一般相対論の数値計算手法

真貝寿明　Hisa-aki Shinkai

大阪工業大学情報科学部　　shinkai@is.oit.ac.jp

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1 Introduction

1.1 一般相対性理論の概略と主要な研究テーマ (Topics in GR)

一般相対論研究者向けに，レビュー論文を更新しているサイト「Living Reviews in Relativity」[1]がある．そのサイトに掲載された論文のテーマ一覧．–published –upcoming の順．2011/12/1 現在．

- 重力波 (Gravitational Waves) 12 本+12
  - The Motion of Point Particles in Curved Spacetime; GW Detection by Interferometry (Ground and Space); GWs from Gravitational Collapse; Interferometer Techniques for GW Detection; On Special Optical Modes and Thermal Issues in Advanced GW Interferometric Detectors; Physics, Astrophysics and Cosmology with GW; The PN Approximation for Relativistic Compact Binaries; Gravitational Radiation from PN Sources and Inspiralling Compact Binaries; Low-Frequency GW Searches Using Spacecraft Doppler Tracking; Time-Delay Interferometry; GW Data Analysis. Formalism and Sample Applications: The Gaussian Case; Analytic BH Perturbation Approach to Gravitational Radiation
  - Advanced Technologies for Space GW Detectors; Extreme and Intermediate Mass-Ratio Inspiral Systems; GW Phenomenology; GW Sources: Binaries (High and Low Frequency); GW Sources: Cosmological Background; GWs from Extreme Mass Ratio Inspiral (EMRI); Interface Between GWs and Astronomy; Pulsar Timing and Low Frequency GW Detection; Quantum Measurement Theory in GW Detection; Rates for Binary Coalescences; The ADM canonical approach to the PN motion of compact binaries; The Square-Kilometre-Array (SKA)
- 数値シミュレーション (Numerical Relativity) 10 本+9
  - Coalescence of BH-Neutron Star Binaries; Characteristic Evolution and Matching; Spectral Methods for NR; Numerical Hydrodynamics and Magnetohydrodynamics in GR; Critical Phenomena in Gravitational Collapse; Event and Apparent Horizon Finders for 3+1 NR; Numerical Hydrodynamics in Special Relativity; Numerical Approaches to Spacetime Singularities; Computational Cosmology: From the Early Universe to the Large Scale Structure; Initial Data for NR
  - Algebraic Computing in GR; Binary Neutron Star Mergers; Boson Stars; Formulations of Einstein's Equations for NR; Interface of PN Theories and NR; Methods of GW Extraction in NR; NR for BHs; Numerical Simulations of Supernovae; Perturbative Interface to the Binary BH Problem
- 数学的な側面 (Mathematical Relativity) 11 本+5
  - The Einstein-Vlasov System/Kinetic Theory; Cosmic Censorship for Gowdy Spacetimes; Null Geodesic Congruences, Asymptotically-Flat Spacetimes and Their Physical Interpretation; Quasi-Local Energy-Momentum and Angular Momentum in GR; Theorems on Existence and Global Dynamics for the Einstein Equations; Isolated and Dynamical Horizons and Their Applications; Gravitational Lensing from a Spacetime Perspective; Conformal Infinity; Speeds of Propagation in Classical and Relativistic Extended Thermodynamics; Stationary BHs: Uniqueness and Beyond; Hyperbolic Methods for Einstein's Equations
  - Continuum and Discrete Initial-Boundary-Value Problems and Einstein's Field Equations; Cosmic Censorship (toolbox); Exact Solutions; Gravitational Lensing from a Spacetime Perspective; The Constraint Problem for Einstein's Equations
- 量子重力 (Quantum General Relativity) 11 本+4
  - Entanglement Entropy of BHs; Quantization of Midisuperspace Models; Loop QG; Loop Quantum Cosmology; Stochastic Gravity: Theory and Applications; The Asymptotic Safety Scenario in QG; QG in 2+1 Dimensions: The Case of a Closed Universe; QG in Everyday Life: GR as an Effective Field Theory; Perturbative QG and its Relation to Gauge Theory; The Thermodynamics of BHs; Discrete Approaches to QG in Four Dimensions

[1] http://relativity.livingreviews.org/

  - Causal Sets; Minimal Length Scale Scenarios for QG; QG Phenomenology; The Spin Foam Approach to QG

- 実験的検証 (Experimental Foundations of Gravitation) 10 本+5
  - Analogue Gravity; Varying Constants, Gravitation and Cosmology; Tests of Gravity Using Lunar Laser Ranging; The Pioneer Anomaly; f(R) Theories; Probes and Tests of Strong-Field Gravity with Observations in the Electromagnetic Spectrum; The Confrontation between GR and Experiment; Modern Tests of Lorentz Invariance; Testing GR with Pulsar Timing; Relativity in the Global Positioning System
  - Experiments in Gravitation with Highly Stable Clocks; Laboratory Measurements of Newtons's Constant, G; MOND; Testing Gravity Using GWs; Tests of Gravity at Short Range
- 宇宙物理現象 (Relativity in Astrophysics) 9 本+5
  - Physics of Neutron Star Crusts; Binary and Millisecond Pulsars; Relativistic Fluid Dynamics: Physics for Many Different Scales; The Evolution of Compact Binary Star Systems; Relativistic Binaries in Globular Clusters; Massive BH Binary Evolution; Rotating Stars in Relativity; Quasi-Normal Modes of Stars and BHs; Gravitational Lensing in Astronomy
  - BH Accretion Disks; Electromagnetic Counterparts to Supermassive BH Mergers; Massive BHs in Galaxies; Microquasars; The Magnetic Fields of Neutron Stars
- 弦理論 (String Theory and Gravitation) 4 本+3
  - Brane-World Gravity; BHs in Higher Dimensions; Spacelike Singularities and Hidden Symmetries of Gravity; Spinning Strings and Integrable Spin Chains in the AdS/CFT Correspondence
  - Brane Actions and Kappa-Symmetry; Classification of Near-Horizon Geometries of Extremal BHs; Solitonic Solutions of Supergravity
- 宇宙論 (Physical Cosmology) 5 本+2
  - The Hubble Constant; Measuring our Universe from Galaxy Redshift Surveys; Experimental Searches for Dark Matter; The Cosmological Constant; The Cosmic Microwave Background
  - Cosmic Evolution of Super Massive BHs in Galactic Centers (the X-Ray view); The Age of the Universe
- 科学史 (History of Relativity) 2 本+3
  - History of Astroparticle Physics and its Components; On the History of Unified Field Theories
  - History of GW Research; On the History of Unified Field Theories (1933-ca 1960); The Hole Argument

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上記の論文タイトルで使用した略語は以下のもの.

BH	Black Hole
GR	General Relativity
GW	Gravitational Wave
NR	Numerical Relativity
PN	Post-Newtonian
QG	Quantum Gravity

1.2 なぜ数値相対論? (Why Numerical Relativity?)

The Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \qquad \kappa = 8\pi G \tag{1.1}$$

What are the difficulties? (# 1)

- for 10-component metric, highly nonlinear partial differential equations.
- completely free to choose coordinates, gauge conditions, and even for decomposition of the space-time.
- mixed with 4 elliptic eqs and 6 dynamical eqs if we apply 3+1 decomposition.
- has singularity in its nature.

How to solve it?

- find exact solutions
  - assume symmetry in space-time, and decomposition of space-time
    spherically symmetric, cylindrical symmetric, ...
  - assume simple situation and matter
    time-dependency, homogeneity, algebraic speciality, ...

  We know many exact solutions ($O(100)$) by this "Spherical Cow" approach.
- approximations
  - weak-field limit, linearization, perturbation, ...

  We know correct prediction in the solar-system, binary neutron stars, ...
  We know post-Newtonian behavior, first-order correction, BH stability, ...

Why don't we solve it using computers?

- dynamical behavior
- strong gravitational field
- no symmetry in space
- gravitational wave!
- higher-dimensional theories, and/or other gravitational theories, ...

The most robust way to study the strong gravitational field. Great.

Numerical Relativity **Box 1.1**

= Solve the Einstein equations numerically.
= Necessary for unveiling the nature of strong gravity. For example:

- gravitational waves from colliding black holes, neutron stars, supernovae, ...
- relativistic phenomena like cosmology, active galactic nuclei, ...
- mathematical feedback to singularity, exact solutions, chaotic behavior, ...
- laboratory for gravitational theories, higher-dimensional models, ...

What are the difficulties? (# 2)

- How to construct a realistic initial data?
- How to treat black-hole singularity?
- We cannot evolve the system stably in long-term evolution. Why?

General and recent introductions
More general and recent introductions to numerical relativity are available, e.g. by Pretorius (2007) [4], Alcubierre (2008) [1], Baumgarte-Shapiro (2010) [2], and Gourgoulhon (2012) [3].

References

[1] M. Alcubierre, *Introduction to 3+1 Numerical Relativity* (International Series of Monographs on Physics), (Oxford University Press, 2008).

[2] T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity: Solving Einstein's Equations on the Computer*, (Cambridge University Press, 2010).

[3] E. Gourgoulhon, *3+1 Formalism in General Relativity: Bases of Numerical Relativity* (Lecture Notes in Physics), (Springer-Verlag, 2012)

[4] F. Pretorius, in *Relativistic Objects in Compact Binaries: From Birth to Coalescence*, Editor: Colpi et al. Pulisher: Springer Verlag, Canopus Publishing Limited, arXiv:0710.1338.

1.3 数値相対論の方法論概略 (Overview of Numerical Relativity Methodology)

Numerical Relativity – Methodology **Box 1.2**

0. How to foliate space-time
   Cauchy $(3+1)$,
   Hyperboloidal $(3+1)$, ⇒ see e.g. [2]
   characteristic $(2+2)$, ⇒ see e.g. [5]
   or combined?

⇒ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data ⇒ see e.g. [1]

   Theoretical: Proper formulation for solving constraints?
   How to prepare realistic initial data?
   Effects of background gravitational waves?
   Connection to the post-Newtonian approximation?

   Numerical: Techniques for solving coupled elliptic equations?
   Appropriate boundary conditions?

2. How to evolve the data

   Theoretical: Free evolution or constrained evolution?
   Proper formulation for the evolution equations? ⇒ see e.g. [4, 3]
   Suitable slicing conditions (gauge conditions)?

   Numerical: Techniques for solving the evolution equations?
   Appropriate boundary treatments?
   Singularity excision techniques?
   Matter and shock surface treatments?
   Parallelization of the code?

3. How to extract the physical information

   Theoretical: Gravitational wave extraction?
   Connection to other approximations?

   Numerical: Identification of black hole horizons?
   Visualization of simulations?

References

[1] G. Cook, Livng Rev. Relativ. **2000-5** at `http://www.livingreviews.org/`

[2] S. Husa, gr-qc/0204043; gr-qc/0204057.

[3] H. Shinkai, J. Korean Phys. Soc. **54** (2009) 2513 (arXiv:0805.0068)

[4] H. Shinkai and G. Yoneda, gr-qc/0209111

[5] J. Winicour, Livng Rev. Relativ. **2009-3** at `http://www.livingreviews.org/`

Notations:

- signature $(-+++)$.
- Covariant derivatives, Christoffel symbol

$$\begin{aligned} \nabla_\mu A^\alpha &\equiv A^\alpha_{;\mu} \equiv A^\alpha{}_{,\mu} + \Gamma^\alpha_{\mu\nu} A^\nu && (1.2)\\ \nabla_\mu A_\alpha &\equiv A_{\alpha;\mu} \equiv A_{\alpha,\mu} - \Gamma^\nu_{\alpha\mu} A_\nu && (1.3)\\ \Gamma^\alpha_{\mu\nu} &= (1/2) g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) && (1.4) \end{aligned}$$

- Riemann tensor, Ricci tensor, Weyl tensor

$$\begin{aligned} R^a{}_{bcd} &\equiv \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} && (1.5)\\ R_{ab} \equiv R^\mu{}_{a\mu b} &\equiv \Gamma^\mu_{ab,\mu} - \Gamma^\mu_{a\mu,b} + \Gamma^\mu{}_{\nu\mu}\Gamma^\nu{}_{ab} - \Gamma^\mu{}_{\nu b}\Gamma^\nu{}_{a\mu} && (1.6)\\ C_{abcd} &= R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} - \frac{1}{3} R g_{a[c}g_{d]b}, && (1.7) \end{aligned}$$

- ADM decomposition, the extrinsic curvature (§2)

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu,\nu = 0,1,2,3)\\ \text{on } \Sigma(t)... \quad d\ell^2 &= \gamma_{ij} dx^i dx^j, \qquad (i,j = 1,2,3) \end{aligned}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{1.8}$$

$$K_{ij} \equiv -\perp^\mu_i \perp^\nu_j n_{\mu;\nu} = -\frac{1}{2} \pounds_n \gamma_{ij}. \tag{1.9}$$