+02	0.580046E+03
4551	1	0.169770E+02	0.584345E+03
4552	1	0.169858E+02	0.588644E+03
4553	1	0.169946E+02	0.592943E+03
4554	1	0.170034E+02	0.597242E+03
4555	1	0.170122E+02	0.601541E+03
4556	1	0.170210E+02	0.605840E+03
4557	1	0.170298E+02	0.610139E+03
4558	1	0.170386E+02	0.614438E+03
4559	1	0.170474E+02	0.618737E+03
4560	1	0.170562E+02	0.623036E+03
4561	1	0.170650E+02	0.627335E+03
4562	1	0.170738E+02	0.631634E+03
4563	1	0.170826E+02	0.635933E+03
4564	1	0.170914E+02	0.640232E+03
4565	1	0.171002E+02	0.644531E+03
4566	1	0.171090E+02	0.648830E+03
4567	1	0.171178E+02	0.653129E+03
4568	1	0.171266E+02	0.657428E+03
4569	1	0.171354E+02	0.661727E+03
4570	1	0.171442E+02	0.666026E+03
4571	1	0.171530E+02	0.670325E+03
4572	1	0.171618E+02	0.674624E+03
4573	1	0.171706E+02	0.678923E+03
4574	1	0.171794E+02	0.683222E+03
4575	1	0.171882E+02	0.687521E+03
4576	1	0.171970E+02	0.691820E+03
4577	1	0.172058E+02	0.696119E+03
4578	1	0.172146E+02	0.700418E+03
4579	1	0.172234E+02	0.704717E+03
4580	1	0.172322E+02	0.709016E+03
4581	1	0.172410E+02	0.713315E+03
4582	1	0.172498E+02	0.717614E+03
4583	1	0.172586E+02	0.721913E+03
4584	1	0.172674E+02	0.726212E+03
4585	1	0.172762E+02	0.730511E+03
4586	1	0.172850E+02	0.734810E+03
4587	1	0.172938E+02	0.739109E+03
4588	1	0.173026E+02	0.743408E+03
4589	1	0.173114E+02	0.747707E+03
4590	1	0.173202E+02	0.752006E+03
4591	1	0.173290E+02	0.756305E+03
4592	1	0.173378E+02	0.760604E+03
4593	1	0.173466E+02	0.764903E+03
4594	1	0.173554E+02	0.769202E+03
4595	1	0.173642E+02	0.773501E+03
4596	1	0.173730E+02	0.777800E+03
4597	1	0.173818E+02	0.782099E+03
4598	1	0.173906E+02	0.786398E+03
4599	1	0.173994E+02	0.790697E+03
4600	1	0.174082E+02	0.794996E+03

we see QNM at 300Hz, 0.003s after the merger.

Table 1. Results of frequency and damping rate of ring-down gravitational wave of GW150914.

data	$f_{\text{real}}$ [Hz]	$f_{\text{imag}}$ [Hz]	mass ( $M/M_{\odot}$ )	Kerr parameter $a/M$
Hanford	$305.94^{+15.68}_{-27.82}$	$43.55^{+13.90}_{-17.99}$	$58.74^{+0.37}_{-0.37}$	$0.75^{+0.15}_{-0.27}$
Livingston	$300.02^{+17.49}_{-27.21}$	$44.94^{+12.86}_{-18.30}$	$58.15^{+0.40}_{-0.53}$	$0.71^{+0.20}_{-0.30}$

LIGO paper says  $62.2^{+3.7}_{-3.4} M_{\odot}$   $a/M = 0.68^{+0.05}_{-0.06}$

### List of Detected GW events

	ref.	M1+M2=Mf, Mdifff/Mtotal	spin a_final	Mpc z	SNR	deg*2
GW150914	PRLL116, 061102 (2016/2/11)	36.2+29.1=62.3+3.0 4.59%	0.68	410Mpc 0.09	23.7	600
LVT151012	(2016/2/11)	23+13=35+1.5 2.78%	0.66	1000Mpc 0.20	9.7	
GW151226	PRLL116, 241103 (2016/6/15)	14.2+7.5=20.8+0.9 4.15%	0.74	440Mpc 0.09	13.0	850
GW170104	PRLL118, 221101 (2017/6/1)	31.2+19.4=48.7+1.9 3.75%	0.64	880Mpc 0.18	13	1300
GW170608	ApJ 851, L35 (2017/12/18)	12+7=18.0+1.0 5.2%	0.69	340Mpc 0.07	13	520
GW170814	PRLL119, 141101 (2017/10/6)	30.5+25.3=53.2+2.6 4.66%	0.70	540Mpc 0.11	18	60
GW170817	PRLL119, 161101 (2017/10/16)	1.36~1.60 + 1.17~1.36 = 2.74 + ?	?	40Mpc	32.4	28

GW150914 (S/N = 23.7)

GW150814 (S/N = 18)

GW170104 (S/N = 13)