# 一般相対論の数値計算手法

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## Contents



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This file and viewgraphs of the seminar are available at http://www.is.oit.ac.jp/~shinkai/

# 1 Introduction

## 1.1 一般相対性理論の概略と主要な研究テーマ (Topics in GR)

一般相対論研究者向けに，レビュー論文を更新しているサイト「Living Reviews in Relativity」[1]がある．そのサイトに掲載された論文のテーマ一覧．–published –upcoming の順．2011/12/1 現在．

- 重力波 (Gravitational Waves) 12 本+12
  - The Motion of Point Particles in Curved Spacetime; GW Detection by Interferometry (Ground and Space); GWs from Gravitational Collapse; Interferometer Techniques for GW Detection; On Special Optical Modes and Thermal Issues in Advanced GW Interferometric Detectors; Physics, Astrophysics and Cosmology with GW; The PN Approximation for Relativistic Compact Binaries; Gravitational Radiation from PN Sources and Inspiralling Compact Binaries; Low-Frequency GW Searches Using Spacecraft Doppler Tracking; Time-Delay Interferometry; GW Data Analysis. Formalism and Sample Applications: The Gaussian Case; Analytic BH Perturbation Approach to Gravitational Radiation
  - Advanced Technologies for Space GW Detectors; Extreme and Intermediate Mass-Ratio Inspiral Systems; GW Phenomenology; GW Sources: Binaries (High and Low Frequency); GW Sources: Cosmological Background; GWs from Extreme Mass Ratio Inspiral (EMRI); Interface Between GWs and Astronomy; Pulsar Timing and Low Frequency GW Detection; Quantum Measurement Theory in GW Detection; Rates for Binary Coalescences; The ADM canonical approach to the PN motion of compact binaries; The Square-Kilometre-Array (SKA)
- 数値シミュレーション (Numerical Relativity) 10 本+9
  - Coalescence of BH-Neutron Star Binaries; Characteristic Evolution and Matching; Spectral Methods for NR; Numerical Hydrodynamics and Magnetohydrodynamics in GR; Critical Phenomena in Gravitational Collapse; Event and Apparent Horizon Finders for 3+1 NR; Numerical Hydrodynamics in Special Relativity; Numerical Approaches to Spacetime Singularities; Computational Cosmology: From the Early Universe to the Large Scale Structure; Initial Data for NR
  - Algebraic Computing in GR; Binary Neutron Star Mergers; Boson Stars; Formulations of Einstein's Equations for NR; Interface of PN Theories and NR; Methods of GW Extraction in NR; NR for BHs; Numerical Simulations of Supernovae; Perturbative Interface to the Binary BH Problem
- 数学的な側面 (Mathematical Relativity) 11 本+5
  - The Einstein-Vlasov System/Kinetic Theory; Cosmic Censorship for Gowdy Spacetimes; Null Geodesic Congruences, Asymptotically-Flat Spacetimes and Their Physical Interpretation; Quasi-Local Energy-Momentum and Angular Momentum in GR; Theorems on Existence and Global Dynamics for the Einstein Equations; Isolated and Dynamical Horizons and Their Applications; Gravitational Lensing from a Spacetime Perspective; Conformal Infinity; Speeds of Propagation in Classical and Relativistic Extended Thermodynamics; Stationary BHs: Uniqueness and Beyond; Hyperbolic Methods for Einstein's Equations
  - Continuum and Discrete Initial-Boundary-Value Problems and Einstein's Field Equations; Cosmic Censorship (toolbox); Exact Solutions; Gravitational Lensing from a Spacetime Perspective; The Constraint Problem for Einstein's Equations
- 量子重力 (Quantum General Relativity) 11 本+4
  - Entanglement Entropy of BHs; Quantization of Midisuperspace Models; Loop QG; Loop Quantum Cosmology; Stochastic Gravity: Theory and Applications; The Asymptotic Safety Scenario in QG; QG in 2+1 Dimensions: The Case of a Closed Universe; QG in Everyday Life: GR as an Effective Field Theory; Perturbative QG and its Relation to Gauge Theory; The Thermodynamics of BHs; Discrete Approaches to QG in Four Dimensions

[1] http://relativity.livingreviews.org/

  - Causal Sets; Minimal Length Scale Scenarios for QG; QG Phenomenology; The Spin Foam Approach to QG

- 実験的検証 (Experimental Foundations of Gravitation) 10 本+5
  - Analogue Gravity; Varying Constants, Gravitation and Cosmology; Tests of Gravity Using Lunar Laser Ranging; The Pioneer Anomaly; f(R) Theories; Probes and Tests of Strong-Field Gravity with Observations in the Electromagnetic Spectrum; The Confrontation between GR and Experiment; Modern Tests of Lorentz Invariance; Testing GR with Pulsar Timing; Relativity in the Global Positioning System
  - Experiments in Gravitation with Highly Stable Clocks; Laboratory Measurements of Newtons's Constant, G; MOND; Testing Gravity Using GWs; Tests of Gravity at Short Range

- 宇宙物理現象 (Relativity in Astrophysics) 9 本+5
  - Physics of Neutron Star Crusts; Binary and Millisecond Pulsars; Relativistic Fluid Dynamics: Physics for Many Different Scales; The Evolution of Compact Binary Star Systems; Relativistic Binaries in Globular Clusters; Massive BH Binary Evolution; Rotating Stars in Relativity; Quasi-Normal Modes of Stars and BHs; Gravitational Lensing in Astronomy
  - BH Accretion Disks; Electromagnetic Counterparts to Supermassive BH Mergers; Massive BHs in Galaxies; Microquasars; The Magnetic Fields of Neutron Stars

- 弦理論 (String Theory and Gravitation) 4 本+3
  - Brane-World Gravity; BHs in Higher Dimensions; Spacelike Singularities and Hidden Symmetries of Gravity; Spinning Strings and Integrable Spin Chains in the AdS/CFT Correspondence
  - Brane Actions and Kappa-Symmetry; Classification of Near-Horizon Geometries of Extremal BHs; Solitonic Solutions of Supergravity

- 宇宙論 (Physical Cosmology) 5 本+2
  - The Hubble Constant; Measuring our Universe from Galaxy Redshift Surveys; Experimental Searches for Dark Matter; The Cosmological Constant; The Cosmic Microwave Background
  - Cosmic Evolution of Super Massive BHs in Galactic Centers (the X-Ray view); The Age of the Universe

- 科学史 (History of Relativity) 2 本+3
  - History of Astroparticle Physics and its Components; On the History of Unified Field Theories
  - History of GW Research; On the History of Unified Field Theories (1933-ca 1960); The Hole Argument

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上記の論文タイトルで使用した略語は以下のもの.

| | |
|---|---|
| BH | Black Hole |
| GR | General Relativity |
| GW | Gravitational Wave |
| NR | Numerical Relativity |
| PN | Post-Newtonian |
| QG | Quantum Gravity |

## 1.2 なぜ数値相対論? (Why Numerical Relativity?)

The Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \qquad \kappa = 8\pi G \tag{1.1}$$

**What are the difficulties? (# 1)**

- for 10-component metric, highly nonlinear partial differential equations.
- completely free to choose coordinates, gauge conditions, and even for decomposition of the space-time.
- mixed with 4 elliptic eqs and 6 dynamical eqs if we apply 3+1 decomposition.
- has singularity in its nature.

**How to solve it?**

- find exact solutions
  - assume symmetry in space-time, and decomposition of space-time
    spherically symmetric, cylindrical symmetric, ...
  - assume simple situation and matter
    time-dependency, homogeneity, algebraic speciality, ...

  We know many exact solutions ($O(100)$) by this "Spherical Cow" approach.
- approximations
  - weak-field limit, linearization, perturbation, ...

  We know correct prediction in the solar-system, binary neutron stars, ...
  We know post-Newtonian behavior, first-order correction, BH stability, ...

**Why don't we solve it using computers?**

- dynamical behavior
- strong gravitational field
- no symmetry in space
- gravitational wave!
- higher-dimensional theories, and/or other gravitational theories, ...

The most robust way to study the strong gravitational field. Great.

**Numerical Relativity** **Box 1.1**

= Solve the Einstein equations numerically.
= Necessary for unveiling the nature of strong gravity. For example:

- gravitational waves from colliding black holes, neutron stars, supernovae, ...
- relativistic phenomena like cosmology, active galactic nuclei, ...
- mathematical feedback to singularity, exact solutions, chaotic behavior, ...
- laboratory for gravitational theories, higher-dimensional models, ...

**What are the difficulties? (# 2)**

- How to construct a realistic initial data?
- How to treat black-hole singularity?
- We cannot evolve the system stably in long-term evolution. Why?

**General and recent introductions**
More general and recent introductions to numerical relativity are available, e.g. by Pretorius (2007) [4], Alcubierre (2008) [1], Baumgarte-Shapiro (2010) [2], and Gourgoulhon (2012) [3].

# References

[1] M. Alcubierre, *Introduction to 3+1 Numerical Relativity* (International Series of Monographs on Physics), (Oxford University Press, 2008).

[2] T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity: Solving Einstein's Equations on the Computer*, (Cambridge University Press, 2010).

[3] E. Gourgoulhon, *3+1 Formalism in General Relativity: Bases of Numerical Relativity* (Lecture Notes in Physics), (Springer-Verlag, 2012)

[4] F. Pretorius, in *Relativistic Objects in Compact Binaries: From Birth to Coalescence*, Editor: Colpi et al. Pulisher: Springer Verlag, Canopus Publishing Limited, arXiv:0710.1338.

### 1.3 数値相対論の方法論概略 (Overview of Numerical Relativity Methodology)

**Numerical Relativity – Methodology** **Box 1.2**

0. How to foliate space-time
   - Cauchy $(3+1)$,
   - Hyperboloidal $(3+1)$, ⇒ see e.g. [2]
   - characteristic $(2+2)$, ⇒ see e.g. [5]
   - or combined?

⇒ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data ⇒ see e.g. [1]

   Theoretical: Proper formulation for solving constraints?
   How to prepare realistic initial data?
   Effects of background gravitational waves?
   Connection to the post-Newtonian approximation?

   Numerical: Techniques for solving coupled elliptic equations?
   Appropriate boundary conditions?

2. How to evolve the data

   Theoretical: Free evolution or constrained evolution?
   Proper formulation for the evolution equations? ⇒ see e.g. [4, 3]
   Suitable slicing conditions (gauge conditions)?

   Numerical: Techniques for solving the evolution equations?
   Appropriate boundary treatments?
   Singularity excision techniques?
   Matter and shock surface treatments?
   Parallelization of the code?

3. How to extract the physical information

   Theoretical: Gravitational wave extraction?
   Connection to other approximations?

   Numerical: Identification of black hole horizons?
   Visualization of simulations?

## References


[1] G. Cook, Livng Rev. Relativ. **2000-5** at `http://www.livingreviews.org/`

[2] S. Husa, gr-qc/0204043; gr-qc/0204057.

[3] H. Shinkai, J. Korean Phys. Soc. **54** (2009) 2513 (arXiv:0805.0068)

[4] H. Shinkai and G. Yoneda, gr-qc/0209111

[5] J. Winicour, Livng Rev. Relativ. **2009-3** at `http://www.livingreviews.org/`

**Notations:**

- signature $(-+++)$.
- Covariant derivatives, Christoffel symbol

$$\nabla_\mu A^\alpha \equiv A^\alpha_{;\mu} \equiv A^\alpha{}_{,\mu} + \Gamma^\alpha_{\mu\nu} A^\nu \tag{1.2}$$
$$\nabla_\mu A_\alpha \equiv A_{\alpha;\mu} \equiv A_{\alpha,\mu} - \Gamma^\nu_{\alpha\mu} A_\nu \tag{1.3}$$
$$\Gamma^\alpha_{\mu\nu} = (1/2) g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \tag{1.4}$$

- Riemann tensor, Ricci tensor, Weyl tensor

$$R^a{}_{bcd} \equiv \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{1.5}$$
$$R_{ab} \equiv R^\mu{}_{a\mu b} \equiv \Gamma^\mu_{ab,\mu} - \Gamma^\mu_{a\mu,b} + \Gamma^\mu{}_{\nu\mu}\Gamma^\nu{}_{ab} - \Gamma^\mu{}_{\nu b}\Gamma^\nu{}_{a\mu} \tag{1.6}$$
$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} - \frac{1}{3} R g_{a[c}g_{d]b}, \tag{1.7}$$

- ADM decomposition, the extrinsic curvature (§2)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu = 0,1,2,3)$$
$$\text{on } \Sigma(t)... \quad d\ell^2 = \gamma_{ij} dx^i dx^j, \quad (i, j = 1,2,3)$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{1.8}$$

$$K_{ij} \equiv -\perp^\mu_i \perp^\nu_j n_{\mu;\nu} = -\frac{1}{2} \pounds_n \gamma_{ij}. \tag{1.9}$$

## 2 時間発展を考えるための時空の分解

ここでは，Einstein 方程式

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad \text{where} \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \quad \text{and} \quad \kappa = 8\pi G \tag{2.1}$$

を「時間発展を追う」形式に書き換える方法を説明する．

### 2.1 ADM 形式 (ADM formulation)

#### 2.1.1 The 3+1 decomposition of space-time

The idea of space-time evolution was first formulated by Arnowitt, Deser, and Misner (ADM) [10]. The formulation was first motivated by a desire to construct a canonical framework in general relativity, but it also gave the community to the fundamental idea of time evolution of space and time: such as foliations of 3-dimensional hypersurface (Figure 2.1). This scheme is often called '3+1 formulation', 'the ADM formulation', or 'Cauchy approach'.

**3-metric, lapse function, shift vectors**
Let us denote the hypersurface $\Sigma(t)$ which is the three-dimensional spatial space with a parameter $t$. The evolution of spacetime is expressed as the dynamics of $\Sigma(t)$. The formulation begins by decomposing the metic as

$$\begin{aligned} ds^2 &= g_{\mu\nu}\, dx^\mu\, dx^\nu, \qquad (\mu, \nu = 0, 1, 2, 3) \\ \text{on } \Sigma(t)\ldots \quad d\ell^2 &= \gamma_{ij}\, dx^i\, dx^j, \qquad (i, j = 1, 2, 3) \end{aligned}$$

Let the unit normal vector of the slices be $n^\mu$, where

$$n_\mu = (-\alpha, 0, 0, 0), \qquad n^\mu = g^{\mu\nu} n_\nu = (1/\alpha, -\beta^i/\alpha).$$

We then have a 3+1 decomposed metric as

$$\begin{aligned} ds^2 &= -\alpha^2 dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)(dx^j + \beta^j dt) \\ &= (-\alpha^2 + \beta_l \beta^l)\, dt^2 + 2\beta_i\, dt\, dx^i + \gamma_{ij}\, dx^i\, dx^j \end{aligned} \tag{2.2}$$

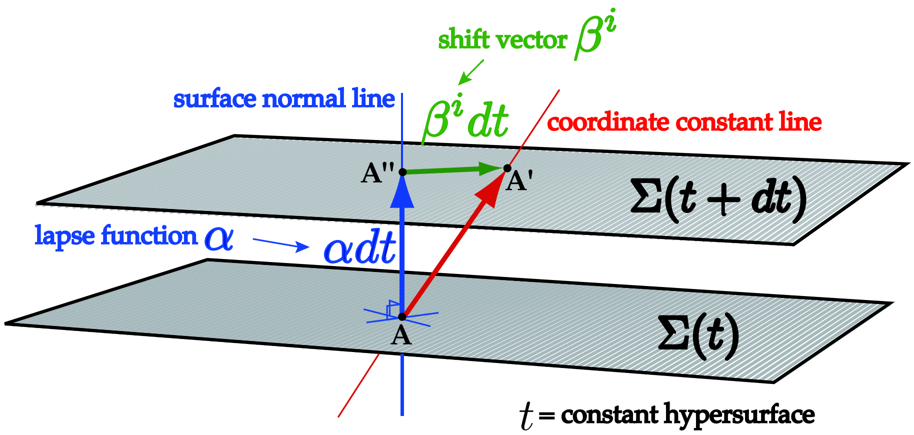


Figure 2.1: Concept of time evolution of space-time: foliations of 3-dimensional hypersurface. The lapse and shift functions are often denoted $\alpha$ or $N$, and $\beta^i$ or $N^i$, respectively.

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_l \beta^l & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}$$

where $\alpha$ and $\beta_j$ are defined as

$$\alpha \equiv 1/\sqrt{-g^{00}}, \qquad \beta_j \equiv g_{0j}. \tag{2.3}$$

and called the lapse function and shift vector, respectively.

**Projection onto $\Sigma$**

In order to decompose the Einstein equation into 3+1, we introduce the projection operator $\perp^\mu_\nu$ normal to $n^\mu$,

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \qquad \gamma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu \equiv \perp^\mu_\nu. \tag{2.4}$$

We also call the spatial components of $\gamma_{ij}$ the intrinsic 3-metric $g_{ij}$.[2]

The projections of the Einstein equation can be the following three:

$$G_{\mu\nu}\, n^\mu\, n^\nu = \kappa\, T_{\mu\nu}\, n^\mu\, n^\nu \equiv \kappa \rho_H \tag{2.5}$$

$$G_{\mu\nu}\, n^\mu \perp^\nu_i = \kappa\, T_{\mu\nu}\, n^\mu \perp^\nu_i \equiv -\kappa J_i \tag{2.6}$$

$$G_{\mu\nu} \perp^\mu_i \perp^\nu_j = \kappa\, T_{\mu\nu} \perp^\mu_i \perp^\nu_j \equiv \kappa S_{ij}, \tag{2.7}$$

where $\rho_H$, $J_i$ and $S_{ij}$ are energy density, momentum density and stress tensor, respectively, defined by an observer moving along $n_\mu = (-\alpha, 0, 0, 0)$. That is, the energy-momentum tensor, $T_{\mu\nu}$, is decomposed as

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}. \tag{2.8}$$

**Extrinsic curvature**

In order to express equations (2.5)-(2.7) tractable, we introduce the extrinsic curvature $K_{ij}$ as

$$K_{ij} \equiv -\perp^\mu_i \perp^\nu_j n_{\mu;\nu} = \cdots = \frac{1}{2\alpha}\left(-\partial_t \gamma_{ij} + \beta_{i|j} + \beta_{j|i}\right) = -\frac{1}{2}\pounds_n \gamma_{ij}. \tag{2.9}$$

Projection of the Einstein equation onto the 3-hypersurface $\Sigma$ is given using the Gauss-Codacci relation: The Gauss equation,

$${}^{(3)}R^\alpha{}_{\beta\gamma\delta} = {}^{(4)}R^\mu{}_{\nu\rho\sigma} \perp_\mu{}^\alpha \perp_\beta{}^\nu \perp_\gamma{}^\rho \perp_\delta{}^\sigma - K^\alpha{}_\gamma K_{\beta\delta} + K^\alpha{}_\delta K_{\beta\gamma}\,, \tag{2.10}$$

and the Codacci equation,

$$D_j K_i{}^j - D_i K = -{}^{(4)}R_{\rho\sigma} n^\sigma \perp_i{}^\rho\,, \tag{2.11}$$

where $K = K^i{}_i$, and $D_\mu$ is the covariant differentiation with respect to $\gamma_{ij}$.

[2] If $n_\mu$ is space-like, then $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$.

### 2.1.2 The Standard ADM formulation

The projections (2.5)-(2.7) can be derived as follows.

**The Standard ADM formulation [63, 75]:** **Box 2.1**

The fundamental dynamical variables are $(\gamma_{ij}, K_{ij})$, the three-metric and extrinsic curvature. The three-hypersurface $\Sigma$ is foliated with gauge functions, $(\alpha, \beta^i)$, the lapse and shift vector.

- The evolution equations:

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, && (2.12)\\
\partial_t K_{ij} &= \alpha\, {}^{(3)}R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - D_iD_j\alpha + (D_i\beta^k)K_{kj} + (D_j\beta^k)K_{ki} + \beta^k D_k K_{ij}\\
&\quad -\alpha\Lambda\gamma_{ij} - \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}, && (2.13)
\end{aligned}$$

  where $K = K^i{}_i$, and ${}^{(3)}R_{ij}$ and $D_i$ denote three-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$\begin{aligned}
\mathcal{H}^{ADM} &:= {}^{(3)}R + K^2 - K_{ij}K^{ij} - 2\kappa\rho - 2\Lambda \approx 0, && (2.14)\\
\mathcal{M}_i^{ADM} &:= D_jK^j{}_i - D_iK - \kappa J_i \approx 0, && (2.15)
\end{aligned}$$

  where ${}^{(3)}R ={}^{(3)} R^i{}_i$: these are called the Hamiltonian (or energy) and momentum constraint equations, respectively.

The formulation has 12 free first-order dynamical variables $(\gamma_{ij}, K_{ij})$, with 4 freedom of gauge choice $(\alpha, \beta_i)$ and with 4 constraint equations, (2.14) and (2.15). The rest freedom expresses 2 modes of gravitational waves.

**What are constraints?**

The ADM formulation is a kind of constrained system, like Maxwell equations.

| | Maxwell eqs. | ADM Einstein eq. |
|---|---|---|
| constraints | div $\mathbf{E} = 4\pi\rho$ | Hamiltonian constraint (2.14) |
| | div $\mathbf{B} = 0$ | Momentum constraints (2.15) |
| evolution eqs. | $\partial_t\mathbf{E} = \mathrm{rot}\ \mathbf{B} - 4\pi\mathbf{j}$ | $\partial_t\gamma_{ij} = \cdots$ (2.12) |
| | $\partial_t\mathbf{B} = -\mathrm{rot}\ \mathbf{E}$ | $\partial_t K_{ij} = \cdots$ (2.13) |

Table 2.1: Maxwell equations and ADM equations.

**Constraint propagations**

In order to see the constraints are conserved during the evolution or not, we have to check how the constraints evolve. The constraint propagation equations, which are the time evolution equations of the Hamiltonian constraint (2.14) and the momentum constraints (2.15), can be written as [33, 59]

**The Constraint Propagations of the Standard ADM:** **Box 2.2**

$$\begin{aligned} \partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha \gamma^{ij}(\partial_i \mathcal{M}_j) \\ &\quad +\alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \quad (2.16) \\ \partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\ &\quad +\alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j. \quad (2.17) \end{aligned}$$

From these equations, we know that **if the constraints are satisfied on the initial slice $\Sigma$, then the constraints are satisfied throughout evolution** (in principle).

**Standard ADM vs Original ADM**

We should remark here the 'original' ADM formulation. The evolution equations in Box 2.1 is the version by Smarr and York which is now the standard convention for numerical relativists. They adapted $K_{ij}$ as a fundamental variable instead of the conjugate momentum $\pi^{ij}$, which was in the original Arnowitt-Deser-Misner's canonical formulation. Note that there is one replacement in (2.13) using (2.14) in the process of conversion from the original ADM to the standard ADM equations.

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More detail description (vacuum case): The Hamiltonian density can be written as

$$\mathcal{H}_{GR} = \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}, \quad \text{where} \quad \mathcal{L} = \sqrt{-g}R = \alpha\sqrt{\gamma}[{}^{(3)}R - K^2 + K_{ij}K^{ij}],$$

where $\pi^{ij}$ is the canonically conjugate momentum to $\gamma_{ij}$,

$$\pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}_{ij}} = -\sqrt{\gamma}(K^{ij} - K\gamma^{ij}),$$

omitting the boundary terms. The variation of $\mathcal{H}_{GR}$ with respect to $\alpha$ and $\beta_i$ yields the constraints, and the dynamical equations are given by $\dot{\gamma}_{ij} = \frac{\delta \mathcal{H}_{GR}}{\delta \pi^{ij}}$ and $\dot{\pi}^{ij} = -\frac{\delta \mathcal{H}_{GR}}{\delta h_{ij}}$.

$$\begin{aligned} \partial_t \gamma_{ij} &= 2\frac{N}{\sqrt{\gamma}}(\pi_{ij} - (1/2)\gamma_{ij}\pi) + 2D_{(i}N_{j)}, \\ \partial_t \pi^{ij} &= -\sqrt{\gamma}N({}^{(3)}R^{ij} - (1/2){}^{(3)}R\gamma^{ij}) + (1/2)\frac{N}{\sqrt{\gamma}}h^{ij}(\pi_{mn}\pi^{mn} - (1/2)\pi^2) - 2\frac{N}{\sqrt{\gamma}}(\pi^{in}\pi_n{}^j - (1/2)\pi\pi^{ij}) \\ &\quad +\sqrt{\gamma}(D^iD^jN - \gamma^{ij}D^mD_mN) + \sqrt{\gamma}D_m(\gamma^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)} \end{aligned}$$

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### 2.1.3 Matter equations

The energy-momentum tensor, $T_{\mu\nu}$, and its evolution equations are model dependent. Let us see two introductory cases briefly.

**Scalar field**
We start from the Lagrangian

$$\mathcal{L} = \sqrt{-g}\left[\frac{R}{2\kappa} - \epsilon\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right)\right] \tag{2.18}$$

where $V(\Phi)$ is a potential of the scalar field. The parameter $\epsilon$ is the signature of the field $\psi$ and takes the value $+1$ (normal field) or $-1$ (ghost field). From the variation of Lagrangian, we get [3]

$$\delta S_g = \delta\int\sqrt{-g}\frac{R}{2\kappa}d^4x = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\delta g^{\mu\nu}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] \tag{2.20}$$

$$\begin{aligned}\delta S_\phi &= \int d^4x\left(-\epsilon\frac{1}{2}\right)\left[-g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right) + \phi_{,\mu}\phi_{,\nu}\right]\sqrt{-g}\delta g^{\mu\nu}, \\ &\quad + \int d^4x\epsilon\left[(\sqrt{-g}g^{\mu\nu}\phi_{,\mu})_{,\nu} - \sqrt{-g}\frac{\partial V}{\partial\phi}\right]\delta\phi.\end{aligned} \tag{2.21}$$

Therefore, we naturally set $T_{\mu\nu}$ as

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \qquad T_{\mu\nu} = \epsilon\left[\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V(\phi)\right)\right]. \tag{2.22}$$

The field equation ( Klein-Gordon equation) for the scalar field becomes

$$\Box\phi = \frac{\partial V}{\partial\phi}, \quad \text{that is} \quad \frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\mu\nu}\phi_{,\mu})_{,\nu} = \frac{\partial V}{\partial\phi}. \tag{2.23}$$

The equation (2.23) can be constructed also in a first-order form. For example, in a plane symmetric spacetime, $ds^2 = -\alpha^2 dt^2 + 2\beta dtdx + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2$, where all metric components are functions of $x$ and $t$, we introduce the conjugate momentum

$$\Pi = \frac{\sqrt{\gamma}}{\alpha}(-\partial_t\phi + \frac{\beta}{\gamma_{11}}\partial_x\phi), \tag{2.24}$$

where $\gamma = \det\gamma_{ij}$, and write down eq.(2.23) into two first-order partial differential equations:

$$\partial_t\phi = \frac{\beta}{\gamma_{11}}\partial_x\phi - \frac{\alpha}{\sqrt{\gamma}}\Pi, \tag{2.25}$$

$$\partial_t\Pi = \alpha\sqrt{\gamma}\frac{dV}{d\phi} + \partial_x\frac{1}{\gamma_{11}}[\beta\Pi - \alpha\sqrt{\gamma}\partial_x\phi]. \tag{2.26}$$

Consequently, the dynamical variables are $\gamma_{ij}$ and $K_{ij}$ (and $\phi$ and $\Pi$, when a scalar field exists).

[3] Note that from $\delta g = gg^{ab}\delta g_{ab} = -gg_{ab}\delta g^{ab}$,

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab} = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab}. \tag{2.19}$$

**Perfect fluid [6]**

We assume the perfect fluid stress-energy tensor,

$$T_{\mu\nu} = (\rho + \rho\varepsilon + p)u_\mu u_\nu + p g_{\mu\nu} \tag{2.27}$$

where $\rho, \varepsilon$ and $p$ are the proper mass density, the specific internal energy and the pressure, respectively, and $u_\mu$ is the 4-velocity of the fluid.

The evolution equation for the fluid is given by the Bianchi identity, $T^{\mu\nu}{}_{;\nu} = 0$. The projections $n^\mu T_\mu{}^\nu{}_{;\nu} = 0$ and $h_i{}^\mu T_{\mu;\nu}^{\ \nu} = 0$ give respectively,

$$\begin{aligned}
\partial_t(\sqrt{\gamma}\rho_H) + \partial_\ell(\sqrt{\gamma}\rho_H V^\ell) &= -\partial_\ell(\sqrt{\gamma}p(V^\ell + \beta^\ell)) + \alpha\sqrt{\gamma}pK \\
&\quad -(\partial_\ell\alpha)\sqrt{\gamma}J^\ell + \frac{\alpha\sqrt{\gamma}J^\ell J^m K_{\ell m}}{\rho_H + p}, && (2.28) \\
\partial_t(\sqrt{\gamma}J_i) + \partial_\ell(\sqrt{\gamma}J_i V^\ell) &= -\alpha\sqrt{\gamma}\,\partial_i p - \sqrt{\gamma}(p+\rho_H)\,\partial_i\alpha \\
&\quad +\frac{1}{2}\alpha\sqrt{\gamma}\,(\partial_i\gamma_{kl})\frac{J^k J^l}{p+\rho_H} + \sqrt{\gamma}J_\ell(\partial_i\beta^\ell), && (2.29)
\end{aligned}$$

where

$$\begin{aligned}
\rho_H &= \rho + \rho\varepsilon, && (2.30) \\
V^i &= \frac{u^i}{u^0} = \frac{\alpha J^i}{p+\rho_H} - \beta^i. && (2.31)
\end{aligned}$$

These represent the energy conservation and the Euler equation. The continuity equation, $(\rho u^\mu)_{;\mu} = 0$ (the GR version of $\partial_t\rho + \partial_i(\rho v_i) = 0$), gives

$$\partial_t(\sqrt{\gamma}\alpha u^0\rho) + \partial_\ell(\sqrt{\gamma}\alpha u^0\rho V^\ell) = 0. \tag{2.32}$$

The normalization of the 4-velocity, $u^\mu u_\mu = -1$, also gives us

$$\alpha u^0 = \frac{p+\rho_H}{\sqrt{(p+\rho_H)^2 - J^\ell J_\ell}}. \tag{2.33}$$

We also need the equation of state,

$$p = p(\varepsilon, \rho). \tag{2.34}$$

- For the perfect fluid, the variables are fluid components $(\rho, \varepsilon, p)$, which are related by (2.34) so that the freedom is 2. We can say the combination $(\rho, \rho_H)$, instead.
- The momentum $J_i$ is also freely speciable. From $(\rho, \rho_H, J_i)$,

$$S_{ij} = \frac{J_i J_j}{\rho + \rho_H} + p\gamma_{ij} \tag{2.35}$$

- For the total 5 variables, we have 5 equations (2.28), (2.29), and (2.32).

### 2.1.4 Numerical Procedures

In numerical relativity, this free-evolution approach is also the standard. This is because solving the constraints (non-linear elliptic equations) is numerically expensive, and because free evolution allows us to monitor the accuracy of numerical evolution.

**The normal numerical scheme (free evolution scheme):**

1. preparation of the initial data
   solve the elliptic constraints for preparing the initial data $(\gamma_{ij}, K_{ij})$.
2. time evolution
   (a) specify the gauge conditions (slicing conditions) for the lapse $\alpha$ and shift $\beta_i$.
   (b) evolve $(\gamma_{ij}, K_{ij})$ by using the evolution equations.
   (c) monitor the accuracy of simulations by checking the constraints.
   (d) extract physical quantities.
3. step back to 2 and repeat.

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## 2.2 Ashtekar 形式 (Ashtekar formulation)

ここでは，Ashtekar による一般相対論の拡張を紹介する．重力場の方程式は電磁気学の理論と非常に似た形式をもっているが，ゲージ理論的な特徴からは完全に対応していない．Ashtekar は，重力場を電場に相当する $E$ と，場の強さに相当する接続量 $A$ という 2 つの基本変数に変更することにより，ゲージ理論と対応する形式を導いた．

**重力場と電磁気場の比較** **Box 2.3**

- 一般相対論は，時空の各点で局所座標系を選択できる（等価原理）とする．
- ゲージ理論は，理論が局所対称性（不変性）をもつ，とする．
  $\Psi(x) \longrightarrow e^{i\alpha}\Psi(x)$：大域対称性（大局的ゲージ不変性）$\Longrightarrow$ 保存則
  $\Psi(x) \longrightarrow e^{i\alpha(x)}\Psi(x)$：局所対称性（局所的ゲージ不変性）$\Longrightarrow$ 相互作用

| | 重力場 | 電磁気場 |
|---|---|---|
| 自由場の方程式 | $\frac{d^2}{dt^2}\boldsymbol{x} = 0$ | $(i\gamma^\mu\partial_\mu - m)\Psi = 0$ |
| 対称性 | 一般座標変換<br>$(x^\mu \longrightarrow x^{\mu'})$ | 局所ゲージ変換<br>$(\Psi \longrightarrow e^{i\alpha(x)}\Psi)$ |
| 共変微分 | $\nabla_\mu = \partial_\mu + \Gamma$ | $D_\mu \equiv \partial_\mu + iqA_\mu$ |
| 接続係数 | ${\Gamma^\mu}_{\alpha\beta}$<br>(局所的に 0 とできる) | $A_\mu$<br>(直接観測できない，ゲージ依存) |
| 相互作用 | $\frac{d^2x^\alpha}{d\tau^2} + {\Gamma^\alpha}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$ | $(i\gamma^\mu(\partial_\mu + iqA_\mu) - m)\,\Psi = 0$ |
| 共変微分の非可換性 | 曲率テンソル ${R^\mu}_{\alpha\beta\gamma}$<br>(観測可能量) | 電磁場テンソル $F_{\mu\nu}$<br>(ゲージ不変量) |

### 2.2.1 From Einstein to Ashtekar; transformation of Lagragians

Here we try to understand Ashtekar's new formulation of general relativity [1] as the steps of rewriting the Lagragian formalism [2, 3]. Note that Ashtekar himself introduced his new variables through a kind of canonical transformation in the Hamiltonian formalism. [4]

**Einstein-Hilbert action (metric $g_{\mu\nu}$)**
First let us start from the Einstein-Hilbert action

$$S_E[g] = \int d^4x\,\sqrt{-g}\,R(g) \quad \sim \quad g\,\partial^2 g + (\partial g)^2 \tag{2.36}$$

which can be put into a canonical theory by means of the ADM method. That is, the metric $g_{\mu\nu}$ is decomposed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad = \quad -N^2\,dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.37}$$

[4]This subsection refers much to H. Ikemori's note in the proceedings of the 1st JGRG workshop at Tokyo (1991).

| Theory | action | | order of $\partial_\mu$ | independent variables |
|---|---|---|---|---|
| Einstein | Einstein-Hilbert action | $S_E$ | 2nd order | metric ($g_{\mu\nu}$) |
| | Palatini action | $S_P$ | 1st order | metric ($g_{\mu\nu}$) & Affine connection ($\Gamma^\lambda_{\mu\nu}$) |
| | Tetrad Palatini action | $S_T$ | 1st order | tetrad (${e^a}_\mu$) & spin connection ($\omega^{ab}_\mu$) |
| AshtekarOriginal | Jacobson-Smolin action | ${}^+S_T$ | 1st order | tetrad (${e^a}_\mu$) & self-dual connection (${}^+\omega^{ab}_\mu$) |

Table 2.2: Steps to the Ashtekar theory via Lagrangian formalism.

| | | |
|---|---|---|
| $S_P$ with the Christoffel condition for $\Gamma$ | $\Longrightarrow$ | $S_E$ |
| $S_T$ with the Levi-Civita condition for $\omega^{ab}$ (torsion free condition) | $\Longrightarrow$ | $S_P$ |
| ${}^+S_T$ with the Bianchi identity for $R_{ab}$ ($R_{\mu[\nu\alpha\beta]} = 0$) | $\Longrightarrow$ | $S_T$ |

Table 2.3: Steps to the Ashtekar theory and their extensions.

where $N$ is the lapse function (the same with $\alpha$) and $N^i$ is the shift vector ($\beta^i$) [5], and $q_{ij}$ is the three metric. That is,

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & q_{ij} \end{pmatrix}. \tag{2.38}$$

The canonical action, then, is given by

$$S_E[q,p] = \int d^4x \, [\dot{q}_{ij}\, p^{ij} - N\, \mathcal{C}_{\mathcal{H}} - N_i\, {\mathcal{C}_{\mathcal{M}}}^i] \tag{2.39}$$

where

$$\mathcal{C}_{\mathcal{H}} := G_{ijkl} p^{ij} p^{kl} - \sqrt{q}^{(3)}R \tag{2.40}$$
$${\mathcal{C}_{\mathcal{M}}}^i := -2\nabla_j p^{ij} \tag{2.41}$$

where $G_{ijkl} = \dfrac{1}{2\sqrt{q}}(q_{ik}q_{jl} + q_{il}q_{jk} - q_{ij}q_{kl})$.

**Palatini action (metric $g_{\mu\nu}$, Affine connection $\Gamma^\alpha_{\mu\nu}$)**
The Einstein-Hilbert action (2.36) consists of the terms with the second-order derivative or the square of the first order derivative of metric $g_{\mu\nu}$. Palatini's idea is to introduce the Affine connection $\Gamma^\alpha_{\mu\nu}(= \Gamma^\alpha_{\nu\mu})$ to be independent to the metric $g_{\mu\nu}$. The Palatini action

$$S_P[g,\Gamma] = \int d^4x \, \sqrt{-g}\, g^{\mu\nu}\, R_{\mu\nu}(\Gamma) \quad \sim \quad g(\partial\Gamma + \Gamma\,\Gamma) \tag{2.42}$$

which is equivalent to the Einstein-Hilbert action (2.36) , $S_P = S_E$, when a connection $\Gamma^\lambda_{\mu\nu}$ satisfies the definition of the Christoffel symbol, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}(g) \sim \partial g$. This condition is derived from the variation with respect to $\Gamma^\alpha_{\mu\nu}$,

$$\frac{\delta}{\delta\Gamma^\alpha_{\mu\nu}} S_P[g,\Gamma] = 0. \tag{2.43}$$

The action (2.42) contains up to the first-order derivatives.

[5] We use $N$ and $N^i$ instead of $\alpha$ and $\beta$, according to the convensions throughout this section.

**テトラド (tetrad)，トライアド (triad)，スピン接続 (spin connection)** **Box 2.4**

- 各時空点ごとに局所的な 4 次元直交座標系を定義する．直交座標の基底ベクトルを $E^I$ として，これを任意の座標系で表したもの $E^I_\mu$ をテトラド（4 脚場）と呼ぶ．

$$g_{\mu\nu} = E^I_\mu\, E^J_\nu\, \eta_{IJ}, \qquad \eta_{IJ} = \text{diag}\ (-1,1,1,1)$$

- 同様に，3 次元空間で局所的に直交座標を導入した基底ベクトルをトライアド（3 脚場）と呼ぶ．

$$g_{ij} = E^a_i\, E^b_j\, \delta_{ab}$$

- 局所直交座標系の成分を持つベクトルに対する共変微分を

$$\nabla_\mu V^I = \partial_\mu V^I + \omega^I_{\mu J} V^J$$

と表すとき，$\omega^I_{\mu J}$ をスピン接続と呼ぶ．具体的には，

$$\omega^{IJ}_\mu \;=\; E^{I\nu}\nabla_\mu E^J_\nu = E^{\nu I}\partial_{[\mu}E^J_{\nu]} - E_{\mu K}E^{\rho I}E^{\nu J}\partial_{[\rho}E^K_{\nu]} + E^{\rho J}\partial_{[\rho}E^I_{\mu]}$$

**Tetrad Palatini action (tetrad $e^a_\mu$, spin connection $\omega^{ab}_\mu$)**
The next step is the introduction of the internal symmetry, that is, to introduce the local Lorentz transformation as a gauge symmetry. We employ the orthonormal tetrad $e^a_\mu$ in stead of the metric $g_{\mu\nu}$, which acts as a basis of the local Lorentz frame. We also employ the spin connection $\omega^{ab}_\mu (= -\omega^{ba}_\mu)$ instead of the Affine connection $\Gamma^\alpha_{\mu\nu}$, which acts as a gauge field of the local Lorentz albebra so(3,1). The internal indices $a, b, \cdots$ are lowering and raising by the metric $\eta_{ab} = \text{diag}(-1,1,1,1)$. The tetrad plays a role of a square root of the metric,

$$g_{\mu\nu} = \eta_{ab}\, e^a_\mu\, e^b_\nu. \tag{2.44}$$

The Palatini action in the tetrad form is written as

$$S_T[e,\omega] \;=\; \int d^4x\, e\, E^\mu_a\, E^\nu_b\, R^{ab}{}_{\mu\nu}(\omega) \tag{2.45}$$

where $e$ is the determinant of $e^a_\mu$, and the $E^\mu_a$ is the inverse tetrad,

$$e := \det\, e^a_\mu = \sqrt{-g}, \qquad E^\mu_a = e^b_\nu\, g^{\mu\nu}\, \eta_{ab}. \tag{2.46}$$

Now that the internal symmetry is taken into account, the Riemann curvature $R^\alpha{}_{\beta\mu\nu}$ will be replaced by the curvature $R^{ab}_{\mu\nu}(\omega)$ of the spin connection $\omega^{ab}_\mu$ defined by

$$R^{ab}_{\mu\nu}(\omega) := \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu + \omega^a_{\mu c}\omega^{cb}_\nu - \omega^a_{\nu c}\omega^{cb}_\mu, \tag{2.47}$$

that is to say, the curvature 2-form $R^{ab}$ is defined from the spin connection 1-form $\omega^{ab}$ by

$$R^{ab}(\omega) := d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \tag{2.48}$$

in the language of the differential forms. The action (2.45), then, can be expressed also as

$$S_T[e,\omega] \;=\; \int \frac{1}{2}\,\varepsilon_{abcd}R^{ab}(\omega)\wedge e^c\wedge e^d. \tag{2.49}$$

The tetrad Palatini action (2.45) is equivalent to $S_P$ only when the spin connection equals to the Levi-Civita connection $\omega^{ab}=\omega^{ab}(e)$, that is the torsion free condition,

$$De^a := de^a + {\omega^a}_b\wedge e^b = 0 \tag{2.50}$$

which is derived from the variation respect to $\omega^{ab}$,

$$\frac{\delta}{\delta\omega^{ab}}S_T[e,\omega]=0. \tag{2.51}$$

**Self-dual action (tetrad $e^a_\mu$, self-dual connection ${}^+\omega^{ab}_\mu$)**
The last step to the Ashtekar's formulation is the introduction of the self-dual connection ${}^+\omega^{ab}_\mu$. Note that the self-duality here is with respect to the internal indices and not with the space-time indices.

**Self-duality, anti-self-duality:** **Box 2.5**
Suppose $F_{ab}$ is an anti-symmetric tensor, then the duality transformation is defined as

$${}^*F_{ab} := \frac{1}{2}{\varepsilon_{ab}}^{cd}F_{cd}, \tag{2.52}$$

making use of the totally anti-symmetric symbol, $\varepsilon^{abcd}$. Note that the dual of dual is equal to the minus of the original,

$${}^*({}^*F_{ab}) = -F_{ab} \tag{2.53}$$

when we choose the Lorentzian signature and use the metric $\eta_{ab}$ for lowering and raising the internal indices. Thus, the duality transformation (2.52) corresponds to $\pm i$ operation. If we suppose the complex combinations

$${}^\pm F_{ab} = \frac{1}{2}(F_{ab}\mp i\,{}^*F_{ab}), \tag{2.54}$$

then this satisfies the eigen-equations

$${}^*({}^\pm F_{ab}) = \pm i\,{}^\pm F_{ab}. \tag{2.55}$$

The notion of self-duality means an eigen-state of the duality transform operation and we call ${}^+F_{ab}$ self-dual part of $F_{ab}$ (and ${}^-F_{ab}$ anti-self-dual part of $F_{ab}$).

The spin connection 1-form $\omega^{ab}$ which has a pair of anti-symmetric internal indices can be uniquely decomposed into the self-dual and anti-self-dual part,

$$\omega^{ab} = {}^+\omega^{ab} + {}^-\omega^{ab}. \tag{2.56}$$

The substitution of this relation into the definition of the curvature 2-form $R^{ab}$ results in

$$R^{ab}(\omega) = R^{ab}({}^+\omega^{ab}+{}^-\omega^{ab}) = R^{ab}({}^+\omega^{ab}) + R^{ab}({}^-\omega^{ab}) := {}^+R^{ab} + {}^-R^{ab}, \tag{2.57}$$

which means that the $R^{ab}$ can also be decomposed additively according to the decomposition with respect to the self-duality.

The previously mentioned tetrad-Palatini action (2.49)

$$S_T[e,\omega] \;=\; \int \frac{1}{2}\varepsilon_{abcd}R^{ab}(\omega)\wedge e^c\wedge e^d = \int {}^*R_{cd}(\omega)\wedge e^c\wedge e^d \tag{2.58}$$

is decomposed as

$$\begin{aligned} S_T[e,\omega] &= \int {}^*R_{ab}({}^+\omega)\wedge e^a\wedge e^b + \int {}^*R_{ab}({}^-\omega)\wedge e^a\wedge e^b \\ &:= {}^+S_T[e,{}^+\omega] + {}^-S_T[e,{}^-\omega] \end{aligned}$$

with regard to the contributions of self-dual and anti-self-dual connections.

Ashtekar's idea is to consider just a self-dual part of the action. The equivalence to the Einstein-Hilbert action is still preserved with regard to just a half of duality components.

When the self-dual connection is equal to the self-dual part of the Levi-Civita connection

$${}^+\omega^{ab} = {}^+\omega^{ab}(e), \tag{2.59}$$

the variation $\frac{\delta}{\delta^+\omega}{}^+S_T[e,{}^+\omega] = 0$ is satisfied. It reduces the self-dual action equals to the Einstein-Hilbert action with a factor half, ${}^+S_T[e,{}^+\omega] = \frac{1}{2}S_T[e,\omega(e)] = \frac{1}{2}S_E[g]$.

The equivalence to the Einstein theory requires additional condition. Since the curvature of self-dual connection is given by its complex combination

$$R^{ab}({}^+\omega) = {}^+R^{ab}(\omega) = \frac{1}{2}\left(R^{ab}(\omega) - i\,{}^*R^{ab}(\omega)\right), \tag{2.60}$$

the action turns out to be

$$\begin{aligned} {}^+S_T[e,{}^+\omega(e)] &= \frac{1}{2}\int {}^*\left(R_{ab}(\omega(e)) - i\,{}^*R_{ab}(\omega(e))\right)\wedge e^a\wedge e^b \\ &= \frac{1}{2}\int \left({}^*R_{ab}(\omega(e)) + iR_{ab}(\omega(e))\right)\wedge e^a\wedge e^b \\ &= \frac{1}{2}S_T[e,\omega(e)] + i\frac{1}{2}\int R_{ab}(\omega(e))\wedge e^a\wedge e^b = \frac{1}{2}S_E[g] + 0, \end{aligned} \tag{2.61}$$

where the last imaginary term is vanished by virtue of the 1st Bianchi identity

$$R^a{}_b(\omega(e))\wedge e^b \equiv 0 \tag{2.62}$$

which is the cyclic identity $R_{\mu[\nu\alpha\beta]} = 0$ in the tensor form.

This means that the self-dual action would lead to the same equation of motion as the Einstein equation so far as the tetrad or equivalently the metric is concerned. The anti-self-dual action can also play the same role with the above discussion.

### 2.2.2 New Variables

The Ashtekar formalism can be regarded as a canonical theory starting from the self-dual action,

$${}^+S_T[e,{}^+\omega] \;=\; \int d^4x\, e\, E^\mu_a\, E^\nu_b\, R^{ab}{}_{\mu\nu}({}^+\omega). \tag{2.63}$$

where $E^\mu_a$ is the inverse tetrad, defined as $E^\mu_a := E^b_\nu g^{\mu\nu}\eta_{ab}$, which makes the inverse space-time metric as $q^{\mu\nu} = \eta^{ab}E^\mu_a E^\nu_b$ as we mentioned before. See notations in Table 2.4.

| | | | | |
|---|---|---|---|---|
| 4-spacetime indices | $\mu, \nu, \rho, \cdots$ | $0, \cdots, 3;$ | raise and lower indices by | $g_{\mu\nu}$ |
| SO(1,3) indices | $I, J, K, \cdots$ | $(1), \cdots, (3)$ | | $\eta^{IJ} = \mathrm{diag}(-1, 1, 1, 1)$ |
| 3-spacetime indices | $i, j, k, \cdots$ | $1, \cdots, 3$ | | $\gamma_{ij}$ |
| SO(3) indices | $a, b, c, \cdots$ | $(1), \cdots, (3)$ | | $\delta_{ab}$ |
| volume forms | $\epsilon_{abc}$ | $\epsilon_{abc}\epsilon^{abc} = 3!$ | | |
| density | $e$ | $\epsilon_{ijk} = e,$ | $\underset{\sim}{\epsilon}_{ijk} := e^{-1}\epsilon_{ijk}$ | $\underset{\sim}{\epsilon}_{123} = 1, \tilde{\epsilon}^{123} = 1$ |
| tetrad (inverse tetrad) | $E^I_\mu \ (E^\mu_I)$ | $g_{\mu\nu} = E^I_\mu E^J_\nu \eta_{IJ}$ | $E^\mu_I := e^J_\nu g^{\mu\nu}\eta_{IJ}$ | |
| spin connection | $\omega^{IJ}_\mu$ | $\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu E^J_\nu.$ | | |
| curvature 2-form | $F^a_{\mu\nu}$ | $F^a_{\mu\nu} := \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu - i\epsilon^a{}_{bc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu$ | | |

Table 2.4: Notations in §2.2.

Let us consider the $3+1$ decomposition of the self-dual theory in the tetrad form after the ADM decomposition. The spatial component of the tetrad, $E^i_I$ acts as an inverse triad since it produces the inverse 3-metric, $q^{ij} = E^i_I E^j_I$. We further impose the gauge condition

$$E^0_I = 0 \tag{2.64}$$

then, the inverse tetrad is expressed as

$$E^\mu_a = \begin{pmatrix} E^0_0 & E^i_0 \\ E^0_I & E^i_I \end{pmatrix} = \begin{pmatrix} 1/N & -N^i/N \\ 0 & E^i_I \end{pmatrix} \tag{2.65}$$

Note that (2.64) allows $E^\mu_0 = (1/N, -N^i/N)$ as a normal vector field to the space like hypersurface spanned by the condition of $t =$const. This gauge choice is not a restriction on the general coordinate transformation but on the local Lorentz transformation.

**New variables (densitized inverse triad $\tilde{E}^i_a$, self-dual connection ${}^+\mathcal{A}^a_\mu$)**
The key feature of Ashtekar's formulation of general relativity [1] is the introduction of a self-dual connection as one of the basic dynamical variables.

**Ashtekar variables (New variables) [1]:** **Box 2.6**

The geometry in the Ashtekar formulation is expressed by the pair of new variables, $(\tilde{E}^i_a, \mathcal{A}^a_i)$.

- self-dual connection (Ashtekar connection)
  We define $so(3,\mathrm{C})$ connections

$$ {}^{\pm}\mathcal{A}^I_\mu := \omega^{0I}_\mu \mp \frac{i}{2}\epsilon^I{}_{JK}\,\omega^{JK}_\mu, \tag{2.66}$$

  where $\omega^{IJ}_\mu$ is a spin connection 1-form (Ricci connection), $\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu E^J_\nu$. Ashtekar's plan is to use only the self-dual part of the connection ${}^+\mathcal{A}^a_\mu$ and to use its spatial part ${}^+\mathcal{A}^a_i$ as a dynamical variable. Hereafter, we simply denote ${}^+\mathcal{A}^a_\mu$ as $\mathcal{A}^a_\mu$.

- densitized inverse triad $\tilde{E}^i_a$

$$ \tilde{E}^i_a := eE^i_a, \tag{2.67}$$

  where $e := \det E^a_i$ is a density.

This pair forms a canonical set.

For later convenience, we denote the relation,

$$ e^2 = \det g_{ij} = \det \tilde{E}^i_a = (\det E^a_i)^2 = (1/6)\epsilon^{abc}\underset{\sim}{\epsilon}_{ijk}\tilde{E}^i_a\tilde{E}^j_b\tilde{E}^k_c, \tag{2.68}$$

where $\epsilon_{ijk} := \epsilon_{abc}E^a_iE^b_jE^c_k$ and $\underset{\sim}{\epsilon}_{ijk} := e^{-1}\epsilon_{ijk}$. [6]

In the case of pure gravitational spacetime with cosmological constant $\Lambda$, the Hilbert action takes the form

$$ {}^+S_A[\tilde{E}, {}^+\mathcal{A}] \;=\; \int d^4x[(\partial_t\mathcal{A}^a_i)\tilde{E}^i_a + \underset{\sim}{N}\,\mathcal{C}_H + N^i\,\mathcal{C}_{Mi} + \mathcal{A}^a_0\,\mathcal{C}_{Ga}], \tag{2.69}$$

where $\underset{\sim}{N} := e^{-1}N$. The latter terms are understood as Lagrange multipliers $(\mathcal{A}^a_0, N^i$, and $\underset{\sim}{N})$ and their accompanied constraints, $\mathcal{C}_H \approx 0, \mathcal{C}_{Mi} \approx 0$ and $\mathcal{C}_{Ga} \approx 0$, which are

$$ \mathcal{C}_H \;:=\; (i/2)\epsilon^{ab}{}_c\,\tilde{E}^i_a\tilde{E}^j_bF^c_{ij} - 2\Lambda\,\det\tilde{E} \tag{2.70}$$

$$ \mathcal{C}_{Mi} \;:=\; F^a_{ij}\tilde{E}^j_a \tag{2.71}$$

$$ \mathcal{C}_{Ga} \;:=\; \mathcal{D}_i\tilde{E}^i_a \tag{2.72}$$

where $F^a_{\mu\nu} := 2\partial_{[\mu}\mathcal{A}^a_{\nu]} - i\epsilon^a{}_{bc}\,\mathcal{A}^b_\mu\mathcal{A}^c_\nu$ is the curvature 2-form, and $\mathcal{D}_i\tilde{E}^j_a := \partial_i\tilde{E}^j_a - i\epsilon_{ab}{}^c\,\mathcal{A}^b_i\tilde{E}^j_c$.

[6] When $(i, j, k) = (1, 2, 3)$, we have $\epsilon_{ijk} = e$, $\underset{\sim}{\epsilon}_{ijk} = 1$, $\epsilon^{ijk} = e^{-1}$, and $\tilde{\epsilon}^{ijk} = 1$.

**The Ashtekar formulation [1]:** **Box 2.7**

The dynamical variables are $(\tilde{E}^i_a, \mathcal{A}^a_i)$.

$$\mathcal{A}^a_i := \omega^{0a}_i - \frac{i}{2}\epsilon^a{}_{bc}\omega^{bc}_i = -K_{ij}E^{ja} - \frac{i}{2}\epsilon^a{}_{bc}\omega^{bc}_i \tag{2.73}$$

$$\tilde{E}^i_a := eE^i_a \tag{2.74}$$

- The evolution equations for a set of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ are

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c, \tag{2.75}$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + 2\Lambda \underset{\sim}{N} \tilde{e}^a_i, \tag{2.76}$$

where $\mathcal{D}_j X^{ji}_a := \partial_j X^{ji}_a - i\epsilon_{ab}{}^c \mathcal{A}^b_j X^{ji}_c$, and $F^a_{ij} := 2\partial_{[i}\mathcal{A}^a_{j]} - i\epsilon^a{}_{bc}\mathcal{A}^b_i \mathcal{A}^c_j$.

- Constraint equations: (Hamiltonian, momentum and Gauss constraints)

$$\mathcal{C}^{\rm ASH}_H := (i/2)\epsilon^{ab}{}_c \tilde{E}^i_a \tilde{E}^j_b F^c_{ij} - 2\Lambda \det \tilde{E} \approx 0, \tag{2.77}$$

$$\mathcal{C}^{\rm ASH}_{Mi} := F^a_{ij}\tilde{E}^j_a \approx 0, \tag{2.78}$$

$$\mathcal{C}^{\rm ASH}_{Ga} := \mathcal{D}_i \tilde{E}^i_a \approx 0. \tag{2.79}$$

- Gauge variables are the lapse function $\underset{\sim}{N}$, the shift vector $N^i$, and the triad lapse $\mathcal{A}^a_0$.

The set of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ forms a canonical relation,

$$\{\tilde{E}^i_a(x), \tilde{E}^j_b(y)\} = 0, \tag{2.80}$$

$$\{\mathcal{A}^a{}_i(x), \tilde{E}^j_b(y)\} = i\delta^j{}_i \delta^a{}_b \delta(x-y), \tag{2.81}$$

$$\{\mathcal{A}^a{}_i(x), \mathcal{A}^b{}_j(y)\} = 0. \tag{2.82}$$

The dynamical degrees of freedom are summarized in Table 2.5.

| covariant vars. | | canonical vars. | gauge conditions | gauge vars. |
|---|---|---|---|---|
| $E^\mu_a$ (16) | $\Longrightarrow$ | $\tilde{E}^i_a$ (9) | $E^0_a = 0$ (3) | $N^i$ (3) + $\underset{\sim}{N}$ (1) |
| $+\omega^{ab}_\mu$ (12) | $\Longrightarrow$ | $\mathcal{A}^a_i$ (9) | | $\mathcal{A}^a_0$ (3) |

Table 2.5: Dynamical degrees of freedom.

### 2.2.3 Einstein vs. Ashtekar

Let us compare the features of Ashtekar's formulation of general relativity with the conventional one. See Table 2.6 for brief summary.

**From the viewpoint of classical dynamics**

If we apply this formulation to the time evolution of Lorenzian space-time, the bottleneck is the additional constraint $\mathcal{C}_G$ and the reality conditions.

| Einstein theory | Ashtekar theory |
|---|---|
| purely geometrical theory | gauge theoretical features |
| 2nd order derivative theory | 1st order derivative theory |
| dynamical eqs are non-polynomial | dynamical els are polynomial |
| | dynamical eqs are (weakly) hyperbolic |
| does contain the inverse of variables | does not contain the inverse of variables |
| does not admit degenerate metric | does admit degenerate metric |
| constraints are $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{M}}$ | additional constraint, $\mathcal{C}_{\mathcal{G}}$ |
| | additional "reality condition" to recover real geometry |

Table 2.6: Einstein vs Ashtekar theories

- Additional gauge variables ($\mathcal{A}^a_0$)
  When we consider the space-time evolution as foliations of space-like hypersurfaces, the ADM formulation says that we have gauge freedoms which are expressed with the lapse function, $\alpha$ or $N$, and with the shift vector, $\beta^i$ or $N^i$. In Ashtekar's theory, there is additional gauge variable, $\mathcal{A}^a_0$, which we named "triad lapse" [7]. This freedom appears due to the introduction of the internal indices. We somehow have to spacify $\mathcal{A}^a_0$ in a proper manner. See Fig. 2.2.

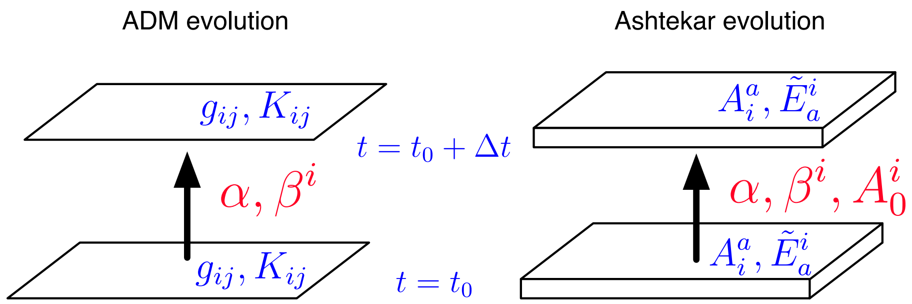


Figure 2.2: Concept of time evolution of space-time: foliations of 3-dimensional hypersurface. The lapse and shift functions are often denoted $\alpha$ or $N$, and $\beta^i$ or $N^i$, respectively.

- Additional "Gauss constraint" ($\mathcal{C}_{\mathcal{G}}$)
  In ADM formulation, we have Hamiltonian (scalar) and momentum (vector) constraint equations. These are the first-class, and we have to solve these 4-equations when we prepare the initial data for time evolutions.

  In Ashtekar's theory, we have additional Gauss constraint ($\mathcal{C}_{\mathcal{G}}$), which has 3 components. The set of constraints forms the first-class, therefore we have to solve them when we prepare the initial data.

- Reality conditions to recover classical GR
  We have to consider the reality conditions when we use this formalism to describe the classical Lorentzian spacetime. The reality conditions are, so far, posed on the metric or the triad.

  Fortunately, the metric will remain on its real-valued constraint surface during time evolution automatically if we prepare initial data which satisfies the reality condition[6].

  More practically, we further can require that triad is real-valued. But again this reality condition appears as a gauge restriction on $\mathcal{A}^a_0$[9], which can be imposed at every time step.

[7] Actually, HS asked Ashtekar to name this variable, and he named it after a minute.

From the fact that the reality of the spacetime is conserved if we solve reality conditions initially, so we propose to prepare ADM initial data for evolution in Ashtekar's variables by transforming variables and introducing internal variables as they satisfy $\mathcal{C}_G$.

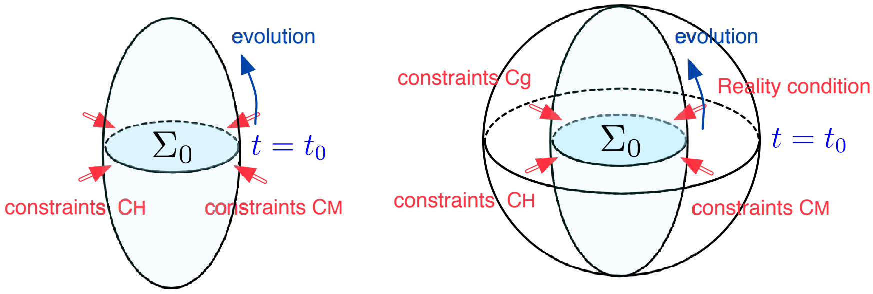


Figure 2.3: Images of constraints, as a solution space in the Einstein manifold. (Left) The ADM approach has two constraints, $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{M}i}$, which specify a solution so as it satisfies the Einstein equations. (Right) The Ashtekar formulation has another constraint, $\mathcal{C}_{\mathcal{G}a}$, and reality condition.

In our actual simulation, we prepare our initial data using the standard ADM approach, so that we have no difficulties in maintaining these reality conditions.

| | ADM formulation | | connection formulation | | | |
|---|---|---|---|---|---|---|
| | | | $Re$(metric) | | $Re$(triad) | |
| | | | | | | $\Sigma_0$ ($\Sigma_t$) |
| variables | $\gamma_{ij}$ | 6 | $\tilde{E}^i_a$ | 18 | $\tilde{E}^i_a$ | 18 (9) |
| | $K_{ij}$ | 6 | $\mathcal{A}^a_i$ | 18 | $\mathcal{A}^a_i$ | 18 (9) |
| gauge | $N$ | 1 | $N$ | 1 | $N$ | 1 (1) |
| | $N^i$ | 3 | $N^i$ | 3 | $N^i$ | 3 (3) |
| | | | $\mathcal{A}^a_0$ | 6 | $\mathcal{A}^a_0$ | 3 (3) |
| constraints | $\mathcal{C}_H$ | 1 | $\mathcal{C}_H$ | 1 | $\mathcal{C}_H$ | 1 (1) |
| | $\mathcal{C}_{Mi}$ | 3 | $\mathcal{C}_{Mi}$ | 3 | $\mathcal{C}_{Mi}$ | 3 (3) |
| | | | $\mathcal{C}_{Ga}$ | 6 | $\mathcal{C}_{Ga}$ | 6 (3) |
| reality condition | | | primary | 6 ($\Sigma_0$) | primary | 9 (0) |
| | | | secondary | 6 ($\Sigma_0$) | secondary | 6 (0) |
| GW freedom | | $2 \times 2$ | | $2 \times 2$ | | $2 \times 2$ |

Table 2.7: Number of components in actual simulations. We here count the numbers of freedom in components, i.e. one complex number has two components.

### 2.2.4 Reality conditions

Notice that the metric in Ashtekar's formulation is not necessary to be real. In order to recover the real metric, we must impose the reality conditions.

To ensure the metric is real-valued, we need to impose two conditions; the primary is that the doubly densitized contravariant metric $\tilde{\tilde{\gamma}}^{ij} := e^2\gamma^{ij}$ is real,

$$\Im(\tilde{E}^i_a \tilde{E}^{ja}) = 0, \tag{2.83}$$

and the secondary condition is that the time derivative of $\tilde{\tilde{\gamma}}^{ij}$ is real,

$$\Im\{\partial_t(\tilde{E}^i_a\tilde{E}^{ja})\} = 0. \tag{2.84}$$

Using the equations of motion for $\tilde{E}^i_a$ (2.75), the Gauss constraint (2.79) and the primary reality condition (2.83), we can replace the secondary condition (2.84) with a different constraint

$$W^{ij} := \Re(\epsilon^{abc}\tilde{E}^k_a\tilde{E}^{(i}_b\mathcal{D}_k\tilde{E}^{j)}_c) \approx 0, \tag{2.85}$$

which fixes six components of $\mathcal{A}^a_i$ and $\tilde{E}^i_a$. Moreover, in order to recover the original lapse function $N := \underset{\sim}{N}e$, we demand $\Im(N/e) = 0$, i.e. the density $e$ be real and positive. This requires that $e^2$ be positive, i.e.

$$\det\tilde{E} > 0. \tag{2.86}$$

The secondary condition of (2.86),

$$\Im[\partial_t(\det\tilde{E})] = 0, \tag{2.87}$$

is automatically satisfied (see [9]). Therefore, in order to ensure that $e$ is real, we only require (2.86).

Rather stronger reality conditions are sometimes useful in Ashtekar's formalism for recovering the real 3-metric and extrinsic curvature. These conditions are

$$\begin{aligned} \Im(\tilde{E}^i_a) &= 0 && (2.88)\\ \text{and}\quad \Im(\dot{\tilde{E}}^i_a) &= 0, && (2.89)\end{aligned}$$

and we call them the "primary triad reality condition" and the "secondary triad reality condition", respectively. Using the equations of motion of $\tilde{E}^i_a$, the Gauss constraint (2.79), the metric reality conditions (2.83), (2.84) and the primary condition (2.88), we see that (2.89) is equivalent to [9]

$$\Re(\mathcal{A}^a_0) = \partial_i(\underset{\sim}{N})\tilde{E}^{ia} + \frac{1}{2}e^{-1}e^b_i\underset{\sim}{N}\tilde{E}^{ja}\partial_j\tilde{E}^i_b + N^i\Re(\mathcal{A}^a_i). \tag{2.90}$$

From this expression we see that the second triad reality condition restricts the three components of "triad lapse" vector $\mathcal{A}^a_0$. Therefore (2.90) is not a restriction on the dynamical variables ($\mathcal{A}^a_i$ and $\tilde{E}^i_a$) but on the slicing, which we should impose on each hypersurface. Thus the second triad reality condition does not restrict the dynamical variables any further than the second metric condition does.

### 2.2.5 Trick for passing a degenerate point

Next, we examine the possibilities of passing a degenerate point. A 'degenerate point', we use here, is defined as the point in the spacetime where the density $e$ of 3-space vanishes. In the Ashtekar formulation, all the equations do not include any inverse of $e$ apparently, so that we expect we can 'pass' such a degenerate point.

In order to say 'pass' degenerate points, we start from requiring the finiteness of the fundmental variables (and their derivatives), $\tilde{E}^i_a, \mathcal{A}^a_i, N/e, N^i, \mathcal{A}^a_0$, and the condition that the calculation must be finished in finite coordinate time. Although these are natural conditions for pursuiting the evolutions of spacetime, we concluded that continuing evolutions including a degenerate point in its foliation of 3-space is generally break one of above conditions. The difficulties are that the term $\omega^{bc}_i$ in $\mathcal{A}^a_i$ diverges generally and a requirement of finite coordinate time fails when we pass a degenerate point. This means generally we face a trouble when we pass a degenerate point directly in Lorentzian spacetime even if we use Ashtekar's variables.

However, since the variables are originally defined as complex numbers, if we are allowed to break the reality condition locally in the neibour of a degenerate point, which we also assume its degeneracy exists only on the real section of spacetime, then we can 'pass' a degenerate point by such a 'deformed slice approach'. Note that, in our proposal, the foliation maintains $3+1$ dimensions $\boldsymbol{R}^3 \times \boldsymbol{R}$ in $\boldsymbol{C}^4$.

In order to recover a real metric spacetime again later, we have to impose 'reality recovering condition' on the foliation, which requires us to determine shooting parameters in complex part of gauge variables. We showed this technique actually works, by demonstrating a numerical evolution for an analytic solution of degenerate point in flat spacetime[8]. We see that the time evolution does work properly in the sense that the real part of evolution recovers the analytic evolutions and the imaginary part of metric vanishes asymptotically.

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## 2.3 高次元の場合 (Higher-dimensional ADM formulation)

### 2.3.1 Application to $N+1$-dimensional space-time

Let us describe how the ADM equations turns to be in higher-dimensional cases. The set of equations are shown in [1] in the context of constraint propagation equations.

**The Standard ADM formulation in $N+1$-dim. [1]** **Box 2.6**

The fundamental dynamical variables are $(\gamma_{ij}, K_{ij})$, the $N$-metric and extrinsic curvature. The $N$-hypersurface $\Sigma$ is foliated with gauge functions, $(\alpha, \beta^i)$, the lapse and shift vector.

- The evolution equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_j\beta_i + D_i\beta_j, \tag{2.91}$$

$$\partial_t K_{ij} = \alpha^{(N)}R_{ij} + \alpha K K_{ij} - 2\alpha K^{\ell}{}_j K_{i\ell} - D_i D_j \alpha + \beta^k(D_k K_{ij}) + (D_j\beta^k)K_{ik} + (D_i\beta^k)K_{kj} - \frac{2\alpha}{N-1}\gamma_{ij}\Lambda, -\kappa\alpha\left(S_{ij} - \frac{1}{N-1}\gamma_{ij}T\right) \tag{2.92}$$

  where $K = K^i{}_i$, and ${}^{(N)}R_{ij}$ and $D_i$ denote N-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$\text{Hamiltonian constr.} \quad \mathcal{H}^{ADM} := {}^{(N)}R + K^2 - K_{ij}K^{ij} - 2\Lambda - 2\kappa\rho \approx 0,$$

$$\text{momentum constr.} \quad \mathcal{M}_i^{ADM} := D_j K^j{}_i - D_i K - \kappa J_i \approx 0,$$

  where ${}^{(N)}R ={}^{(N)} R^i{}_i$.

### 2.3.2 $N+1$-formalism in Einstein-Gauss-Bonnet gravity

As one of the application to an alternative gravity model, Gauss-Bonnet gravity is extensively studied. Since dynamical studies have not yet been done, we first set up the ADM-type decomposition of the equations[2].

**Gauss-Bonnet action**

Einstein-Gauss-Bonnet action is given by

$$S = \int_{\mathcal{M}} d^{N+1}X\sqrt{-g}\left[\frac{1}{2\kappa^2}(\mathcal{R} - 2\Lambda + \alpha_{GB}\mathcal{L}_{GB}) + \mathcal{L}_{\text{matter}}\right] \tag{2.93}$$

$$\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

where $\kappa^2$ is the $(N+1)$-dimensional gravitational constant, $\mathcal{R}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}_{\mu\nu\rho\sigma}$ and $\mathcal{L}_{\text{matter}}$ are the $(N+1)$-dimensional scalar curvature, Ricci tensor, Riemann curvature and the matter Lagrangian, respectively. This action will reproduce the standard $(N+1)$-dimensional Einstein gravity, if we set the coupling constant $\alpha_{GB}(\geq 0)$ equals to zero. [8]

[8] The Greek indices $(\mu, \nu, \cdots)$ move $1, \cdots, N+1$, while the Latin indices $(i, j, \cdots)$ move $1, \cdots, N$.

The action gives the gravitational equation

$$\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu} = \kappa^2\,\mathcal{T}_{\mu\nu} \tag{2.94}$$

where

$$\begin{aligned}
\mathcal{G}_{\mu\nu} &= \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu}, \\
\mathcal{H}_{\mu\nu} &= 2\Big[\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}{}^{\alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}\Big] - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB}, \\
\mathcal{T}_{\mu\nu} &\equiv -2\frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\rm matter}.
\end{aligned}$$

**Projections to Hypersurface $\Sigma_N$ (spacelike or timelike)**
The projection operator,

$$\perp_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu, \quad n_\mu n^\mu = \varepsilon \tag{2.95}$$

where $n_\mu$ is the unit-normal vector to $\Sigma$ with $n_\mu$ is timelike (if $\varepsilon = -1$) or spacelike (if $\varepsilon = 1$). $\Sigma$ is spacelike (timelike) if $n_\mu$ is timelike (spacelike).

The induced $N$-dimensional metric $\gamma_{ij}$ is defined by $\gamma_{ij} = \perp_{ij}$.

The projections of the gravitational equation:

$$(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\,n^\mu\,n^\nu = \kappa^2\,T_{\mu\nu}\,n^\mu\,n^\nu =: \kappa^2\rho_H, \tag{2.96}$$

$$(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\,n^\mu\perp^\nu_\rho = \kappa^2\,T_{\mu\nu}\,n^\mu\perp^\nu_\rho =: -\kappa^2 J_\rho, \tag{2.97}$$

$$(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\perp^\mu_\rho\perp^\nu_\sigma = \kappa^2\,T_{\mu\nu}\perp^\mu_\rho\perp^\nu_\sigma =: \kappa^2 S_{\rho\sigma}, \tag{2.98}$$

where we defined

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}, \;\; T = -\rho_H + S^\ell{}_\ell$$

Introduce the extrinsic curvature $K_{ij}$

$$K_{ij} := -\frac{1}{2}\pounds_n h_{ij} = -\perp^\alpha_i\perp^\beta_j\nabla_\alpha n_\beta, \tag{2.99}$$

where $\pounds_n$ denotes the Lie derivative in the $n$-direction and $\nabla$ and $D_i$ is the covariant differentiation with respect to $g_{\mu\nu}$ and $\gamma_{ij}$, respectively.

- Projection of the $(N+1)$-dimensional Riemann tensor onto $\Sigma_N$

Gauss eq.
$$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha_i\perp^\beta_j\perp^\gamma_k\perp^\delta_l = R_{ijkl} - \varepsilon K_{ik}K_{jl} + \varepsilon K_{il}K_{jk}, \tag{2.100}$$

Codacci eq.
$$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha_i\perp^\beta_j\perp^\gamma_k n^\delta = -2D_{[i}K_{j]k}, \tag{2.101}$$

$$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha_i\perp^\gamma_k n^\beta n^\delta = \pounds_n K_{ik} + K_{i\ell}K^\ell{}_k\,, \tag{2.102}$$

- Curvature relations

$$\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - \varepsilon(K_{\mu\rho}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\rho} - n_\mu D_\rho K_{\nu\sigma} + n_\mu D_\sigma K_{\rho\nu} + n_\nu D_\rho K_{\sigma\mu} - n_\nu D_\sigma K_{\rho\mu} \\
&\quad - n_\rho D_\mu K_{\nu\sigma} + n_\rho D_\nu K_{\mu\sigma} + n_\sigma D_\mu K_{\nu\rho} - n_\sigma D_\nu K_{\mu\rho}) \\
&\quad + n_\mu n_\rho K_{\nu\alpha}K^\alpha{}_\sigma - n_\mu n_\sigma K_{\nu\alpha}K^\alpha{}_\rho - n_\nu n_\rho K_{\mu\alpha}K^\alpha{}_\sigma + n_\nu n_\sigma K_{\mu\alpha}K^\alpha{}_\rho \\
&\quad + n_\mu n_\rho\pounds_n K_{\nu\sigma} - n_\mu n_\sigma\pounds_n K_{\nu\rho} - n_\nu n_\rho\pounds_n K_{\mu\sigma} + n_\nu n_\sigma\pounds_n K_{\mu\rho},
\end{aligned} \tag{2.103}$$

$$\begin{aligned}
\mathcal{R}_{\mu\nu} &= R_{\mu\nu} - \varepsilon\Big[KK_{\mu\nu} - 2K_{\mu\alpha}K^\alpha{}_\nu + n_\mu\left(D_\alpha K^\alpha{}_\nu - D_\nu K\right) + n_\nu\left(D_\alpha K^\alpha{}_\mu - D_\mu K\right)\Big] \\
&\quad + n_\mu n_\nu K_{\alpha\beta}K^{\alpha\beta} + \varepsilon\pounds_n K_{\mu\nu} + n_\mu n_\nu\gamma^{\alpha\beta}\pounds_n K_{\alpha\beta},
\end{aligned} \tag{2.104}$$

$$\mathcal{R} = R - \varepsilon(K^2 - 3K_{\alpha\beta}K^{\alpha\beta} - 2\gamma^{\alpha\beta}\pounds_n K_{\alpha\beta}). \tag{2.105}$$

**$N+1$ Einstein-Gauss-Bonnet equations [2]** **Box 2.7**
Substituting (2.103)-(2.105) into (2.95) or (2.5)-(2.7), we find:

(a) dynamical equations for $\gamma_{ij}$:

$$\begin{aligned} & M_{ij} - \frac{1}{2}M\gamma_{ij} - \varepsilon(-K_{ia}K^a{}_j + \gamma_{ij}K_{ab}K^{ab} - \pounds_n K_{ij} + \gamma_{ij}\gamma^{ab}\pounds_n K_{ab}) \\ & +2\alpha_{GB}\Big[H_{ij} + \varepsilon(M\pounds_n K_{ij} - 2M_i{}^a\pounds_n K_{aj} - 2M_j{}^a\pounds_n K_{ai} - W_{ij}{}^{ab}\pounds_n K_{ab})\Big] = \kappa^2 \mathcal{T}_{\mu\nu}\gamma^\mu{}_i\gamma^\nu{}_j \end{aligned}$$

(b) Hamiltonian constraint equation:

$$M + \alpha_{GB}(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}) = -2\varepsilon\kappa^2\mathcal{T}_{\mu\nu}n^\mu n^\nu$$

(c) momentum constraint equation:

$$N_i + 2\alpha_{GB}\left(MN_i - 2M_i{}^aN_a + 2M^{ab}N_{iab} - M_i{}^{cab}N_{abc}\right) = -\kappa^2\mathcal{T}_{\mu\nu}n^\mu\gamma^\nu{}_i$$

$$\begin{aligned} M_{ijkl} &= R_{ijkl} - \varepsilon(K_{ik}K_{jl} - K_{il}K_{jk}) \\ M_{ij} &= \gamma^{ab}M_{iajb} = R_{ij} - \varepsilon(KK_{ij} - K_{ia}K^a{}_j) \\ M &= \gamma^{ab}M_{ab} = R - \varepsilon(K^2 - K_{ab}K^{ab}) \\ N_{ijk} &= D_iK_{jk} - D_jK_{ik} \\ N_i &= \gamma^{ab}N_{aib} = D_aK_i{}^a - D_iK \\ W_{ij}{}^{kl} &= M\gamma_{ij}\gamma^{kl} - 2M_{ij}\gamma^{kl} - 2\gamma_{ij}M^{kl} + 2M_{iajb}\gamma^{ak}\gamma^{bl} \\ H_{ij} &= MM_{ij} - 2(M_{ia}M^a{}_j + M^{ab}M_{iajb}) + M_{iabc}M_j{}^{abc} \\ & -2\varepsilon\Big[-K_{ab}K^{ab}M_{ij} - \frac{1}{2}MK_{ia}K^a{}_j + K_{ia}K^a{}_bM^b{}_j + K_{ja}K^a{}_bM^b{}_i + K^{ac}K_c{}^bM_{iajb} \\ & \qquad +N_iN_j - N^a(N_{aij} + N_{aji}) - \frac{1}{2}N_{abi}N^{ab}{}_j - N_{iab}N_j{}^{ab}\Big] \\ & -\frac{1}{4}\gamma_{ij}[M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}] \\ & -\varepsilon\gamma_{ij}[K_{ab}K^{ab}M - 2M_{ab}K^{ac}K_c{}^b - 2N_aN^a + N_{abc}N^{abc}] \end{aligned}$$

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# 3 数値相対論の標準的手法

## 3.1 どのように初期値を準備するか

**Initial Data Construction Problem** **Box 3.1**

Prepare all metric and matter components by solving the two constraints:

- The Hamiltonian constraint equation

$$^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho + 2\Lambda \tag{3.1}$$

- The momentum constraint equations

$$D_j(K^{ij} - \gamma^{ij}\mathrm{tr}K) = \kappa J^i \tag{3.2}$$

### 3.1.1 Conformal Approach – York-ÓMurchadha (1974)

**Conformal transformation**

The idea by ÓMurchadha and York [1] is

$$\text{solution} \quad \gamma_{ij} = \psi^4\hat{\gamma}_{ij} \quad \text{trial metric} \tag{3.3}$$

We introduce the decomposition of $K_{ij}$,

$$K_{ij} \Rightarrow \begin{cases} \mathrm{tr}K = \gamma^{ij}K_{ij} & \text{trace part} \\ A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}\mathrm{tr}K & \text{trace-free part} \end{cases} \tag{3.4}$$

Then, other conformal transformations as consistent with (3.3) are:

$$\gamma_{ij} = \psi^4\hat{\gamma}_{ij}, \qquad \gamma^{ij} = \psi^{-4}\hat{\gamma}^{ij}, \tag{3.5}$$
$$A^{ij} = \psi^{-10}\hat{A}^{ij}, \quad A_{ij} = \psi^{-2}\hat{A}_{ij}, \tag{3.6}$$
$$\rho = \psi^{-n}\hat{\rho}, \qquad J^i = \psi^{-10}\hat{J}^i, \tag{3.7}$$

and we suppose

$$\mathrm{tr}K = \hat{\mathrm{tr}}\hat{K}, \quad \mathrm{tr}A = \hat{\mathrm{tr}}\hat{A} = 0. \tag{3.8}$$

From (3.5), we get

$$\Gamma^i{}_{jk} = \hat{\Gamma}^i{}_{jk} + 2\psi^{-1}(\delta^i{}_j\hat{D}_k\psi + \delta^i{}_k\hat{D}_j\psi - \hat{\gamma}_{jk}\hat{\gamma}^{im}\hat{D}_m\psi), \tag{3.9}$$
$$R = \psi^{-4}\hat{R} - 8\psi^{-5}\hat{\Delta}\psi. \tag{3.10}$$

where $\hat{\Delta} = \hat{\gamma}^{jk}\hat{D}_j\hat{D}_k$ and $\hat{R} = R(\hat{\gamma})$, and also $D_jA^{ij} = \psi^{-10}\hat{D}_j\hat{A}^{ij}$.

We further decompose $\hat{A}^{ij}$ to divergence-free (transverse-traceless, TT) part and longitudinal part:

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + (\hat{\mathbb{l}}W)^{ij}, \tag{3.11}$$

where we suppose

$$\hat{D}_j\hat{A}^{ij}_{TT} = 0 \qquad \text{and} \qquad \hat{\mathrm{tr}}\hat{A}_{TT} = 0. \tag{3.12}$$

and

$$(\hat{\mathbb{L}}W)^{ij} = \hat{D}^i W^j + \hat{D}^j W^i - \frac{2}{3}\hat{\gamma}^{ij}\hat{D}_k W^k. \tag{3.13}$$

Using these terms, we can write

$$\begin{aligned} \hat{D}_j \hat{A}^{ij} &= \hat{D}_j(\hat{\mathbb{L}}W)^{ij} \equiv (\hat{\Delta}_{\mathbb{L}}W)^i, \\ &= (\hat{\Delta}W)^i + \frac{1}{3}\hat{D}^i(\hat{D}_j W^j) + \hat{R}^i{}_j W^j. \end{aligned} \tag{3.14}$$

With above transformation, the two constraints, (3.1) and (3.2), can be expressed as follows.

- The Hamiltonian constraint equation

$$8\hat{\Delta}\psi = \hat{R}\psi - (\hat{A}_{ij}\hat{A}^{ij})\psi^{-7} + [\frac{2}{3}(\mathrm{tr}K)^2 - 2\Lambda]\psi^5 - 16\pi G\hat{\rho}\psi^{5-n} \tag{3.15}$$

- The momentum constraint equations

$$\hat{\Delta}W^i + \frac{1}{3}\hat{D}^i\hat{D}_k W^k + \hat{R}^i{}_k W^k = \frac{2}{3}\psi^6\hat{D}^i \mathrm{tr}K + 8\pi G\hat{J}^i \tag{3.16}$$

**Equations to solve**

**Conformal approach (York-ÓMurchadha, 1974)** **Box 3.2**

One way to set up the metric and matter components $(\gamma_{ij}, K_{ij}, \rho, J^i)$ so as to satisfy the constraints (3.1) and (3.2) is as follows.

1. Specify metric components $\hat{\gamma}_{ij}$, $\mathrm{tr}K$, $\hat{A}^{TT}_{ij}$, and matter distribution $\hat{\rho}$, $\hat{J}$ in the conformal frame.
2. Solve the next equations for $(\psi,\ W^i)$

$$8\hat{\Delta}\psi = \hat{R}\psi - (\hat{A}_{ij}\hat{A}^{ij})\psi^{-7} + [\frac{2}{3}(\mathrm{tr}K)^2 - 2\Lambda]\psi^5 - 16\pi G\hat{\rho}\psi^{5-n} \tag{3.15}$$

$$\hat{\Delta}W^i + \frac{1}{3}\hat{D}^i\hat{D}_k W^k + \hat{R}^i{}_k W^k = \frac{2}{3}\psi^6\hat{D}^i \mathrm{tr}K + 8\pi G\hat{J}^i \tag{3.16}$$

   where

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + \hat{D}^i W^j + \hat{D}^j W^i - \frac{2}{3}\hat{\gamma}^{ij}\hat{D}_k W^k. \tag{3.17}$$

3. Apply the inverse conformal transformation and get the metric and matter components $\gamma_{ij}$, $K_{ij}$, $\rho$, $J^i$ in the physical frame:

$$\gamma_{ij} = \psi^4\hat{\gamma}_{ij}, \tag{3.18}$$

$$K_{ij} = \psi^{-2}[\hat{A}^{TT}_{ij} + (\hat{\mathbb{L}}W)_{ij}] + \frac{1}{3}\psi^4\hat{\gamma}_{ij}\mathrm{tr}K, \tag{3.19}$$

$$\rho = \psi^{-n}\hat{\rho}, \tag{3.20}$$

$$J^i = \psi^{-10}\hat{J}^i \tag{3.21}$$

**Comments**

- Using the idea of conformal rescaling, we have a way to fix 12 components of $(\gamma_{ij}, K_{ij})$ that satisfy 4 constraints.
- The Hamiltonian constraint, (3.15), is a non-linear elliptic equation for $\psi$, so that we have to solve it by an iterative method.
- The momentum constraints, (3.16), are PDEs for $W^i$ and coupled with (3.15). If we assume $\mathrm{tr}K = 0$, then two constraints are decoupled. Normally people assume $\mathrm{tr}K = 0$ (maximal slicing condition) or $(\mathrm{tr}K)$ =const. (constant mean curvature slicing) for this purpose.
- For simplicity, people assume the background metric $\hat{\gamma}_{ij}$ is conformally flat $\hat{\gamma}_{ij} = \delta_{ij}$. The physical appropriateness of conformal flatness is often debatable.
- Two freedom of $\hat{A}^{TT}_{ij}$ corresponds to the one of gravitational wave. However, there have been no systematic discussion how to specify them, except applying tensor harmonics in a linearized situation.

**Solving the Hamiltonian constraint – Several tips**
Two Methods:

1. Solve the non-linear equation (3.15) directly.
2. Solve the linearized equation $\psi = \psi_0 + \delta\psi$ iteratively.

$$
\begin{aligned}
8\hat{\Delta}\psi &= E\,\psi + F\,\psi^{-7} + G\,\psi^5 + H\,\psi^{-3} + I\,\psi^{-1} \\
&= [E - 7F\psi_0^{-8} + 5G\psi_0^4 - 3H\psi_0^{-4} - 2I\psi_0^{-2}]\,\psi + [8F\psi_0^{-7} - 4G\psi_0^5 + 4H\psi_0^{-3} + 2I\psi_0^{-1}]
\end{aligned}
$$

Under an appropriate boundary condition, such as Robin BC $\psi = 1 + const./r$, or Dirichlet BC $\psi = 1 + M_{total}/2r$.

**Solve the momentum constraints – Several tips**
A couple of methods:

1. Solve the non-linear equations (3.16) directly.
2. Bowen's method for conformally flat case [GRG14(1982)1183]
   Under the $(\nabla^i K = 0)$ condition, (3.16) becomes

$$\Delta W^i + \frac{1}{3}\nabla^i\nabla_j W^j = 8\pi S^i.$$

By introducing a decomposition of $W^i$ into vector and gradient terms

$$W^i = V^i - \frac{1}{4}\nabla^i\theta,$$

the equations to solve are:

$$\Delta V^i = 8\pi S^i, \tag{3.22}$$

$$\Delta\theta = \nabla_i V^i, \tag{3.23}$$

If the source is of finite extent, then the the asymptotic behavior of $V^i$ and $\theta$ are given by

$$V^i = -2\sum_{l=0}^{\infty} Q^{ij_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}}, \tag{3.24}$$

$$\begin{aligned}\theta &= -\sum_{l=1}^{\infty} Q^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l-1}} + \sum_{l=0}^{\infty}\frac{2(l+1)}{(2l+1)(2l+3)} Q_k^{kj_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}} \\ &+\sum_{l=1}^{\infty}\frac{2l-1}{2l+1} M^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l+1}}\end{aligned} \tag{3.25}$$

where $n^i = x^i r^{-1}$ in the Cartesian cordinate, the multipoles $Q$ and $M$ are defined as

$$Q^{ij_1\cdots j_l} \equiv \frac{(2l-1)!!}{l!}\int S^i(\mathbf{r}) x^{\{j_1}x^{j_2}\cdots x^{j_l\}} dV,$$
$$M^{ij_1\cdots j_l} \equiv \frac{(2l-1)!!}{l!}\int r^2 S^i(\mathbf{r}) x^{\{j_1}x^{j_2}\cdots x^{j_l\}} dV,$$

and where brackets denote the completely symmetric trace-free part

$$Z^{\{ij_1\cdots j_l\}} = Z^{(ij_1\cdots j_l)} - \frac{l}{2l+1} Z_k^{k(j_1\cdots j_{l-1}}\delta^{j_l i)}$$

### 3.1.2 Conformal Approach : $N$-dimensional case

We generalized the above conformal approach by York and ÓMurchadha (1974) to $N$-dimensional version and also for Gauss-Bonnet gravity. Here, we show only for the $N$-dimensional equations.

We start from the conformal transformation

| solution | $\gamma_{ij} = \psi^{2m}\hat\gamma_{ij}, \ \ \gamma^{ij} = \psi^{-2m}\hat\gamma^{ij}$ | trial metric |
|---|---|---|

this gives

$$\begin{aligned}R &= \psi^{-2m}\Big\{\hat R - 2(N-1)m\psi^{-1}(\hat D^a\hat D_a\psi) + (N-1)[2-(N-2)m]m\psi^{-2}(\hat D\psi)^2\Big\},\\ R_{ij} &= \hat R_{ij} - m\hat\gamma_{ij}\psi^{-1}\hat D_a\hat D^a\psi - (N-2)m\psi^{-1}\hat D_i\hat D_j\psi \\ &\quad + (N-2)m(m+1)\psi^{-2}\hat D_i\psi\hat D_j\psi - m[(N-2)m-1]\psi^{-2}(\hat D\psi)^2\hat\gamma_{ij},\\ R_{ijkl} &= \psi^{2m}\Big\{\hat R_{ijkl} + m\psi^{-1}\hat\gamma_{il}[\hat D_j\hat D_k\psi - (m+1)\psi^{-1}\hat D_j\psi\hat D_k\psi] \\ &\quad - m\psi^{-1}\hat\gamma_{ik}[\hat D_j\hat D_l\psi - (m+1)\psi^{-1}\hat D_j\psi\hat D_l\psi] \\ &\quad + m\psi^{-1}\hat\gamma_{jk}[\hat D_i\hat D_l\psi - (m+1)\psi^{-1}\hat D_i\psi\hat D_l\psi] \\ &\quad - m\psi^{-1}\hat\gamma_{jl}[\hat D_i\hat D_k\psi - (m+1)\psi^{-1}\hat D_i\psi\hat D_k\psi] + m^2\psi^{-2}(\hat D\psi)^2(\hat\gamma_{il}\hat\gamma_{jk} - \hat\gamma_{ik}\hat\gamma_{jl})\Big\}.\end{aligned}$$

Decompose the extrinsic curvature $K_{ij}$ as $K_{ij} \equiv A_{ij} + \frac{1}{N}\gamma_{ij}K$, and assume

$$A_{ij} = \psi^{\ell}\hat A_{ij}, \ \ A^{ij} = \psi^{\ell-4m}\hat A^{ij}, \ \ \text{and} \ \ K = \psi^{\tau}\hat K.$$

Conformal transformation of the divergence $D_jA^{ij}$ becomes

$$D_jA^{ij} = \psi^{-4m+\ell}\hat D_j\hat A^{ij} + \psi^{-4m+\ell-1}[\ell + m(N-2)]\hat A^{ij}\hat D_j\psi, \tag{3.26}$$

which indicates to set $\ell = -m(N-2) = -2$ for simplifying the equation. and for the components in the constraints,

$$K_{ij}K^{ij} = A_{ij}A^{ij} + \frac{1}{N}K^2 = \psi^{-12}\hat{A}_{ij}\hat{A}^{ij} + \frac{1}{N}\hat{K}^2, \tag{3.27}$$

$$D_jA^{ij} = \psi^{-10}\hat{D}_j\hat{A}^{ij}. \tag{3.28}$$

Decompose $A_{ij}$ into the divergence-free (transverse-traceless, TT) part, $A^{ij}_{TT}$, and the rest (longitudinal part), such as

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + \hat{A}^{ij}_{L}, \quad \text{where} \quad \hat{D}_j\hat{A}^{ij}_{TT} = 0. \tag{3.29}$$

The latter part can be expressed using a vector potential, $W^i$, as $\hat{A}^{ij}_L = \hat{D}^iW^j + \hat{D}^jW^i - \frac{2}{N}\hat{\gamma}^{ij}\hat{D}_kW^k$. When matter exists, define also the conformal transformation

$$\rho = \psi^{-p}\hat{\rho}, \qquad J^i = \psi^{-q}\hat{J}^i.$$

- Hamiltonian constraint equation, then, becomes

$$2(N-1)m\hat{D}_a\hat{D}^a\psi - (N-1)[2-(N-2)m]m(\hat{D}\psi)^2\psi^{-1} \\ = \hat{R}\psi - \frac{N-1}{N}\varepsilon\psi^{2m+2\tau+1}\hat{K}^2 + \varepsilon\psi^{-2m+2\ell+1}\hat{A}_{ab}\hat{A}^{ab} + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} \tag{3.30}$$

  We found that the combination $\ell = 2/(N-2)$ and $p = -1$ makes the RHS of (3.30) linear. If we choose $\ell = -2$, which will make the momentum constraint simpler as we see later, (3.30) also remains as a simple equation.

- We obtain the momentum constraint equation as

$$\hat{D}_a\hat{D}^aW_i + \frac{N-2}{N}\hat{D}_i\hat{D}_kW^k + \hat{R}_{ik}W^k \\ +\psi^{-1}[\ell + (N-2)m]\Big(\hat{D}^aW^b + \hat{D}^bW^a - \frac{2}{N}\hat{\gamma}^{ab}\hat{D}_kW^k\Big)\hat{\gamma}_{bi}\hat{D}_a\psi \\ -\psi^{2m-\ell}\frac{N-1}{N}\hat{D}_i(\psi^\tau\hat{K}) = \kappa^2\psi^{4m-\ell-q}\hat{J}_i \tag{3.31}$$

  We found that the choice of $\ell = -2$ cancels the mixing term between $\psi$ and $W^i$. The decoupling feature between two constraints is available when $\hat{K}$ =const. and $q = 8/(N-2) + 2$.

**Conformal approach for solving constraints in $(N+1)$-dim. [2]** **Box 3.3**

One way to set up $(\gamma_{ij}, K_{ij}, \rho, J^i)$ so as to satisfy the constraints:

1. Specify metric components $\hat{\gamma}_{ij}$, tr$K$, $\hat{A}_{ij}^{TT}$, and matter distribution $\hat{\rho}$, $\hat{J}$ in the conformal frame.
2. Solve the next equations for $(\psi,\ W^i)$

   (A) Hamiltonian constraint

   $$\frac{4(N-1)}{N-2}\hat{\Delta}\psi = \hat{R}\psi - \varepsilon\psi^{2\ell+1-4/(N-2)}(\hat{K}^2 - \hat{K}_{ab}\hat{K}^{ab}) + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} \quad (3.32)$$

   (B) momentum constraint

   $$\hat{\Delta}W_i + \frac{N-2}{N}\hat{D}_i\hat{D}_kW^k + \hat{R}_{ik}W^k + \psi^{-1}(\ell+2)\Big(\hat{D}^aW^b + \hat{D}^bW^a - \frac{2}{N}\hat{\gamma}^{ab}\hat{D}_kW^k\Big)\hat{\gamma}_{bi}\hat{D}_a\psi$$
   $$-\frac{N-1}{N}\Big[\Big(\ell - \frac{4}{N-2}\Big)(\hat{D}_i\psi)\hat{K} + \hat{D}_i\hat{K}\Big] = \kappa^2\psi^{8/(N-2)-\ell-q}\hat{J}_i \quad (3.33)$$

3. Apply the inverse conformal transformation and get the metric and matter components $\gamma_{ij}$, $K_{ij}$, $\rho$, $J^i$ in the physical frame:

   $$\begin{aligned}\gamma_{ij} &= \psi^{4/(N-2)}\hat{\gamma}_{ij},\\ K_{ij} &= \psi^{\ell}[\hat{A}_{ij}^{TT} + (\hat{\mathbb{l}}W)_{ij}] + \frac{1}{N}\psi^{\ell-4/(N-2)}\hat{\gamma}_{ij}\mathrm{tr}K,\\ \rho &= \psi^{-p}\hat{\rho},\\ J^i &= \psi^{-q}\hat{J}^i\end{aligned}$$

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### 3.2 どのようにゲージを設定するか

The standard 3+1 formulation allows us to choose gauge conditions (slicing conditions) for every time step. The fundamental guidelines for fixing the lapse function $\alpha$ and the shift vector $\beta_i$:

- to avoid the foliation hitting the physical and coordinate singularity in its evolution.
- to make system suitable for physical situation.
- to make the evolution system as simple as possible.
- to enable the gravitational wave extraction easy.

I list several essential slicing conditions below. The notations hereafter follows those of §2.1 (ADM formulation).

#### 3.2.1 Lapse conditions

| | | | | |
|---|---|---|---|---|
| geodesic slice | $\alpha = 1$ | `GOOD`<br>`BAD` | simple, easy to understand<br>no singularity avoidance | |
| harmonic slice | $\nabla_a \nabla^a x^b = 0$ | `GOOD`<br>`GOOD`<br>`BAD` | simplify eqs.,<br>easy to compare analytical investigations<br>no singularity avoidance or coordinate pathologies | [2]-[7] |
| maximal slice | $K = 0$ | `GOOD`<br>`BAD` | singularity avoidance<br>have to solve an elliptic eq. | [1],[8]-[15] |
| maximal slice (K-driver) | $\partial_t K = -c^2 K$ | `G&B`<br>`GOOD` | same with maximal slice,<br>easy to maintain $K = 0$ | [12] |
| constant mean curvature | $K = const.$ | `G&B`<br>`GOOD` | same with maximal slice,<br>suitable for cosmological situation | [16]-[18] |
| polar slicing | $K_\theta^\theta + K_\varphi^\varphi = 0$, or $K = K_r^r$ | `GOOD`<br>`BAD` | singularity avoidance in isotropic coord.<br>trouble in Schwarzschild coord. | [19]-[21] |
| algebraic | $\alpha \sim \sqrt{\gamma}$,<br>$\alpha \sim 1 + \log \gamma$ | `GOOD`<br>`BAD` | easy to implement<br>not avoiding singularity | |

**Maximal slicing**
This is always the first one to be mentioned as a singularity avoiding gauge condition. The name of 'maximal' comes from the fact that the deviation of the 3-volume $V = \int \sqrt{\gamma} d^3x$ along to the normal line becomes maximal when we set $K = 0$. This is simply written as

$$K = 0 \qquad \text{on} \quad \Sigma(t). \tag{3.34}$$

Pioneering idea can be seen in Lichnerowicz [8], and it was extended by York [1]. This condition is supposed to be applied in simulations that a singularity will appear during evolutions such as gravitational collapses. The actual equation for determining the lapse function $\alpha$ can be obtained from $\partial_t K = \partial_t (K_{ij}\gamma^{ij}) = 0$. By substituting the evolution equations, we get

$$D^i D_i \alpha = \{ {}^{(3)}R + K^2 + 4\pi G(S - 3\rho_H) - 3\Lambda\}\alpha, \tag{3.35}$$

or by using the Hamiltonian constraint further,

$$D^i D_i \alpha = \{K_{ij} K^{ij} + 4\pi G(S + \rho_H) - \Lambda\}\alpha. \tag{3.36}$$

This is an elliptic equation. When the curvature is strong (i.e. close to the appearance of a singularity), the RHS of equation become larger, hence the lapse becomes smaller. Therefore the foliation near the singularity evolves slowly.

For Schwarzschild black-hole space-time, Estabrook *et al.* [10] showed that the maximal slicing condition allows the 3-surface to reach into $r = 1.5M$ in the limit $t \to \infty$, that is inside of the event horizon, $r = 2M$. However, it is also reported that the difference of $\alpha$-evolution causes the grid-stretching problem.

### 3.2.2 Shift conditions

| | | | | |
|---|---|---|---|---|
| geodesic slice | $\beta^i = 0$ | `GOOD` | simple, easy to understand | |
| | | `BAD` | too simple | |
| minimal distortion | $\min \Sigma^{ij}\Sigma_{ij}$ | `GOOD` | geometrical meaning | [1] |
| | | `BAD` | elliptic eqs., hard to solve | |
| minimal strain | $\min \Theta^{ij}\Theta_{ij}$ | `G&B` | same with minimal distortion | [1] |

**Minimal distortion condition, minimal strain condition**
Any singularity avoiding slice conditions causes the grid stretching problem. Smarr and York [1] proposed the condition which minimize the distortion in a global sense.

Let us define the expansion tensor $\Theta_{\mu\nu}$ and the distortion tensor $\Sigma_{ij}$. Let the normal direction to the surface $n^\mu$, and the coordinate-constant congruence $t_\mu = \alpha n_\mu + \beta_\mu$. By projecting $t^\mu$ onto the hypersurface using the projection operator $\perp^a_b = \delta^a_b + n^a n_b$,

$$\Theta_{\mu\nu} = \perp \nabla_{(\nu} t_{\mu)} = -\alpha K_{\mu\nu} + \frac{1}{2} D_{(\mu}\beta_{\nu)} \tag{3.37}$$

We then extract this traceless part and define,

$$\Sigma_{ij} = \Theta_{ij} - \frac{1}{3}\Theta\gamma_{ij} = -2\alpha\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) + \frac{1}{2}\left(D_{(i}\beta_{j)} - \frac{1}{3}D^k\beta_k\right). \tag{3.38}$$

The minimal distortion condition is to choose $\beta^i$ which minimize the action

$$\delta S[\beta] = \delta\{\frac{1}{2}\int \Sigma_{ij}\Sigma^{ij} d^3x\} = 0. \tag{3.39}$$

This condition can be written as $D^j\Sigma_{ij} = 0$, or

$$D^jD_j\beta_i + D^jD_i\beta_j - \frac{2}{3}D_iD_j\beta^j = D^j\left[2\alpha\left(K_{ij} - \frac{1}{3}trK\gamma_{ij}\right)\right], \tag{3.40}$$

$$\text{or} \quad \Delta\beta_i + \frac{1}{3}D_i(D^j\beta_j) + R^j_i\beta_j = D^j\left[2\alpha\left(K_{ij} - \frac{1}{3}trK\gamma_{ij}\right)\right], \tag{3.41}$$

where $\Delta = D^iD_i$.

Similarly, we can define the minimal strain condition by minimizing $\Theta^{ij}\Theta_{ij}$.

The both requires non-linear elliptic equations and hard to solve. Several group solves “pseudo”-minimal distortion condition by replacing the covariant derivatives to the partial derivatives [22]. This simplification also works for inspiral binary neutron star evolution.

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## 3.3 Ashtekar 形式を用いた数値相対論

### 3.3.1 History

Ashtekar's formulation of general relativity[11] has many attractive features comparing to the conventional ADM formulation. Therefore, an application to numerical simulations was suggested [2] soon after Ashtekar completed his formulation, but had not yet been completed more than a decade. Historically, an application to numerical relativity of the connection formulation was also suggested [3, 4] using Capovilla-Dell-Jacobson's version of the connection variables [5], which produce an direct relation to Newman-Penrose's $\Psi$s.

The first full numerical application was reported by Shinkai and Yoneda [58, 71]. They developed a plane symmetric evolution code, and showed comparisons of numerical stability due to the different hyperbolicity in the context of formulation problem (§4, in this lecture note). They also showed that their new formulation called $\lambda$-system, which makes the evolution system asymptotically constrained, works as desired.

In this subsection, we only look at how they realized numerical experiments from the viewpoint of methodology.

### 3.3.2 Numerical treatments by Shinkai-Yoneda

Shinkai and Yoneda coded up the program so as to compare the evolutions of spacetime with three different sets of dynamical equations (Ashtekar's original, and two modified sets) but with the common conditions: the same initial data, the same boundary conditions, the same slicing condition and the same evolution scheme.

They considered the plane symmetric vacuum spacetime without cosmological constant. This spacetime has the true freedom of gravitational waves of two polarized (+ and $\times$) modes. They applied the periodic boundary conditions to remove any difficulties caused by numerical treatment of the boundary conditions. The initial data are given by solving constraint equations in ADM variables, using the standard conformal approach by York and O'Murchadha (Box 3.2 in this lecture note).

When we use Ashtekar's variables for evolution, we transform the ADM initial data in terms of Ashtekar's variables. The results are analyzed by monitoring the violation of the constraint equations which are expressed using the same (or transformed if necessary) variables.

#### Reformulation of the Ashtekar evolution equations

They constructed three variations of Ashtekar's evolution system (see Table 3.1 for summary).

(a) The original set of dynamical equations (2.75) and (2.76) [the *original* equations] already forms a weakly hyperbolic system [12]. So that we regard the mathematical structure of the original equations as one step advanced from the standard ADM.

| | system | variables | Eqs of motion | remark |
|---|---|---|---|---|
| I | Ashtekar (weakly hyp.) | $(\tilde{E}^i_a, \mathcal{A}^a_i)$ | (2.75), (2.76) (original) | "original" eqs. |
| II | Ashtekar (strongly hyp.) | $(\tilde{E}^i_a, \mathcal{A}^a_i)$ | (3.42), (3.43) (with $\kappa = 1$) | (3.45) required |
| III | Ashtekar (symmetric hyp.) | $(\tilde{E}^i_a, \mathcal{A}^a_i)$ | (3.42), (3.43) (with $\kappa = 1$) | (3.44) required |
| adj | Ashtekar (adjusted) | $(\tilde{E}^i_a, \mathcal{A}^a_i)$ | (3.42), (3.43) (with $\kappa \neq 1$) | |
| $\lambda$ | Ashtekar-$\lambda$-system | $(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a)$ | | controls $\mathcal{C}_H, \mathcal{C}_{Mi}, \mathcal{C}_{Ga}$ |

Table 3.1: List of Ashtekar evolution systems that applied in [58].

(b) Further, we can construct higher levels of hyperbolic systems by restricting the gauge condition and/or by adding constraint terms, $\mathcal{C}_H^{\rm ASH}, \mathcal{C}_{Mi}^{\rm ASH}$ and $\mathcal{C}_{Ga}^{\rm ASH}$, to the original equations.

- by requiring additional gauge conditions *or* adding constraints to the dynamical equations, we can obtain a strongly hyperbolic system [12],
- by requiring additional gauge conditions *and* adding constraints to the dynamical equations, we can obtain a symmetric hyperbolic system [70, 12].

(c) Based on the above symmetric hyperbolic system, we can construct an Ashtekar version of the $\lambda$-system [20] which is robust against perturbative errors for both constraints and reality conditions [57].

In order to obtain a symmetric hyperbolic system, we add constraint terms to the right-hand-side of (2.75) and (2.76). The adjusted dynamical equations,

$$\begin{aligned}
\partial_t \tilde{E}^i_a &= -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c + \kappa_1 P^i{}_{ab}\, \mathcal{C}_G^{\mathrm{ASH}b}, \quad (3.42)\\
&\quad \text{where} \qquad P^i{}_{ab} \equiv N^i\delta_{ab} + i\underset{\sim}{N}\epsilon_{ab}{}^c \tilde{E}^i_c,\\
\partial_t \mathcal{A}^a_i &= -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \kappa_2 Q^a_i \mathcal{C}_H^{\mathrm{ASH}} + \kappa_3 R_i{}^{ja}\, \mathcal{C}_{Mj}^{\mathrm{ASH}}, \quad (3.43)\\
&\quad \text{where} \qquad Q^a_i \equiv e^{-2}\underset{\sim}{N}\tilde{E}^a_i, \qquad R_i{}^{ja} \equiv ie^{-2}\underset{\sim}{N}\epsilon^{ac}{}_b \tilde{E}^b_i \tilde{E}^j_c
\end{aligned}$$

form a symmetric hyperbolicity if we further require $\kappa_1 = \kappa_2 = \kappa_3 = 1$ and the gauge conditions,

$$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i, \qquad \partial_i N = 0. \tag{3.44}$$

We remark that the adjusted coefficients, $P^i{}_{ab}, Q^a_i, R_i{}^{ja}$, for constructing the symmetric hyperbolic system are uniquely determined, and there are no other additional terms (say, no $\mathcal{C}_H^{\rm ASH}, \mathcal{C}_M^{\rm ASH}$ for $\partial_t \tilde{E}^i_a$, no $\mathcal{C}_G^{\rm ASH}$ for $\partial_t \mathcal{A}^a_i$) [12]. The gauge conditions, (3.44), are consequences of the consistency with (triad) reality conditions.

We can also construct a strongly (or diagonalizable) hyperbolic system by restricting to a gauge $N^l \neq 0, \pm N\sqrt{\gamma^{ll}}$ (where $\gamma^{ll}$ is the three-metric and we do not sum indices here) for the original equations (2.75), (2.76). Or we can also construct from the adjusted equations, (3.42) and (3.43), together with the gauge condition

$$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i. \tag{3.45}$$

As for the strongly hyperbolic system, we hereafter take the latter expression.

**metric and the initial data construction**

We consider the plane symmetric metric,

$$ds^2 = (-N^2 + N_x N^x)dt^2 + 2N_x dxdt + \gamma_{xx}dx^2 + \gamma_{yy}dy^2 + \gamma_{zz}dz^2 + 2\gamma_{yz}dydz \tag{3.46}$$

where the components are the function of $N(x,t), N_x(x,t), \gamma_{xx}(x,t), \gamma_{yy}(x,t), \gamma_{zz}(x,t), \gamma_{yz}(x,t)$. $N$ and $N^x$ are called the lapse function and the shift vector.

We prepare our initial data by solving the ADM constraint equations, (2.14) and (**??**), using the conformal approach (Box 3.2). Since we consider only the vacuum spacetime, the input quantities are the initial guess of the 3-metric $\hat{\gamma}_{ij}$, the trace part of the extrinsic curvature tr$K$, and the transverse traceless part of the extrinsic curvature $\hat{A}_{TT}$. For simplicity, we impose $\hat{A}_{TT} = 0$ and tr$K = K_0$ (constant). The Hamiltonian constraint, then, becomes an equation for the conformal factor, $\psi$:

$$8\hat{\Delta}\psi := 8\frac{1}{\sqrt{\hat{\gamma}}}\partial_i(\hat{\gamma}^{ij}\sqrt{\hat{\gamma}}\partial_j\psi) = \hat{R}\psi + \frac{2}{3}(K_0)^2\psi^5, \tag{3.47}$$

where $\hat{\gamma} = \det \hat{\gamma}_{ij}$. The momentum constraint is automatically satisfied by assumption. The initial dynamical quantities $\gamma_{ij}$, $K_{ij}$ are given by the conformal transformation,

$$\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}, \qquad K_{ij} = \frac{1}{3}\psi^4 \hat{\gamma}_{ij} K_0. \tag{3.48}$$

We solve (3.47) under the periodic boundary conditions using the incomplete Cholesky conjugate gradient (ICCG) method.

We can set two different modes of gravitational waves. One is the $+$-mode waves, which is given by setting a conformal guess metric as (in a matrix form)

$$\hat{\gamma}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ sym. & 1 + a\exp(-b(x-c)^2) & 0 \\ sym. & sym. & 1 - a\exp(-b(x-c)^2) \end{pmatrix} \tag{3.49}$$

where $a, b, c$ are parameters. The other is the $\times$-mode waves, given by

$$\hat{\gamma}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ sym. & 1 & a\exp(-b(x-c)^2) \\ sym. & sym. & 1 \end{pmatrix} \tag{3.50}$$

where $a, b, c$ are parameters again. Both cases, we expect non-linear behavior when wave's curvature becomes quite large compared to the background. In the collision of a $+$-mode wave and a $\times$-mode wave, we also expect to see the mode-mixing phenomena which is known as gravitational Faraday effect. These effects are confirmed in our numerical simulations.

**Transformation of variables: From ADM to Ashtekar**

We need to transform the dynamical variables on the initial data when we evolve them in the connection variables. We list the procedure to obtain $(\tilde{E}^i_a, \mathcal{A}^a_i)$ from $(\gamma_{ij}, K_{ij})$. This procedure is used also when we evaluate the constraints, $\mathcal{C}^{\rm ASH}_H, \mathcal{C}^{\rm ASH}_{Mi}, \mathcal{C}^{\rm ASH}_{Ga}$ for the data evolved using ADM variables.

From the three-metric $\gamma_{ij}$ to $\tilde{E}^i_a$:

1. Define the triad $E^a_i$ corresponding to the three-metric $\gamma_{ij}$. We take

$$E^a_i = \begin{bmatrix} E^1_x & E^1_y & E^1_z \\ E^2_x & E^2_y & E^2_z \\ E^3_x & E^3_y & E^3_z \end{bmatrix} = \begin{bmatrix} \sqrt{\gamma_{xx}} & 0 & 0 \\ 0 & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{bmatrix}. \tag{3.51}$$

and set simply $e_{23} = e_{32}$. The relation between the metric and the triad becomes

$$e_{22}^2 + e_{33}^2 = \gamma_{yy}, \qquad e_{23}^2 + e_{33}^2 = \gamma_{zz}, \qquad (e_{22} + e_{33})e_{23} = \gamma_{yz}. \tag{3.52}$$

For the case of $+$-mode waves, we define naturally, $e_{22} = \sqrt{\gamma_{yy}}, e_{33} = \sqrt{\gamma_{zz}}, e_{23} = 0$. For $\times$-mode waves, we also take a natural set of definitions, $e_{22} = e_{33} = [(\gamma_{yy} + (\gamma_{yy}^2 - \gamma_{yz}^2)^{1/2})/2]^{1/2}$ and $e_{23} = \gamma_{yz}/2e_{22}$ which are given by solving $e_{22}^2 + e_{33}^2 = \gamma_{yy}$ and $2e_{22}e_{23} = \gamma_{yz}$.

2. obtain the inverse triad $E^i_a$ from triad $E^a_i$.

3. calculate the density, $e$, as $e = \det E^a_i$.

4. obtain the densitized triad, $\tilde{E}^i_a = eE^i_a$.

From three-metric $(\gamma_{ij}, K_{ij})$ to $\mathcal{A}^a_i$:

1. prepare the triad $E^a_i$ and its inverse $E^i_a$.

2. calculate the connection 1-form $\omega_i^{bc} = E^{b\mu}\nabla_i E^c_\mu$. This is expressed only using partial derivatives as [9]

$$\omega_i^{bc} = E^{jb}\partial_{[i}E^c_{j]} - E_{id}E^{kb}E^{jc}\partial_{[k}E^d_{j]} + E^{jc}\partial_{[j}E^b_{i]}. \quad (3.53)$$

3. $\mathcal{A}^a_i = -K_{ij}E^{ja} - \frac{i}{2}\epsilon^a{}_{bc}\omega_i^{bc}$.

**Transformation of variables: From Ashtekar to ADM**

In contrast to the previous transformation, we also need to obtain $(\gamma_{ij}, K_{ij})$ from $(\tilde{E}^i_a, \mathcal{A}^a_i)$ when we evaluate the metric output or ADM constraints when we evolve the spacetime using connection variables. This process is only required at an evaluation times, not required at every time step (unless we use the gauge condition which is primarily defined using ADM quantities).

From densitized inverse triad $\tilde{E}^i_a$ to three-metric $\gamma_{ij}$:

1. calculate the density $e$ as $e = (\det \tilde{E}^i_a)^{1/2}$.
2. get the three inverse metric as $\gamma^{ij} = \tilde{E}^i_a \tilde{E}^j_a / e^2$.
3. obtain $\gamma_{ij}$.

From $(\tilde{E}^i_a, \mathcal{A}^a_i)$ to the extrinsic curvature $K_{ij}$:

1. prepare the un-densitized inverse triad, $E^i_a = \tilde{E}^i_a / e$.
2. prepare triad $E^a_i$.
3. calculate the connection 1-form $\epsilon^a{}_{bc}\omega_i^{bc}$.
4. calculate $Z^a_i$, which is defined as [10] $Z^a_i := -\mathcal{A}^a_i + \frac{i}{2}\epsilon^a{}_{bc}\omega_i^{bc} (= K_{ij}E^{ja})$, and get $K_{ij} = Z^a_i E_{ja}$.

**Gauge conditions**

Their choice of the slicing (gauge) condition was the simplest one.

(1) the simplest geodesic slicing condition for the lapse function,

(2) the simplest zero shift vector $N^x = 0$, and

(3) the natural choice of triad lapse function $\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$ [$= 0$ if $N^x = 0$, which is suggested from (3.44) or (3.45)].

However, in the Ashtekar formalism, the densitized lapse function $\underset{\sim}{N}$ is the fundamental gauge quantity (rather than $N$). Therefore we try two conditions for the lapse,

(1a) the standard geodesic slicing condition $N = 1$, which will be transformed to $\underset{\sim}{N} = 1/e$ when we apply this condition in Ashtekar's evolution system, and

[9] This is from the definitions, $\omega_i^{bc} := E^{jb}\nabla_i E^c_j$ and $\omega^{abc} := E^{ja}\omega_j^{bc}$, and a relation

$$3\omega^{[abc]} - 2\omega^{[bc]a} = \omega^{a[bc]} + \omega^{b[ca]} + \omega^{c[ab]} - \omega^{abc} + \omega^{cba} = \omega^{abc}.$$

Using the densitized triad, eq. (3.53) can be also expressed as

$$\omega_i^{bc} = \frac{2}{e^2}\tilde{E}^{jb}(\partial_{[i}\tilde{E}^c_{j]}) + \frac{1}{e^4}\tilde{E}^{jb}\tilde{E}^c_i\tilde{E}^a_k(\partial_j\tilde{E}^k_a) + \frac{1}{4e^4}\tilde{E}_{ia}\tilde{E}^{kb}\tilde{E}^j_c(\partial_j\tilde{E}^a_k), \quad \text{taking } [bc].$$

[10] This is from the original definition of $\mathcal{A}^a_i$, $\mathcal{A}^a_i := \omega_i^{0a} - (i/2)\epsilon^a{}_{bc}\,\omega_i^{bc}$.

(1b) the densitized geodesic slicing condition $\underset{\sim}{N} = 1$, which will be transformed to $N = e$ when we evolve the system using ADM equations.

In practice, such a transformation using the density $e$ will not guarantee that the Courant condition holds if we fix the time evolution step $\Delta t$ [11]. Therefore we need to rescale the transformed lapse [ $\underset{\sim}{N}$ in (1a), $N$ in (1b)] so that it has a maximum value of unity, in order to keep our evolution system stable.

If we apply the standard geodesic slice, then we can compare the weakly hyperbolic system with the symmetric hyperbolic one. Similarly if we apply the densitized geodesic slice, then we can compare the (original) weakly hyperbolic system with the strongly hyperbolic one.

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[11] We here remind the reader of the stability condition, $N\Delta t \leq \Delta x$ for a standard forward-time centered-space (FTCS) scheme for a simple wave equation, in a $(\Delta t,\, \Delta x)$-spaced numerical grid. Note that this condition will be changed due to the choice of the evolution scheme and the equations of the system.

# 4 数値相対論の定式化問題

ADM 形式は，一般相対論における時空分解の基本であるが，現在の数値相対論では ADM 変数を用いるのは主流ではない．長時間の積分に対して不安定だからである (Fig. 4.1)．数値シミュレーションに適した方程式は何か，という問題を「定式化問題 (formulation problem)」という．

2005 年に，ブラックホール連星の合体に関するシミュレーションの成功が報告され，世界中の研究グループがその処方箋にしたがって計算を行っている．しかし，現在の定式化がベストなものなのかは，いまだ不明である．

ここでは，定式化問題を概観する．まとまったレビューは，[60, 56] などを参照されたい．

## 4.1 Overview

Up to a couple of years ago, the "standard ADM" decomposition (§2.1) of the Einstein equation was taken as the standard formulation for numerical relativists. However, numerical simulations were often interrupted by unexplained blow-ups. This was thought due to the lack of resolution, or inappropriate gauge choice, or the particular numerical scheme which was applied. However, after the accumulation of much experience, people have noticed the importance of the formulation of the evolution equations, since there are apparent differences in numerical stability although the equations are mathematically equivalent Figures 4.2 are chronological maps of the research. See Column 1 for the meaning of "stability".

At this moment, there are three major ways to obtain longer time evolutions:

(1) modifications of the standard Arnowitt-Deser-Misner equations initiated by the Kyoto group,

(2) rewriting of the evolution equations in hyperbolic form, and

(3) construction of an "asymptotically constrained" system. Of course, the ideas, procedures, and problems are mingled with each other. The purpose of this section is to review all three approaches and to introduce our idea to view them in a unified way.

The third idea has been generalized by us as an asymptotically constrained system. The main procedure is to adjust the evolution equations using the constraint equations [71, 72, 59]. The method is also applied to explain why the above approach (1) works, and also to propose alternative systems based on the ADM [72, 59] and BSSN [73] equations.

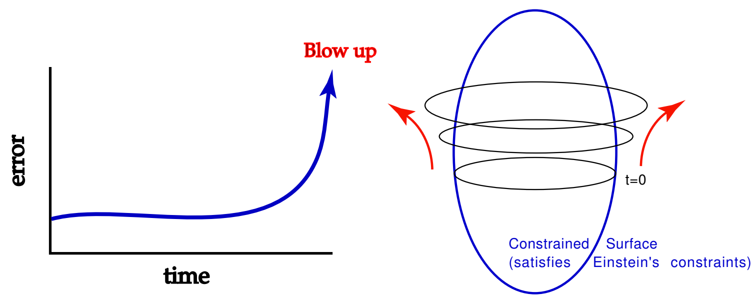


Figure 4.1: Origin of the problem for numerical relativists: Numerical evolutions depart from the constraint surface.

**Column 1**

The word **stability** is used quite different ways in the community.

- We mean by *numerical stability* a numerical simulation which continues without any blow-ups and in which data remains on the constrained surface.
- *Mathematical stability* is defined in terms of the well-posedness in the theory of partial differential equations, such that the norm of the variables is bounded by the initial data. See eq. (4.14) and around.
- For numerical treatments, there is also another notion of *stability*, the stability of finite differencing schemes. This means that numerical errors (truncation, round-off, etc) are not growing by evolution, and the evaluation is obtained by von Neumann's analysis. Lax's equivalence theorem says that if a numerical scheme is consistent (converging to the original equations in its continuum limit) and stable (no error growing), then the simulation represents the right (converging) solution. See [24] for the Einstein equations.

## 4.2 The standard way and the three other roads

### 4.2.1 Strategy 0: The ADM formulation

As we see in §2.1, we know that *if the constraints are satisfied on the initial slice $\Sigma$, then the constraints are satisfied throughout evolution.* The normal numerical scheme is to solve the elliptic constraints for preparing the initial data, and to apply the free evolution (solving only the evolution equations). The constraints are used to monitor the accuracy of simulations.

The origin of the problem was that the above statement in *Italics* is true in principle, but is not always true in numerical applications. A long history of trial and error began in the early 90s. Shinkai and Yoneda showed that the standard ADM equations *has* a constraint violating mode in its constraint propagation equations even for a single black-hole (Schwarzschild) spacetime [59].

### 4.2.2 Strategy 1: Modified ADM formulation by Nakamura et al

Up to now, the most widely used formulation for large scale numerical simulations is a modified ADM system, which is now often cited as the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation. This reformulation was first introduced by Nakamura *et al.* [49, 48, 55]. The usefulness of this reformulation was re-introduced by Baumgarte and Shapiro [13], then was confirmed by other groups to show a long-term stable numerical evolution [3, 4].

**Basic variables and equations** The widely used notation[13] introduces the variables $(\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$ instead of $(\gamma_{ij}, K_{ij})$, where

$$\varphi = (1/12)\log(\det\gamma_{ij}), \qquad \tilde{\gamma}_{ij} = e^{-4\varphi}\gamma_{ij}, \qquad K = \gamma^{ij}K_{ij}, \tag{4.1}$$

$$\tilde{A}_{ij} = e^{-4\varphi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i = \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk}. \tag{4.2}$$

The new variable $\tilde{\Gamma}^i$ was introduced in order to calculate Ricci curvature more accurately. In BSSN formulation, Ricci curvature is not calculated as $R^{ADM}_{ij} = \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^l_{ij}\Gamma^k_{lk} - \Gamma^l_{kj}\Gamma^k_{li}$, but as $R^{BSSN}_{ij} = R^{\varphi}_{ij} + \tilde{R}_{ij}$, where the first term includes the conformal factor $\varphi$ while the second term does

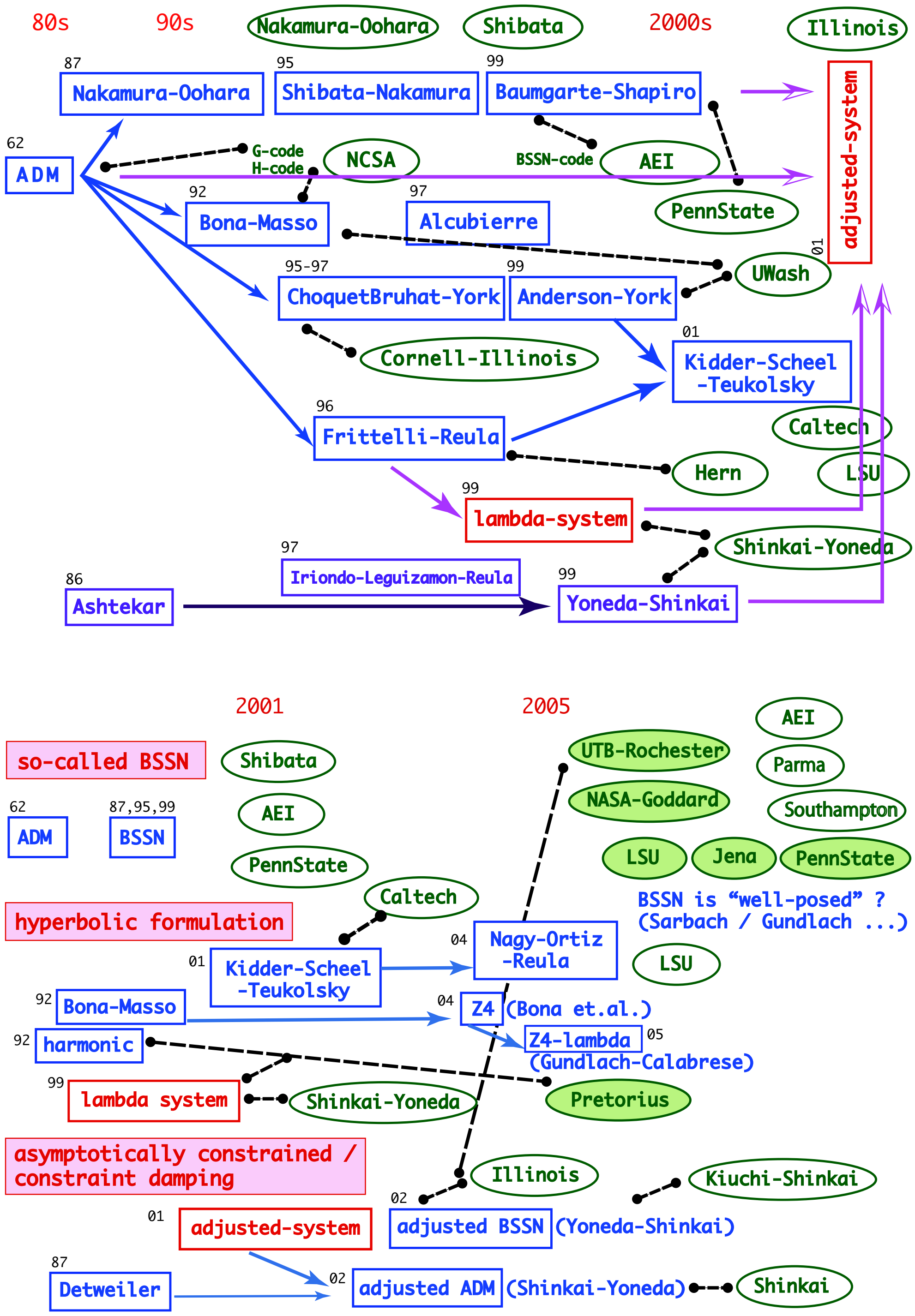


Figure 4.2: Chronological table of formulations and their numerical tests (∼ 2001) and (2001 ∼). Boxed ones are proposals of formulations, and circled ones are related numerical experiments. Please refer to Table 1 in Ref. [60] or [56] for each reference.

not. These are approximately equivalent, but $R_{ij}^{BSSN}$ does have wave operator apparently in the flat background limit, so that we can expect more natural wave propagation behavior.

Additionally, the BSSN requires us to impose the conformal factor as $\tilde{\gamma}(:= \det\tilde{\gamma}_{ij}) = 1$, during evolution. This is a kind of definition, but can also be treated as a constraint.

**The BSSN formulation [49, 48, 55, 13]:** **Box 4.1**

The fundamental dynamical variables are $(\varphi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$.

The three-hypersurface $\Sigma$ is foliated with gauge functions, $(\alpha, \beta^i)$, the lapse and shift vector.

- The evolution equations:

$$
\begin{aligned}
\partial_t^B \varphi &= -(1/6)\alpha K + (1/6)\beta^i(\partial_i \varphi) + (\partial_i \beta^i), && (4.3)\\
\partial_t^B \tilde{\gamma}_{ij} &= -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik}(\partial_j \beta^k) + \tilde{\gamma}_{jk}(\partial_i \beta^k) - (2/3)\tilde{\gamma}_{ij}(\partial_k \beta^k) + \beta^k(\partial_k \tilde{\gamma}_{ij}), && (4.4)\\
\partial_t^B K &= -D^i D_i \alpha + \alpha \tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 + \beta^i(\partial_i K), && (4.5)\\
\partial_t^B \tilde{A}_{ij} &= -e^{-4\varphi}(D_i D_j \alpha)^{TF} + e^{-4\varphi}\alpha(R_{ij}^{BSSN})^{TF} + \alpha K \tilde{A}_{ij} - 2\alpha \tilde{A}_{ik}\tilde{A}^k{}_j \\
&\quad +(\partial_i \beta^k)\tilde{A}_{kj} + (\partial_j \beta^k)\tilde{A}_{ki} - (2/3)(\partial_k \beta^k)\tilde{A}_{ij} + \beta^k(\partial_k \tilde{A}_{ij}), && (4.6)\\
\partial_t^B \tilde{\Gamma}^i &= -2(\partial_j \alpha)\tilde{A}^{ij} + 2\alpha(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - (2/3)\tilde{\gamma}^{ij}(\partial_j K) + 6\tilde{A}^{ij}(\partial_j \varphi)) \\
&\quad -\partial_j(\beta^k(\partial_k \tilde{\gamma}^{ij}) - \tilde{\gamma}^{kj}(\partial_k \beta^i) - \tilde{\gamma}^{ki}(\partial_k \beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k \beta^k)). && (4.7)
\end{aligned}
$$

- Constraint equations:

$$
\begin{aligned}
\mathcal{H}^{BSSN} &= R^{BSSN} + K^2 - K_{ij}K^{ij}, && (4.8)\\
\mathcal{M}_i^{BSSN} &= \mathcal{M}_i^{ADM}, && (4.9)\\
\mathcal{G}^i &= \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, && (4.10)\\
\mathcal{A} &= \tilde{A}_{ij}\tilde{\gamma}^{ij}, && (4.11)\\
\mathcal{S} &= \tilde{\gamma} - 1. && (4.12)
\end{aligned}
$$

(4.8) and (4.9) are the Hamiltonian and momentum constraints (the "kinematic" constraints), while the latter three are "algebraic" constraints due to the requirements of BSSN formulation.

**Remarks, pros and cons** Why is the BSSN better than the standard ADM? Together with numerical comparisons with the standard ADM case[4], this question has been studied by many groups using different approaches.

- Using numerical test evolutions, Alcubierre *et al.* [3] found that the essential improvement is in the process of replacing terms by the momentum constraints. They also pointed out that the eigenvalues of the BSSN *evolution equations* have fewer "zero eigenvalues" than those of ADM, and they conjectured that the instability might be caused by these "zero eigenvalues."

- Miller[46] reported that the BSSN had a wider range of parameters that gave stable evolutions in the von Neumann's stability analysis.

- An effort was made to understand the advantage of the BSSN from the point of hyperbolization of the equations in the linearized limit [3, 52] or with a particular combination of slicing conditions plus auxiliary variables[40]. If we define the 2nd-order symmetric hyperbolic form, then the principal part of the BSSN can be one of them[38].

As we discussed in Ref. [73], the stability of the BSSN formulation is due not only to the introductions of new variables but also to the replacement of terms in the evolution equations by using constraints. Further, we can show several additional adjustments to the BSSN equations, which give us more stable numerical simulations. We will devote Section 4.3 to this fundamental idea.

The current binary black-hole simulations apply the BSSN formulations with several implementations. For example,

tip-1 Alcubierre *et al.* [4] reported that the trace-out $A_{ij}$ technique at every time-step helped the stability.

tip-2 Campanelli *et al.* [23] reported that in their codes $\tilde{\Gamma}^i$ was replaced by $-\partial_j \tilde{\gamma}^{ij}$ where it was not differentiated.

tip-3 Baker *et al.* [12] modified the $\tilde{\Gamma}^i$-equation, Eq. (4.7), as suggested by Yo *et al.* [69].

These technical tips are again explained by using the constraint propagation analysis as we will do in Section 4.3.3.

These studies provide evidence regarding the advantage of the BSSN while it is also shown an example of an ill-posed solution in the BSSN (as well in the ADM) by Frittelli and Gomez [34]. Recently, the popular combination, BSSN with Bona-Masso type slicing condition, was investigated. Garfinkle *et al.* [36] speculated that the reason for gauge shocks being missing in the current 3-dimensional black-hole simulations is simply the lack of resolution.

### 4.2.3 Strategy 2: Hyperbolic reformulations

**Definitions, properties, mathematical backgrounds** The second effort to re-formulate the Einstein equations is to make the evolution equations reveal a first-order hyperbolic form explicitly. This is motivated by the expectation that the symmetric hyperbolic system has well-posed properties in its Cauchy treatment in many systems and also that the boundary treatment can be improved if we know the characteristic speed of the system.

**Hyperbolic formulations** **Box 4.2**

We say that the system is *a first-order (quasi-linear) partial differential equation system*, if a certain set of (complex-valued) variables $u_\alpha$ $(\alpha = 1, \cdots, n)$ forms

$$\partial_t u_\alpha = \mathcal{M}^{l\beta}{}_\alpha(u)\, \partial_l u_\beta + \mathcal{N}_\alpha(u), \tag{4.13}$$

where $\mathcal{M}$ (the characteristic matrix) and $\mathcal{N}$ are functions of $u$ but do not include any derivatives of $u$. Further we say the system is

- *a weakly hyperbolic system*, if all the eigenvalues of the characteristic matrix are real.
- *a strongly hyperbolic system* (or a diagonalizable / symmetrizable hyperbolic system), if the characteristic matrix is diagonalizable (has a complete set of eigenvectors) and has all real eigenvalues.
- *a symmetric hyperbolic system*, if the characteristic matrix is a Hermitian matrix.

Writing the system in a hyperbolic form is a quite useful step in proving that the system is *well-posed*. The mathematical well-posedness of the system means (1°) local existence (of at least one solution $u$), (2°) uniqueness (i.e., at most solutions), and (3°) stability (or continuous dependence of solutions $\{u\}$ on the Cauchy data) of the solutions. The resultant statement expresses the existence of the energy inequality on its norm,

$$||u(t)|| \leq e^{\alpha\tau}||u(t=0)||, \qquad \text{where } 0 < \tau < t, \quad \alpha = const. \tag{4.14}$$

This indicates that the norm of $u(t)$ is bounded by a certain function and the initial norm. Remark that this mathematical boundness does not mean that the norm $u(t)$ decreases along the time evolution.

The inclusion relation of the hyperbolicities is,

$$\text{symmetric hyperbolic} \subset \text{strongly hyperbolic} \subset \text{weakly hyperbolic}. \tag{4.15}$$

The Cauchy problem under weak hyperbolicity is not, in general, $C^\infty$ well-posed. At the strongly hyperbolic level, we can prove the finiteness of the energy norm if the characteristic matrix is independent of $u$ (cf [65]), that is one step definitely advanced over a weakly hyperbolic form. Similarly, the well-posedness of the symmetric hyperbolic is guaranteed if the characteristic matrix is independent of $u$, while if it depends on $u$ we have only limited proofs for the well-posedness.

From the point of numerical applications, to hyperbolize the evolution equations is quite attractive, not only for its mathematically well-posed features. The expected additional advantages are the following.

(a) It is well known that a certain flux conservative hyperbolic system is taken as an essential formulation in the computational Newtonian hydrodynamics when we control shock wave formations due to matter.

(b) The characteristic speed (eigenvalues of the principal matrix) is supposed to be the propagation speed of the information in that system. Therefore it is naturally imagined that these magnitudes are equivalent to the physical information speed of the model to be simulated.

(c) The existence of the characteristic speed of the system is expected to give us an improved treatment of the numerical boundary, and/or to give us a new well-defined Cauchy problem within a finite region (the so-called initial boundary value problem, IBVP).

These statements sound reasonable, but have not yet been generally confirmed in actual numerical simulations. But we are safe in saying that the formulations are not yet well developed to test these issues.

**Hyperbolic formulations of the Einstein equations** Most physical systems can be expressed as symmetric hyperbolic systems. In order to prove that the Einstein's theory is a well-posed system, to hyperbolize the Einstein equations is a long-standing research area in mathematical relativity.

The standard ADM system does not form a first order hyperbolic system. This can be seen immediately from the fact that the ADM evolution equation (2.13) has Ricci curvature in RHS. So far, several first order hyperbolic systems of the Einstein equation have been proposed. In constructing hyperbolic systems, the essential procedures are (1°) to introduce new variables, normally the spatially derivatived metric, (2°) to adjust equations using constraints. Occasionally, (3°) to restrict the gauge conditions, and/or (4°) to rescale some variables. Due to process (1°), the number of fundamental dynamical variables is always larger than that of ADM.

Due to the limitation of space, we can only list several hyperbolic systems of the Einstein equations.

- The Bona-Massó formulation [17, 18]
- The Einstein-Ricci system [25, 1] / Einstein-Bianchi system [8]
- The Einstein-Christoffel system [9]

- The Ashtekar formulation [11, 70]
- The Frittelli-Reula formulation [35, 65]
- The Conformal Field equations [30]
- The Bardeen-Buchman system [15]
- The Kidder-Scheel-Teukolsky (KST) formulation [42]
- The Alekseenko-Arnold system [7]
- The general-covariant Z4 system [16]
- The Nagy-Ortiz-Reula (NOR) formulation [47]
- The Weyl system [31, 29]

Note that there are no apparent differences between the word 'formulation' and 'system' here.

**Remarks** When we discuss hyperbolic systems in the context of numerical stability, the following questions should be considered:

> Q From the point of the set of evolution equations, does hyperbolization actually contribute to numerical accuracy and stability? Under what conditions/situations will the advantages of hyperbolic formulation be observed?

Unfortunately, we do not have conclusive answers to these questions, but many experiences are being accumulated. Several earlier numerical comparisons reported the stability of hyperbolic formulations [18, 19, 53, 54]. But we have to remember that this statement went against the standard ADM formulation, which has a constraint-violating mode for Schwarzschild spacetime as has been shown recently[59].

These partial numerical successes encouraged the community to formulate various hyperbolic systems. Recently, Calabrese et al [22] reported there is a certain differences in the long-term convergence features between weakly and strongly hyperbolic systems on the Minkowskii background space-time. However, several numerical experiments also indicate that this direction is not a complete success.

<u>Objections from numerical experiments</u>

- Above earlier numerical successes were also terminated with blow-ups.
- If the gauge functions are evolved according to the hyperbolic equations, then their finite propagation speeds may cause pathological shock formations in simulations [2, 5].
- There are no drastic differences in the evolution properties *between* hyperbolic systems (weakly, strongly and symmetric hyperbolicity) by systematic numerical studies by Hern [39] based on Frittelli-Reula formulation [35], and by the authors [58] based on Ashtekar's formulation [11, 70].
- Proposed symmetric hyperbolic systems were not always the best ones for numerical evolution. People are normally still required to reformulate them for suitable evolution. Such efforts are seen in the applications of the Einstein-Ricci system [54], the Einstein-Christoffel system [15], and so on.

Of course, these statements only casted on a particular formulation, and therefore we have to be careful not to over-emphasize the results. In order to figure out the reasons for the above objections, it is worth stating the following cautions:

<u>Remarks on hyperbolic formulations</u>

(a) Rigorous mathematical proofs of well-posedness of PDE are mostly for simple symmetric or strongly hyperbolic systems. If the matrix components or coefficients depend on dynamical variables (as in all any versions of hyperbolized Einstein equations), almost nothing was proved in more general situations.

(b) The statement of "stability" in the discussion of well-posedness refers to the bounded growth of the norm, and does not indicate a decay of the norm in time evolution.

(c) The discussion of hyperbolicity only uses the characteristic part of the evolution equations, and ignores the rest.

We think the origin of confusion in the community results from over-expectation on the above issues. Mostly, point (c) is the biggest problem. The above numerical claims from Ashtekar and Frittelli-Reula formulations were mostly due to the contribution (or interposition) of non-principal parts in evolution. Regarding this issue, the recent KST formulation finally opens the door. KST's "kinematic" parameters enable us to reduce the non-principal part, so that numerical experiments are hopefully expected to represent predicted evolution features from PDE theories. At this moment, the agreement between numerical behavior and theoretical prediction is not yet perfect but close [45].

If further studies reveal the direct correspondences between theories and numerical results, then the direction of hyperbolization will remain as the essential approach in numerical relativity, and the related IBVP researches will become a main research subject in the future. Meanwhile, it will be useful if we have an alternative procedure to predict stability including the effects of the non-principal parts of the equations. Our proposal of adjusted system in the next subsection may be one of them.

#### 4.2.4 Strategy 3: Asymptotically constrained systems

The third strategy is to construct a robust system against the violation of constraints, such that the constraint surface is an attractor. The idea was first proposed as "$\lambda$-system" by Brodbeck et al [20], and then developed in more general situations as "adjusted system" by the authors [71].

**The "$\lambda$-system"** Brodbeck et al [20] proposed a system which has additional variables $\lambda$ that obey artificial dissipative equations. The variable $\lambda$s are supposed to indicate the violation of constraints and the target of the system is to get $\lambda = 0$ as its attractor.

**The "$\lambda$-system" (Brodbeck-Frittelli-Hübner-Reula) [20]:** **Box 4.3**

For a symmetric hyperbolic system, add additional variables $\lambda$ and artificial force to reduce the violation of constraints.
The procedure:

| | | |
|---|---|---|
| 1. | Prepare a symmetric hyperbolic evolution system | $\partial_t u = M\partial_i u + N$ |
| 2. | Introduce $\lambda$ as an indicator of violation of constraint which obeys dissipative eqs. of motion | $\partial_t \lambda = \alpha C - \beta\lambda$ $(\alpha \neq 0, \beta > 0)$ |
| 3. | Take a set of $(u, \lambda)$ as dynamical variables | $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$ |
| 4. | Modify evolution eqs so as to form a symmetric hyperbolic system | $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$ |

Since the total system is designed to have symmetric hyperbolicity, the evolution is supposed to be unique. Brodbeck et al showed analytically that such a decay of $\lambda$s can be seen for sufficiently small $\lambda(>0)$ with a choice of appropriate combinations of $\alpha$s and $\beta$s.

Brodbeck et al presented a set of equations based on Frittelli-Reula's symmetric hyperbolic formulation [35]. The version of Ashtekar's variables was presented by the authors [57] for controlling the constraints or reality conditions or both. The numerical tests of both the Maxwell-$\lambda$-system and the Ashtekar-$\lambda$-system were performed [71], and confirmed to work as expected. Although it is questionable whether the recovered solution is true evolution or not [62], we think the idea is quite attractive. To enforce the decay of errors in its initial perturbative stage seems the key to the next improvements, which are also developed in the next section on "adjusted systems".

However, there is a high price to pay for constructing a $\lambda$-system. The $\lambda$-system can not be introduced generally, because (i) the construction of $\lambda$-system requires the original evolution equations to have a symmetric hyperbolic form, which is quite restrictive for the Einstein equations, (ii) the final system requires many additional variables and we also need to evaluate all the constraint equations at every time step, which is a hard task in computation. Moreover, (iii) it is not clear that the $\lambda$-system is robust enough for non-linear violation of constraints, or that $\lambda$-system can control constraints which do not have any spatial differential terms.

**The "adjusted system"** Next, we propose an alternative system which also tries to control the violation of constraint equations actively, which we named "adjusted system". We think that this system is more practical and robust than the previous $\lambda$-system.

**The Adjusted system (procedures):** **Box 4.4**

| | | |
|---|---|---|
| 1. | Prepare a set of evolution eqs. | $\partial_t u = J\partial_i u + K$ |
| 2. | Add constraints in RHS | $\partial_t u = J\partial_i u + K \underbrace{+\kappa C}$ |
| 3. | Choose the coeff. $\kappa$ so as to make the eigenvalues of the homogenized adjusted $\partial_t C$ eqs negative reals or pure imaginary. | $\partial_t C = D\partial_i C + EC$<br>$\partial_t C = D\partial_i C + EC \underbrace{+F\partial_i C + GC}$ |

The process of adjusting equations is a common technique in other re-formulating efforts as we reviewed. However, we try to employ the evaluation process of constraint amplification factors as an alternative guideline to hyperbolization of the system. We will explain these issues in the next section.

## 4.3 A unified treatment: Adjusted System

This section is devoted to present our idea of an "asymptotically constrained system." The original references can be found in Refs. [71], [72], [59], [73], [67] and [68].

### 4.3.1 Procedures: Constraint Propagation Equations and Proposals

Suppose we have a set of dynamical variables $u^a(x^i, t)$, and their evolution equations

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots), \tag{4.16}$$

and the (first class) constraints

$$C^\alpha(u^a, \partial_i u^a, \cdots) \approx 0. \tag{4.17}$$

Note that we do not require that Eq. (4.16) form a first-order hyperbolic form. We propose to investigate the evolution equation of $C^\alpha$ (constraint propagation),

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots), \tag{4.18}$$

for predicting the violation behavior of the constraints in time evolution. We do not mean to integrate Eq. (4.18) numerically together with the original evolution equations, Eq. (4.16), but mean to evaluate them analytically in advance in order to re-formulate Eq. (4.16).

There may be two major analyses of Eq. (4.18): (a) the hyperbolicity of Eq. (4.18) when Eq. (4.18) is a first-order system, and (b) the eigenvalue analysis of the whole RHS in Eq. (4.18) after a suitable homogenization. As we mentioned in Section 4.2.3, one of the problems in the hyperbolic analysis is that it only discusses the principal part of the system. Thus, we prefer to proceed down the road (b).

**Constraint Amplification Factors (CAFs):** **Box.4.5**

We propose to homogenize Eq. (4.18) by using a Fourier transformation, e.g.,

$$\partial_t \hat{C}^\alpha = \hat{g}(\hat{C}^\alpha) = M^\alpha{}_\beta \hat{C}^\beta, \quad \text{where } C(x,t)^\rho = \int \hat{C}(k,t)^\rho \exp(ik\cdot x) d^3k, \tag{4.19}$$

and then to analyze the set of eigenvalues, say $\Lambda$'s, of the coefficient matrix $M^\alpha{}_\beta$ in Eq. (4.19). We call the $\Lambda$'s the constraint amplification factors (CAFs) of Eq. (4.18).

The CAFs predict the evolutions of the constraint violations. We, therefore, can discuss the "distance" to the constraint surface by using the "norm" or "compactness" of the constraint violations (although we do not have exact definitions of these "$\cdots$" words).

The next conjecture seems to be quite useful to predict the evolution features of the constraints:

**Conjecture on CAFs** **Box.4.6**

(A) If CAF has a *negative real-part* (the constraints are forced to be diminished), then we see a more stable evolution than a system which has positive CAF.

(B) If CAF has a *non-zero imaginary-part* (the constraints are propagating away), then we see a more stable evolution than a system which has zero CAF.

We found that the system became more stable when more $\Lambda$'s satisfied the above criteria. (The first observations were in the Maxwell and Ashtekar formulations [58, 71].) Actually, supporting mathematical proofs are available when we classify the fate of the constraint propagations as follows.

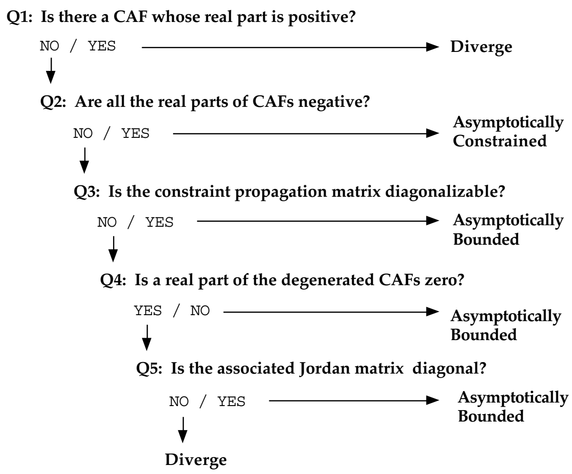


Figure 4.3: Flowchart to classify the constraint propagations.

**Classification of Constraint Propagations:** **Box.4.7**

If we assume that avoiding the divergence of the constraint norm is related to the numerical stability, the next classifications would be useful:

- *Asymptotically constrained:* All the constraints decay and converge to zero.
  This case can be obtained if and only if all the real parts of the CAFs are negative.
- *Asymptotically bounded:* All the constraints are bounded at a certain value. (This includes the above *asymptotically constrained* case.)
  This case can be obtained if and only if (a) all the real parts of CAFs are not positive and the constraint propagation matrix $M^\alpha{}_\beta$ is diagonalizable, or (b) all the real parts of the CAFs are not positive and the real part of the degenerated the CAFs is not zero.
- *Diverge:* At least one constraint will diverge.

The details are shown in Ref. [74].

A practical procedure for this classification is drawn in Fig. 4.3.

The above features of the constraint propagation, Eq. (4.18), will differ when we modify the original evolution equations. Suppose we add (adjust) the evolution equations by using the constraints

$$\partial_t u^a = f(u^a, \partial_i u^a, \cdots) + F(C^\alpha, \partial_i C^\alpha, \cdots); \tag{4.20}$$

then, Eq. (4.18) will also be modified as

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \cdots) + G(C^\alpha, \partial_i C^\alpha, \cdots). \tag{4.21}$$

Therefore, the problem is how to adjust the evolution equations so that their constraint propagations satisfy the above criteria as much as possible.

### 4.3.2 Applications 1: Adjusted ADM Formulations

Generally, we can write the adjustment terms to Eqs. (2.12) and (2.13) using Eqs. (2.14) and (2.15) with the following combinations (using up to the first derivatives of constraints for simplicity):

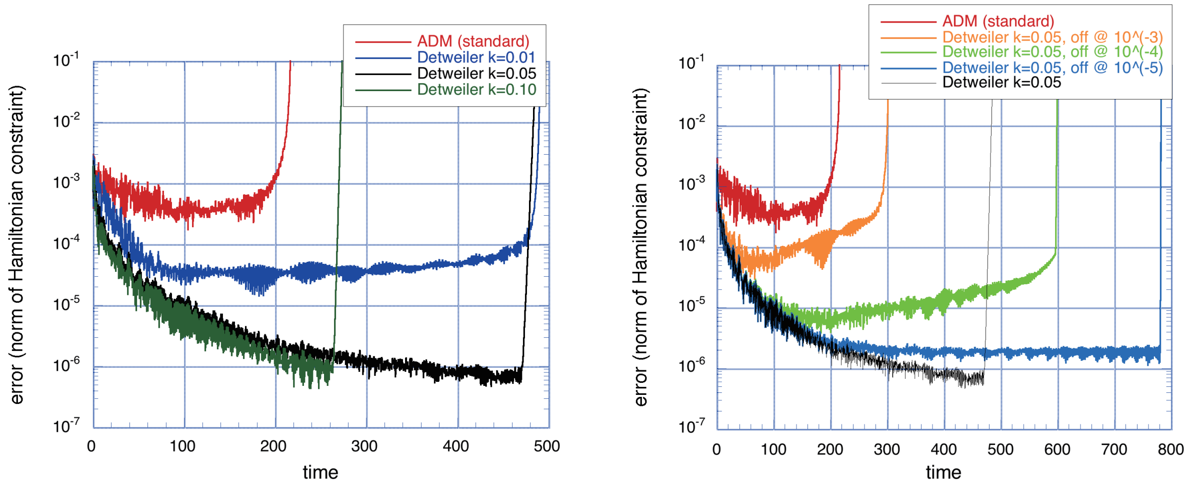


Figure 4.4: Demonstration of numerical evolutions between adjusted ADM systems: especially the standard ADM system and Detweiler's modified ADM system. The L2 norm of the constraints $\mathcal{H}^{ADM}$ is plotted as a function of time. The model is the propagation of a Teukolsky wave in a periodical 3-dimensional box. $k$ is the parameter in Detweiler's adjustment [$k_L$ in Eq.(4.27)-(4.30)], with fixed-$k$ cases (left panel) and with fixed-and-turnoff-$k$ cases (right panel). We see that the life-time of the simulation becomes four-times longer than that of the standard ADM by tuning the parameter $k$.

**The adjusted ADM formulation [59]:** **Box.4.8**

Modify the evolution equations $(\gamma_{ij}, K_{ij})$ by using constraints $\mathcal{H}$ and $\mathcal{M}_i$, i.e.,

$$\partial_t \gamma_{ij} = (2.12) \quad +P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k\mathcal{H}) + q^{kl}{}_{ij}(\nabla_k\mathcal{M}_l), \tag{4.22}$$

$$\partial_t K_{ij} = (2.13) \quad +R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k\mathcal{H}) + s^{kl}{}_{ij}(\nabla_k\mathcal{M}_l), \tag{4.23}$$

where $P, Q, R, S$ and $p, q, r, s$ are multipliers. According to this adjustment, the constraint propagation equations are also modified as

$$\partial_t \mathcal{H} = (2.16) + \text{additional terms}, \tag{4.24}$$

$$\partial_t \mathcal{M}_i = (2.17) + \text{additional terms}. \tag{4.25}$$

We show two examples of adjustments here. Several others are shown in Table 3 of Ref. [59].

1. *The standard ADM vs. original ADM*
   The first comparison is to show the differences between the standard ADM [75] and the original ADM system [10] (see Section 4.2.1). In the notation of Eqs. (4.22) and (4.23), the adjustment

$$R_{ij} = \kappa_F \alpha \gamma_{ij}, \tag{4.26}$$

   (and set the other multipliers zero) will distinguish the two, where $\kappa_F$ is a constant. Here $\kappa_F = 0$ corresponds to the standard ADM (no adjustment), and $\kappa_F = -1/4$ corresponds to the original ADM (without any adjustment to the canonical formulation by ADM). As one can check by using Eqs. (4.24) and (4.25), adding the $R_{ij}$ term keeps the constraint propagation in a first-order form. Frittelli [33] (see also Ref. [72]) pointed out that the hyperbolicity of the constraint propagation equations is better in the standard ADM system. This stability feature is also confirmed numerically, and we set our CAF conjecture so as to satisfy this difference.

2. *Detweiler type*
Detweiler [26] found that with a particular combination, the evolution of the energy norm of the constraints, $\mathcal{H}^2 + \mathcal{M}^2$, can be negative definite when we apply the maximal slicing condition, $K = 0$. His adjustment can be written in our notation in Eqs. (4.22) and (4.23) as

$$P_{ij} = -\kappa_L \alpha^3 \gamma_{ij}, \tag{4.27}$$

$$R_{ij} = \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}), \tag{4.28}$$

$$S^k{}_{ij} = \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], \tag{4.29}$$

$$s^{kl}{}_{ij} = \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}], \tag{4.30}$$

and everything else is zero, where $\kappa_L$ is a multiplier. Detweiler's adjustment, Eqs. (4.27)-(4.30), does not put a constraint propagation equation to a first-order form, so we cannot discuss hyperbolicity or the characteristic speed of the constraints. From a perturbation analysis on the Minkowskii and Schwarzschild space-time, we confirmed that Detweiler's system provides better accuracy than the standard ADM, but only for small positive $\kappa_L$.

We made various predictions how additional adjusted terms will change the constraint propagation [72, 59]. In that process, we applied the CAF analysis for Schwarzschild spacetime and found when and where the negative real or non-zero imaginary eigenvalues of the homogenized constraint propagation matrix appear and how they depend on the choice of coordinate system and adjustments. We found that there *was* a constraint-violating mode near the horizon for the standard ADM formulation and that the constraint-violating mode could be suppressed by adjusting equations and by choosing an appropriate gauge conditions.

**Numerical demonstrations and remarks** Systematic numerical comparisons are in progress, and we show two sample plots here. Fig. 4.4 is the case of a Teukolsky wave [66] propagating under a 3-dimensional periodic boundary condition. Both the standard ADM system and the Detweiler system [one of the adjusted ADM systems with adjustments Eqs. (4.27)-(4.30)] are compared with the same numerical parameters. Plots are the L2 norm of the Hamiltonian constraint $\mathcal{H}^{ADM}$, i.e., the violation of constraints, and we see the life-time of the standard ADM evolution ends at $t = 200$. However, if we chose a particular value of $\kappa_L$ [multiplier in Eqs. (4.27)-(4.30)], we observe that violation of constraints is reduced compared to the standard ADM case and that the simulation can continue longer than that (left panel). If we further tuned $\kappa_L$, say turn-off to $\kappa_L = 0$ when the total L2 norm of $\mathcal{H}^{ADM}$ is small, then we can see that the constraint violation is somewhat maintained at a small level, and a more long-term stable simulation is available (right panel).

During the comparisons of adjustments, we found that it is necessary to create a time asymmetric structure of the evolution equations in order to force the evolution onto the constraint surface. There are an infinite number of ways to adjust the equations, but we found that if we followed the next guideline, then such an adjustment would give us a time-asymmetric evolution.

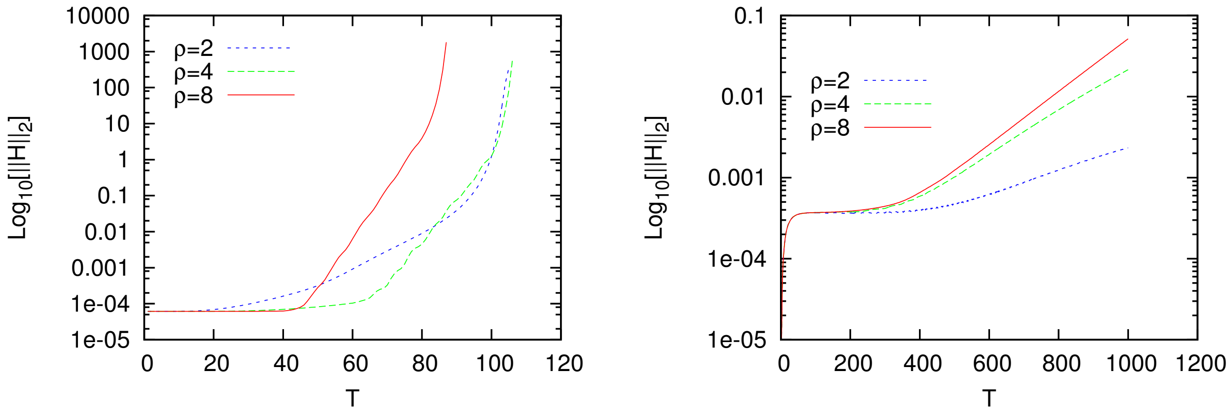


Figure 4.5: One-dimensional gauge-wave test with the BSSN system (left) and the adjusted BSSN system (right) in the $\tilde{A}$-equation, Eq. (4.31). The L2 norm of $\mathcal{H}$, rescaled by the resolution parameter $\rho^2/4$, is plotted as a function of the crossing-time. The wave amplitude is set to 0.01, and we choose the adjustment parameter $\kappa_A = 0.005$. The BSSN system loses convergence at an early time, near the 20 crossing-time, and it will produce blow-ups of the calculation in the end, while in the adjusted version we see that the higher resolution runs show longer convergence, i.e., the 300 crossing-time in $\mathcal{H}$, and that all runs can stably evolve up to the 1000 crossing-time.

**Trick to obtain asymptotically constrained system**: *Box*.4.9

= *Break the time reversal symmetry (TRS) of the evolution equation.*

1. Evaluate the parity of the evolution equation.
   By reversing the time ($\partial_t \to -\partial_t$), there are variables that change their signatures (parity $(-)$) [e.g., $K_{ij}, \partial_t\gamma_{ij}, \mathcal{M}_i, \cdots$], while not (parity $(+)$) [e.g., $g_{ij}, \partial_t K_{ij}, \mathcal{H}, \cdots$].

2. Add adjustments that have different parities of that equation.
   For example, for the parity $(-)$ equation $\partial_t\gamma_{ij}$, add a parity $(+)$ adjustment $\kappa\mathcal{H}$.

One of our criteria, the negative real CAFs, requires breaking the time-symmetric features of the original evolution equations. Such CAFs are obtained by adjusting the terms that break the TRS of the evolution equations, and this is available even for the standard ADM system.

### 4.3.3 Applications 2: Adjusted BSSN formulations

**Constraint propagation analysis of the BSSN equations** In order to understand the stability property of the BSSN system, we studied the structure of the evolution equations, Eqs. (4.3)-(4.7), in detail, especially how the modifications using the constraints, Eqs. (4.8)-(4.12), affect the stability [73]. We investigated the signature of the eigenvalues of the constraint propagation equations and showed that the standard BSSN dynamical equations were balanced from the viewpoint of constrained propagations, including a clarification of the effect of the replacement by using the momentum constraint equation, which was reported by Alcubierre *et al.* [3].

Moreover, we predicted that several combinations of modifications had a constraint-damping nature and named them the *adjusted BSSN systems*. Several adjusted BSSN systems are proposed in Table II of Ref. [73].

Yo *et al.* [69] immediately applied one of our proposals to their simulations of a stationary rotating black hole and reported that one adjustment contributed to maintaining their evolution of the Kerr black hole ($J/M$ up to $0.9M$) for a long time ($t \sim 6000M$). Their results also indicate that the evolved solution is closer to the exact one, that is, the constrained surface.

Now, let us make clear some current technical tips listed in Section 4.2.2 by using a constraint propagation analysis.

tip-1 The trace-out $A_{ij}$ technique can be explained that the violation of the $\mathcal{A}$-constraint, Eq. (4.11), affects all other constraint violations. (See the full set of constraint propagation equations in the Appendix of Ref. [73].)

tip-2 The replacement of $\tilde{\Gamma}^i$ enables to maintain the $\mathcal{G}$-constraint, Eq. (4.10), that delays the violation of $\mathcal{H}^{BSSN}$ and $\mathcal{M}_i^{BSSN}$. (Again, the statement comes from the full set of constraint propagation equations. )

**Numerical demonstrations** We recently presented our numerical comparisons of the three kinds of adjusted BSSN formulation[43]. We performed the three testbeds: gauge-wave, linear wave, and Gowdy-wave tests, proposed by the Mexico workshop [6] on the formulation problem of the Einstein equations. We observed that the signature of the proposed Lagrange multipliers were always right and that the adjustments improved the convergence and the stability of the simulations. When the original BSSN system already shows satisfactory good evolutions (e.g., linear wave test), the adjusted versions also coincide with those evolutions while in some cases (e.g., gauge-wave or Gowdy-wave tests), the simulations using the adjusted systems last 10 times as long as those using the original BSSN equations.

Fig. 4.5 show a comparison between the (plain) BSSN system and the adjusted BSSN system in the $\tilde{A}$-equation by using the momentum constraint

$$\partial_t \tilde{A}_{ij} = \partial_t^B \tilde{A}_{ij} + \kappa_A \alpha \tilde{D}_{(i} \mathcal{M}_{j)}, \tag{4.31}$$

where $\kappa_{\mathcal{A}}$ is predicted (from the eigenvalue analysis) to be positive in order to damp the constraint violations. The testbed is a one-dimensional gauge-wave, the trivial Minkowski space-time, but sliced with the time-dependent 3-metric. The poor performance of the plain BSSN system for this test has been already reported [41], and one remedy is to apply a 4th-order finite differencing scheme [76]. The plots show that our adjusted system also improved the life-time of the plain BSSN simulation by at least 10 times with better convergence.

### 4.3.4 Applications 3: $C^2$-adjusted formulations

The above applications to ADM and BSSN equations are somewhat straightforward, which are all-inclusive but not a far-sighted. Here, we further specify the adjusted terms from another idea.

Fiske[28] proposed an adjustment which uses the norm of constraints, $C^2$, and does not require the background metric for specifying effective Lagrange multipliers. An advantage of his method is what the stability of the numerical simulation can be expected without depending on background metric. We apply his method to the ADM and BSSN formulations, and showed that this adjustment actually performs constraint-dumping by numerical simulations.

**$C^2$-adjusted Systems** For variables $u^i$ and constraint values $C^i$, evolution equations with constraint equations are generally written as

$$\partial_t u^i = f(u^i, \partial_j u^i, \cdots), \quad \text{and} \quad C^i(u^i, \partial_j u^i. \cdots) \approx 0. \tag{4.32}$$

Suppose we adjust $\partial_t u^i$-equation with $C^2 \equiv C^i C_i$, and evaluate constraint propagation as

$$\partial_t C^2 = \frac{\delta C^2}{\delta u^i}(\partial_t u^i). \tag{4.33}$$

There exists various combinations of this adjustment. Fiske[28] proposed an adjusted term as

$$\partial_t u^i = [\textbf{Original} \quad \textbf{Terms}] - \kappa^{ij}\frac{\delta C^2}{\delta u^j}, \tag{4.34}$$

with $\kappa^{ij}$ of positive definite. The constraint propagation, then, becomes

$$\partial_t C^2 = [\textbf{Original} \quad \textbf{Terms}] - \kappa^{ij}\frac{\delta C^2}{\delta u^i}\frac{\delta C^2}{\delta u^j}, \tag{4.35}$$

which clearly shows the dumping of constraints. If we set $\kappa^{ij}$ so that the second term becomes more dominant of (4.35) than the first term in evolution, then $C^2$ dumps because of $\partial_t C^2 < 0$. Fiske presented an numerical example in the Maxwell system.

**Application to the ADM equations** Now we apply Fiske's method to the ADM formulation, which can be written as

$$\begin{aligned}
\partial_t \gamma_{ij} &= [\textbf{Original} \quad \textbf{Terms}] - \kappa_{\gamma ijmn}\frac{\delta (C^A)^2}{\delta \gamma_{mn}}, &(4.36)\\
\partial_t K_{ij} &= [\textbf{Original} \quad \textbf{Terms}] - \kappa_{Kijmn}\frac{\delta (C^A)^2}{\delta K_{mn}}, &(4.37)
\end{aligned}$$

where $(C^A)^2$ is the norm of the constraints,

$$(C^A)^2 \equiv (\mathcal{H}^A)^2 + (\mathcal{M}^A)^i(\mathcal{M}^A)_i, \tag{4.38}$$

and both of $\kappa_{\gamma ijmn}, \kappa_{Kijmn}$ are positive definite.

For the modified ADM equations, (4.36)-(4.37), we confirm that this system has better stablility than the standard ADM system by the CAFs analysis. We find that all the real parts of eigenvalues are negative, when we set $\kappa_{\gamma ijmn} = \kappa_{Kijmn} = \delta_{im}\delta_{jn}$ and the background metric to Minkowski metric. Therefore the system is expected to dump the violation of constraints.

**Application to the BSSN equations** For the BSSN formulation, evolution equations with Fiske-type adjustment are:

$$\begin{aligned}
\partial_t \varphi &= [\textbf{Original} \quad \textbf{Terms}] - \lambda_{\varphi}\frac{\delta (C^B)^2}{\delta \varphi}, &(4.39)\\
\partial_t K &= [\textbf{Original} \quad \textbf{Terms}] - \lambda_{K}\frac{\delta (C^B)^2}{\delta K}, &(4.40)\\
\partial_t \tilde{\gamma}_{ij} &= [\textbf{Original} \quad \textbf{Terms}] - \lambda_{\tilde{\gamma} ijmn}\frac{\delta (C^B)^2}{\delta \tilde{\gamma}_{mn}}, &(4.41)\\
\partial_t \tilde{A}_{ij} &= [\textbf{Original} \quad \textbf{Terms}] - \lambda_{\tilde{A} ijmn}\frac{\delta (C^B)^2}{\delta \tilde{A}_{mn}}, &(4.42)\\
\partial_t \tilde{\Gamma}^i &= [\textbf{Original} \quad \textbf{Terms}] - \lambda^{ij}_{\tilde{\Gamma}}\frac{\delta (C^B)^2}{\delta \tilde{\Gamma}^j}, &(4.43)
\end{aligned}$$

where

$$\begin{aligned}
(C^B)^2 &\equiv (\mathcal{H}^B)^2 + (\mathcal{M}^B)^i(\mathcal{M}^B)_i + \mathcal{A}^2 + \mathcal{G}^i\mathcal{G}_i + \mathcal{S}^2, &(4.44)\\
\mathcal{A} &\equiv \tilde{\gamma}^{ij}\tilde{A}_{ij}, \quad \mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\Gamma}^i{}_{mn}\tilde{\gamma}^{mn}, \quad \mathcal{S} \equiv -1 + \det(\tilde{\gamma}_{ij}),
\end{aligned}$$

and all of $\lambda_{\varphi}, \lambda_K, \lambda_{\tilde{\gamma} ijmn}, \lambda_{\tilde{A} ijmn}$ and $\lambda^{ij}_{\tilde{\Gamma}}$ are positive definite. We find again that all the CAFs suggest the dumping feature of the constraint violations.

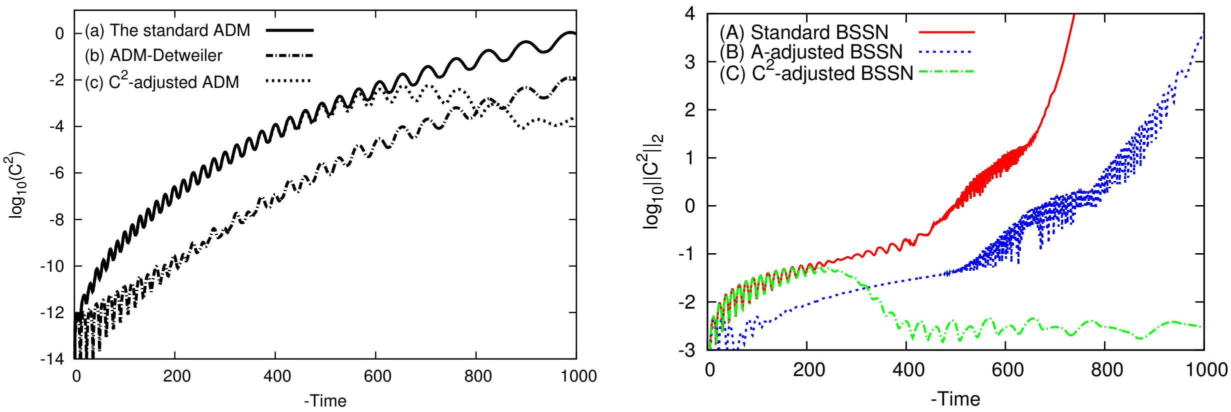


Figure 4.6: The L2 norm of the constraints, $C^2$, of the polarized Gowdy-wave tests with (Left) ADM and two adjusted formulations, and (Right) BSSN and two adjusted formulations. The vertical axis is the logarithm of the $C^2$ and the horizontal axis is backward time.
(Left) (a) The standard ADM formulation, (b) Detweiler's ADM with $L = -10^{+1.9}$, and (c) the $C^2$-adjusted ADM system. with $\kappa_\gamma = -10^{-9.0}$ and $\kappa_K = -10^{-3.5}$. We see the lines (a) and (c) almost overlap until $t = -500$, then the case (c) keeps the L2 norm at the level $\leq 10^{-3}$, while the lines of (a) and (b) monotonically grow larger with oscillations. We confirmed this behavior up to $t \simeq -1700$. From Fig.2 in [67].
(Right) (A) The standard BSSN formulation, (B) $\widetilde{A}$-adjusted BSSN formulation with $\kappa_A = -10^{-0.2}$, and (C) the $C^2$-adjusted BSSN formulation. with $\lambda_\varphi = -10^{-10}$, $\lambda_K = -10^{-4.6}$, $\lambda_{\widetilde{\gamma}} = -10^{-11}$, $\lambda_{\widetilde{A}} = -10^{-1.2}$, and $\lambda_{\widetilde{\Gamma}} = -10^{-14.3}$. We see that lines (A) and (C) are identical until $t = -200$. Line (C) then decreases and maintains its magnitude under $O(10^{-2})$ after $t = -400$. We confirm this behavior until $t = -1500$. From Fig.8 in [68].

**Numerical Examples** We demonstrate numerical simulations of above systems with polarized Gowdy wave:

$$ds^2 = t^{-1/2}\mathrm{e}^{\lambda/2}(-dt^2 + dx^2) + t(\mathrm{e}^P dy^2 + \mathrm{e}^{-P} dz^2). \tag{4.45}$$

which is one of the so-called Apples-with-Apples tests [6], setting all of the numerical parameters to the same. (Fig. 4.6)

## 4.4 Outlook

**What we have achieved** We reviewed recent efforts to the *formulation problem* of numerical relativity, the problem to find a robust system against constraint violations. We categorized the approaches into

(0) The standard ADM formulation (Section 4.2.1),

(1) The BSSN formulation (Section 4.2.2),

(2) Hyperbolic formulations (Section 4.2.3), and

(3) Asymptotically constrained formulations (Section 4.2.4).

Most numerical relativity groups now use the BSSN set of equations, which are obtained empirically. A dramatic announcement of the success of binary black-hole simulations has caused the community to follow that recipe. Actually, we do not yet completely understand why the current set of BSSN equations, together with particular combinations of gauge conditions, works well. Several explanations are applied based on the hyperbolic formulation scheme, but as we viewed, they are not yet satisfactory.

Our approach, on the other hand, tries to construct an evolution system that has its constraint surface as an attractor. Our unified view is to understand the evolution system by evaluating its constraint propagation. Especially, we propose to analyze the constraint amplification factors that are the eigenvalues of the homogenized constraint propagation equations. We analyzed the system based on our conjecture whether the constraint amplification factors suggest a constraint to decay/propagate or not. We concluded that

- The constraint propagation features become different by simply adding constraint terms to the original evolution equations (we call this an *adjustment* of the evolution equations).
- There *is* a constraint-violating mode in the standard ADM evolution system when we apply it to a single non-rotating black hole space-time, and its growth rate is larger near the black-hole horizon.
- Such a constraint-violating mode can be killed if we adjust the evolution equations with a particular modification using constraint terms. An effective guideline is to adjust terms as they break the time-reversal symmetry of the equations.
- Our expectations are borne out in simple numerical experiments using the Maxwell, the Ashtekar, and the ADM systems. However, the modifications are not yet perfect to prevent non-linear growth of the constraint violation.
- We understand why the BSSN formulation works better than the ADM one in a limited case (perturbation analysis in the flat background); further, we propose modified evolution equations along the lines of our previous procedure. Some of these proposed adjusted systems are numerically confirmed to work better than the standard BSSN system.

The common key to the problem is how to adjust the evolution equations with constraints. Any adjusted systems are mathematically equivalent if the constraints are completely satisfied, but this is not the case for numerical simulations. Replacing terms with constraints is one of the normal steps when people re-formulate equations in a hyperbolic form.

In summary, let me answer the following three questions:

- What is the guiding principle for selecting the evolution equations for simulations in GR?
  –The key is to analyze the constraint propagation equation of the system.
- Why do many groups use the BSSN equations?
  –Because people just rush, not to be behind others.
- Is there an alternative formulation better than the BSSN?
  –Yes, there is, but we do not know which is the best one yet.

**Future directions** If we say the final goal of this project is to find a robust evolution system against violation of constraints, then the recipe should be a combination of (a) formulations of the evolution equations, (b) choice of gauge conditions, (c) treatment of boundary conditions, and (d) numerical integration methods. We are now in the stages of solving these mixed puzzles. Recent attention to higher dimensional space-time studies is waiting for numerical research, but it is known that the formulation problem also exists in higher-dimensional cases [61].

We have written this review from the viewpoint that general relativity is a constrained dynamical system. This is not a proper problem in general relativity, but it is in many physical systems, such as electrodynamics, magnetohydrodynamics, molecular dynamics, and mechanical dynamics. Therefore, sharing and discussing thoughts between different fields will definitely accelerate the progress. The ideal algorithm to solve all the problems may not exist, but the author believes that our final numerical recipe is somewhat an *automatic* system and hopes that numerical relativity turns to be an easy *toolkit* for everyone in the near future.

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# A 高次元時空における特異点形成

最後に具体的な数値シミュレーションの例として，現在大学院生と進めている高次元時空の数値解析の話と，以前のワームホールの話を紹介します．

**Black Objects and Hoop Conjecture in Five-dimensional Space-time[1]**
We numerically investigated the sequences of initial data of thin spindle and thin ring in five-dimensional space-time in the context of the cosmic censorship conjecture. We modeled the matter in non-rotating homogeneous spheroidal or toroidal configurations under the momentarily static assumption, solved the Hamiltonian constraint equation, and searched the apparent horizon. We found both $S^3$ (black hole) and $S^2 \times S^1$ (black ring) horizons (”black objects”), only when the matter configuration is not sharp. By monitoring the location of the maximum Kretchmann invariant, an appearance of ‘naked singularity’ or ‘naked ring’ under the special situations is suggested. We also discuss the validity of the ”hyper-hoop” conjecture using minimum ”area” around the object, and show that the appearance of the ring horizon does not match with this hoop.

**Formation of naked singularities in five-dimensional space-time[2]**
We numerically investigate the gravitational collapse of collisionless particles in spheroidal configurations both in four and five-dimensional (5D) space-time. We repeat the simulation performed by Shapiro and Teukolsky (1991) that announced an appearance of a naked singularity, and also find that the similar results in 5D version. That is, in a collapse of a highly prolate spindle, the Kretschmann invariant blows up outside the matter and no apparent horizon forms. We also find that the collapses in 5D proceed rapidly than in 4D, and the critical prolateness for appearance of apparent horizon in 5D is loosened compared to 4D cases. We also show how collapses differ with spatial symmetries comparing 5D evolutions in single-axisymmetry, SO(3), and those in double-axisymmetry, $U(1)\times U(1)$.

**Fate of the first traversible wormhole: black-hole collapse or inflationary expansion[3]**
We study numerically the stability of Morris amp; Thorne’s first traversible wormhole, shown previously by Ellis to be a solution for a massless ghost Klein-Gordon field. Our code uses a dual-null formulation for spherically symmetric space-time integration, and the numerical range covers both universes connected by the wormhole. We observe that the wormhole is unstable against Gaussian pulses in either exotic or normal massless Klein-Gordon fields. The wormhole throat suffers a bifurcation of horizons and either explodes to form an inflationary universe or collapses to a black hole, if the total input energy is respectively negative or positive. As the perturbations become small in total energy, there is evidence for critical solutions with a certain black-hole mass or Hubble constant. The collapse time is related to the initial energy with an apparently universal critical exponent. For normal matter, such as a traveller traversing the wormhole, collapse to a black hole always results. However, carefully balanced additional ghost radiation can maintain the wormhole for a limited time. The black-hole formation from a traversible wormhole confirms the recently proposed duality between them. The inflationary case provides a mechanism for inflating, to macroscopic size, a Planck-sized wormhole formed in space-time foam.

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# B Unsolved Problems

真貝は，2009 年 7 月の「天文天体若手夏の学校」で，若手に対して何か熱いメッセージを，と頼まれ，『相対論のテーマ探し』という題で話をしました．結論としては,『そんな方法論があったら，私が知りたい．答えがあるなら，誰にも教えず，せっせと論文を書く.』というオチで終わるのですが，その中で紹介した Penrose による「未解決問題のリスト」を最後に紹介します.

**"Some Unsolved Problem in Classical GR" by Penrose (1982)**

R. Penrose, in *Seminar on Differential Geometry* (Princeton U. Press, 1982)

1. Find a suitable quasi-local definition of energy-momentum in GR.
2. Find a suitable quasi-local definition of angular-momentum in GR.
3. Find an asymptotically simple Ricci-flat space-time which is not flat – or at least prove that such space-times exist.
4. Are there restrictions on k for non-stationary k-asymptotically simple space-times, with non-zero mass, which are vacuum near $\mathcal{I}$?
5. Find conditions on the Ricci tensor $R_{ab}$ throughout $M$ which ensure that the generators of $\mathcal{I}$ are infinitely long.
6. Show that if a cut $C$ of $\mathcal{I}^+$ [or $\mathcal{I}^-$] can be spanned by a spacelike hypersurface along which an appropriate energy condition holds, then the Bondi-Sachs mass defined at $C$ is non-negative.
7. Does the Bondi-Sachs mass defined on cuts of $\mathcal{I}^+$ have a well-defined limit as the cuts recede into the past along $\mathcal{I}^+$, this limit agreeing with the mass defined at spacelike infinity?
8. Show that if the dominant energy condition holds, then the Bondi-Sachs energy-momentum, and also the energy-momentum defined at spacelike infinity, are future-timelike, the space-time being assumed not to be flat everywhere in the region of an appropriate spacelike hypersurface.
9. In an asymptotically simple space-time which is vacuum near $\mathcal{I}$ and for which outgoing radiation is present and falls off suitably near $i^0$ and $i^+$, is it necessarily the case that $i^0$ and $i^+$ are non-trivially related? (At least, are there some examples $i$ which $i^0$ and $i^+$ are non-trivially related?)
10. Find a good definition of angular momentum for asymptotically simple space-times.
11. If there is no incoming radiation and no outgoing radiation and the space-time $M$ is vacuum near $\mathcal{I}$ and (in some suitable sense) near $i^0$, is $M$ necessarily stationary near $\mathcal{I}$ ?
12. Is Cosmic Censorship a valid principle in classical GR?
13. Let $S$ be a spacelike hypersurface in $M$ which is compact with boundary, the boundary consisting of a cut $C$ of $\mathcal{I}^+$ together with a trapped surface $T$. Let $m$ be the Bondi-Sacks mass evaluated at $C$ and let $A$ be the area of $T$. Show that

$$A \leq 16\pi m^2$$

provided that the dominant energy condition holds throughout some neighbourhood of $S$. Show that there is no vacuum equilibrium configuration involving more than one black hole.