3 数値相対論の標準的手法

3.1 どのように初期値を準備するか

Initial Data Construction Problem **Box 3.1**

Prepare all metric and matter components by solving the two constraints:

- The Hamiltonian constraint equation

$$ {}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho + 2\Lambda \tag{3.1} $$

- The momentum constraint equations

$$ D_j(K^{ij} - \gamma^{ij}\mathrm{tr}K) = \kappa J^i \tag{3.2} $$

3.1.1 Conformal Approach – York-ÓMurchadha (1974)

Conformal transformation

The idea by ÓMurchadha and York [1] is

$$ \text{solution} \quad \gamma_{ij} = \psi^4 \hat{\gamma}_{ij} \quad \text{trial metric} \tag{3.3} $$

We introduce the decomposition of K_{ij},

$$ K_{ij} \Rightarrow \begin{cases} \mathrm{tr}K = \gamma^{ij}K_{ij} & \text{trace part} \\ A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}\mathrm{tr}K & \text{trace-free part} \end{cases} \tag{3.4} $$

Then, other conformal transformations as consistent with (3.3) are:

$$ \gamma_{ij} = \psi^4 \hat{\gamma}_{ij}, \qquad \gamma^{ij} = \psi^{-4}\hat{\gamma}^{ij}, \tag{3.5} $$

$$ A^{ij} = \psi^{-10}\hat{A}^{ij}, \quad A_{ij} = \psi^{-2}\hat{A}_{ij}, \tag{3.6} $$

$$ \rho = \psi^{-n}\hat{\rho}, \qquad J^i = \psi^{-10}\hat{J}^i, \tag{3.7} $$

and we suppose

$$ \mathrm{tr}K = \hat{\mathrm{tr}}\hat{K}, \quad \mathrm{tr}A = \hat{\mathrm{tr}}\hat{A} = 0. \tag{3.8} $$

From (3.5), we get

$$ \Gamma^i{}_{jk} = \hat{\Gamma}^i{}_{jk} + 2\psi^{-1}(\delta^i{}_j\hat{D}_k\psi + \delta^i{}_k\hat{D}_j\psi - \hat{\gamma}_{jk}\hat{\gamma}^{im}\hat{D}_m\psi), \tag{3.9} $$

$$ R = \psi^{-4}\hat{R} - 8\psi^{-5}\hat{\Delta}\psi. \tag{3.10} $$

where $\hat{\Delta} = \hat{\gamma}^{jk}\hat{D}_j\hat{D}_k$ and $\hat{R} = R(\hat{\gamma})$, and also $D_jA^{ij} = \psi^{-10}\hat{D}_j\hat{A}^{ij}$.

We further decompose $\hat{A}^{ij}$ to divergence-free (transverse-traceless, TT) part and longitudinal part:

$$ \hat{A}^{ij} = \hat{A}^{ij}_{TT} + (\hat{\mathbb{l}}W)^{ij}, \tag{3.11} $$

where we suppose

$$ \hat{D}_j\hat{A}^{ij}_{TT} = 0 \qquad \text{and} \qquad \hat{\mathrm{tr}}\hat{A}_{TT} = 0. \tag{3.12} $$

and

$$(\hat{\mathbb{l}}W)^{ij} = \hat{D}^i W^j + \hat{D}^j W^i - \frac{2}{3}\hat{\gamma}^{ij}\hat{D}_k W^k. \tag{3.13}$$

Using these terms, we can write

$$\begin{aligned} \hat{D}_j \hat{A}^{ij} &= \hat{D}_j(\hat{\mathbb{l}}W)^{ij} \equiv (\hat{\Delta}_{\mathbb{l}} W)^i, \\ &= (\hat{\Delta}W)^i + \frac{1}{3}\hat{D}^i(\hat{D}_j W^j) + \hat{R}^i{}_j W^j. \end{aligned} \tag{3.14}$$

With above transformation, the two constraints, (3.1) and (3.2), can be expressed as follows.

- The Hamiltonian constraint equation

$$8\hat{\Delta}\psi = \hat{R}\psi - (\hat{A}_{ij}\hat{A}^{ij})\psi^{-7} + [\frac{2}{3}(\mathrm{tr}K)^2 - 2\Lambda]\psi^5 - 16\pi G\hat{\rho}\psi^{5-n} \tag{3.15}$$

- The momentum constraint equations

$$\hat{\Delta}W^i + \frac{1}{3}\hat{D}^i\hat{D}_k W^k + \hat{R}^i{}_k W^k = \frac{2}{3}\psi^6 \hat{D}^i \mathrm{tr}K + 8\pi G\hat{J}^i \tag{3.16}$$

Equations to solve

Conformal approach (York-ÓMurchadha, 1974) **Box 3.2**

One way to set up the metric and matter components $(\gamma_{ij}, K_{ij}, \rho, J^i)$ so as to satisfy the constraints (3.1) and (3.2) is as follows.

1. Specify metric components $\hat{\gamma}_{ij}$, $\mathrm{tr}K$, $\hat{A}^{TT}_{ij}$, and matter distribution $\hat{\rho}$, $\hat{J}$ in the conformal frame.

2. Solve the next equations for $(\psi,\ W^i)$

$$8\hat{\Delta}\psi = \hat{R}\psi - (\hat{A}_{ij}\hat{A}^{ij})\psi^{-7} + [\frac{2}{3}(\mathrm{tr}K)^2 - 2\Lambda]\psi^5 - 16\pi G\hat{\rho}\psi^{5-n} \tag{3.15}$$

$$\hat{\Delta}W^i + \frac{1}{3}\hat{D}^i\hat{D}_k W^k + \hat{R}^i{}_k W^k = \frac{2}{3}\psi^6 \hat{D}^i \mathrm{tr}K + 8\pi G\hat{J}^i \tag{3.16}$$

   where

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + \hat{D}^i W^j + \hat{D}^j W^i - \frac{2}{3}\hat{\gamma}^{ij}\hat{D}_k W^k. \tag{3.17}$$

3. Apply the inverse conformal transformation and get the metric and matter components γ_{ij}, K_{ij}, ρ, J^i in the physical frame:

$$\gamma_{ij} = \psi^4\hat{\gamma}_{ij}, \tag{3.18}$$

$$K_{ij} = \psi^{-2}[\hat{A}^{TT}_{ij} + (\hat{\mathbb{l}}W)_{ij}] + \frac{1}{3}\psi^4\hat{\gamma}_{ij}\mathrm{tr}K, \tag{3.19}$$

$$\rho = \psi^{-n}\hat{\rho}, \tag{3.20}$$

$$J^i = \psi^{-10}\hat{J}^i \tag{3.21}$$

Comments

- Using the idea of conformal rescaling, we have a way to fix 12 components of (γ_{ij}, K_{ij}) that satisfy 4 constraints.
- The Hamiltonian constraint, (3.15), is a non-linear elliptic equation for ψ, so that we have to solve it by an iterative method.
- The momentum constraints, (3.16), are PDEs for W^i and coupled with (3.15). If we assume $\mathrm{tr}K = 0$, then two constraints are decoupled. Normally people assume $\mathrm{tr}K = 0$ (maximal slicing condition) or $(\mathrm{tr}K)$ =const. (constant mean curvature slicing) for this purpose.
- For simplicity, people assume the background metric $\hat{\gamma}_{ij}$ is conformally flat $\hat{\gamma}_{ij} = \delta_{ij}$. The physical appropriateness of conformal flatness is often debatable.
- Two freedom of $\hat{A}^{TT}_{ij}$ corresponds to the one of gravitational wave. However, there have been no systematic discussion how to specify them, except applying tensor harmonics in a linearized situation.

Solving the Hamiltonian constraint – Several tips
Two Methods:

1. Solve the non-linear equation (3.15) directly.
2. Solve the linearized equation $\psi = \psi_0 + \delta\psi$ iteratively.

$$\begin{aligned} 8\hat{\Delta}\psi &= E\,\psi + F\,\psi^{-7} + G\,\psi^5 + H\,\psi^{-3} + I\,\psi^{-1} \\ &= [E - 7F\psi_0^{-8} + 5G\psi_0^4 - 3H\psi_0^{-4} - 2I\psi_0^{-2}]\,\psi + [8F\psi_0^{-7} - 4G\psi_0^5 + 4H\psi_0^{-3} + 2I\psi_0^{-1}] \end{aligned}$$

Under an appropriate boundary condition, such as Robin BC $\psi = 1 + const./r$, or Dirichlet BC $\psi = 1 + M_{total}/2r$.

Solve the momentum constraints – Several tips
A couple of methods:

1. Solve the non-linear equations (3.16) directly.
2. Bowen's method for conformally flat case [GRG14(1982)1183]
   Under the $(\nabla^i K = 0)$ condition, (3.16) becomes

$$\Delta W^i + \frac{1}{3}\nabla^i\nabla_j W^j = 8\pi S^i.$$

By introducing a decomposition of W^i into vector and gradient terms

$$W^i = V^i - \frac{1}{4}\nabla^i\theta,$$

the equations to solve are:

$$\Delta V^i = 8\pi S^i, \qquad (3.22)$$

$$\Delta\theta = \nabla_i V^i, \qquad (3.23)$$

If the source is of finite extent, then the the asymptotic behavior of V^i and θ are given by

$$V^i = -2\sum_{l=0}^{\infty} Q^{ij_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}}, \tag{3.24}$$

$$\theta = -\sum_{l=1}^{\infty} Q^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l-1}} + \sum_{l=0}^{\infty}\frac{2(l+1)}{(2l+1)(2l+3)} Q_k^{kj_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}}$$
$$+\sum_{l=1}^{\infty}\frac{2l-1}{2l+1} M^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l+1}} \tag{3.25}$$

where $n^i = x^i r^{-1}$ in the Cartesian cordinate, the multipoles Q and M are defined as

$$Q^{ij_1\cdots j_l} \equiv \frac{(2l-1)!!}{l!}\int S^i(\mathbf{r}) x^{\{j_1}x^{j_2}\cdots x^{j_l\}} dV,$$
$$M^{ij_1\cdots j_l} \equiv \frac{(2l-1)!!}{l!}\int r^2 S^i(\mathbf{r}) x^{\{j_1}x^{j_2}\cdots x^{j_l\}} dV,$$

and where brackets denote the completely symmetric trace-free part

$$Z^{\{ij_1\cdots j_l\}} = Z^{(ij_1\cdots j_l)} - \frac{l}{2l+1} Z_k^{k(j_1\cdots j_{l-1}}\delta^{j_l i)}$$

3.1.2 Conformal Approach : N-dimensional case

We generalized the above conformal approach by York and ÓMurchadha (1974) to N-dimensional version and also for Gauss-Bonnet gravity. Here, we show only for the N-dimensional equations.

We start from the conformal transformation

solution $\boxed{\gamma_{ij} = \psi^{2m}\hat{\gamma}_{ij}, \quad \gamma^{ij} = \psi^{-2m}\hat{\gamma}^{ij}}$ trial metric

this gives

$$R = \psi^{-2m}\Big\{\hat{R} - 2(N-1)m\psi^{-1}(\hat{D}^a\hat{D}_a\psi) + (N-1)[2-(N-2)m]m\psi^{-2}(\hat{D}\psi)^2\Big\},$$
$$R_{ij} = \hat{R}_{ij} - m\hat{\gamma}_{ij}\psi^{-1}\hat{D}_a\hat{D}^a\psi - (N-2)m\psi^{-1}\hat{D}_i\hat{D}_j\psi$$
$$+ (N-2)m(m+1)\psi^{-2}\hat{D}_i\psi\hat{D}_j\psi - m[(N-2)m-1]\psi^{-2}(\hat{D}\psi)^2\hat{\gamma}_{ij},$$
$$R_{ijkl} = \psi^{2m}\Big\{\hat{R}_{ijkl} + m\psi^{-1}\hat{\gamma}_{il}[\hat{D}_j\hat{D}_k\psi - (m+1)\psi^{-1}\hat{D}_j\psi\hat{D}_k\psi]$$
$$- m\psi^{-1}\hat{\gamma}_{ik}[\hat{D}_j\hat{D}_l\psi - (m+1)\psi^{-1}\hat{D}_j\psi\hat{D}_l\psi]$$
$$+ m\psi^{-1}\hat{\gamma}_{jk}[\hat{D}_i\hat{D}_l\psi - (m+1)\psi^{-1}\hat{D}_i\psi\hat{D}_l\psi]$$
$$- m\psi^{-1}\hat{\gamma}_{jl}[\hat{D}_i\hat{D}_k\psi - (m+1)\psi^{-1}\hat{D}_i\psi\hat{D}_k\psi] + m^2\psi^{-2}(\hat{D}\psi)^2(\hat{\gamma}_{il}\hat{\gamma}_{jk} - \hat{\gamma}_{ik}\hat{\gamma}_{jl})\Big\}.$$

Decompose the extrinsic curvature K_{ij} as $K_{ij} \equiv A_{ij} + \frac{1}{N}\gamma_{ij}K$, and assume

$$A_{ij} = \psi^{\ell}\hat{A}_{ij}, \quad A^{ij} = \psi^{\ell-4m}\hat{A}^{ij}, \quad \text{and} \quad K = \psi^{\tau}\hat{K}.$$

Conformal transformation of the divergence D_jA^{ij} becomes

$$D_jA^{ij} = \psi^{-4m+\ell}\hat{D}_j\hat{A}^{ij} + \psi^{-4m+\ell-1}[\ell + m(N-2)]\hat{A}^{ij}\hat{D}_j\psi, \tag{3.26}$$

which indicates to set $\ell = -m(N-2) = -2$ for simplifying the equation. and for the components in the constraints,

$$K_{ij}K^{ij} = A_{ij}A^{ij} + \frac{1}{N}K^2 = \psi^{-12}\hat{A}_{ij}\hat{A}^{ij} + \frac{1}{N}\hat{K}^2, \tag{3.27}$$

$$D_jA^{ij} = \psi^{-10}\hat{D}_j\hat{A}^{ij}. \tag{3.28}$$

Decompose A_{ij} into the divergence-free (transverse-traceless, TT) part, A^{ij}_{TT}, and the rest (longitudinal part), such as

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + \hat{A}^{ij}_{L}, \quad \text{where} \quad \hat{D}_j\hat{A}^{ij}_{TT} = 0. \tag{3.29}$$

The latter part can be expressed using a vector potential, W^i, as $\hat{A}^{ij}_L = \hat{D}^iW^j + \hat{D}^jW^i - \frac{2}{N}\hat{\gamma}^{ij}\hat{D}_kW^k$. When matter exists, define also the conformal transformation

$$\rho = \psi^{-p}\hat{\rho}, \qquad J^i = \psi^{-q}\hat{J}^i.$$

- Hamiltonian constraint equation, then, becomes

$$2(N-1)m\hat{D}_a\hat{D}^a\psi - (N-1)[2-(N-2)m]m(\hat{D}\psi)^2\psi^{-1}$$
$$= \hat{R}\psi - \frac{N-1}{N}\varepsilon\psi^{2m+2\tau+1}\hat{K}^2 + \varepsilon\psi^{-2m+2\ell+1}\hat{A}_{ab}\hat{A}^{ab} + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} \tag{3.30}$$

  We found that the combination $\ell = 2/(N-2)$ and $p = -1$ makes the RHS of (3.30) linear. If we choose $\ell = -2$, which will make the momentum constraint simpler as we see later, (3.30) also remains as a simple equation.

- We obtain the momentum constraint equation as

$$\hat{D}_a\hat{D}^aW_i + \frac{N-2}{N}\hat{D}_i\hat{D}_kW^k + \hat{R}_{ik}W^k$$
$$+\psi^{-1}[\ell + (N-2)m]\Big(\hat{D}^aW^b + \hat{D}^bW^a - \frac{2}{N}\hat{\gamma}^{ab}\hat{D}_kW^k\Big)\hat{\gamma}_{bi}\hat{D}_a\psi$$
$$-\psi^{2m-\ell}\frac{N-1}{N}\hat{D}_i(\psi^\tau\hat{K}) = \kappa^2\psi^{4m-\ell-q}\hat{J}_i \tag{3.31}$$

  We found that the choice of $\ell = -2$ cancels the mixing term between ψ and W^i. The decoupling feature between two constraints is available when $\hat{K}$ =const. and $q = 8/(N-2) + 2$.

Conformal approach for solving constraints in $(N+1)$-dim. [2] **Box 3.3**

One way to set up $(\gamma_{ij}, K_{ij}, \rho, J^i)$ so as to satisfy the constraints:

1. Specify metric components $\hat{\gamma}_{ij}$, tr$K$, $\hat{A}^{TT}_{ij}$, and matter distribution $\hat{\rho}$, $\hat{J}$ in the conformal frame.
2. Solve the next equations for $(\psi,\ W^i)$

   (A) Hamiltonian constraint

$$\frac{4(N-1)}{N-2}\hat{\Delta}\psi = \hat{R}\psi - \varepsilon\psi^{2\ell+1-4/(N-2)}(\hat{K}^2 - \hat{K}_{ab}\hat{K}^{ab}) + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} \tag{3.32}$$

   (B) momentum constraint

$$\hat{\Delta}W_i + \frac{N-2}{N}\hat{D}_i\hat{D}_kW^k + \hat{R}_{ik}W^k + \psi^{-1}(\ell+2)\Big(\hat{D}^aW^b + \hat{D}^bW^a - \frac{2}{N}\hat{\gamma}^{ab}\hat{D}_kW^k\Big)\hat{\gamma}_{bi}\hat{D}_a\psi$$
$$-\frac{N-1}{N}\left[\Big(\ell - \frac{4}{N-2}\Big)(\hat{D}_i\psi)\hat{K} + \hat{D}_i\hat{K}\right] = \kappa^2\psi^{8/(N-2)-\ell-q}\hat{J}_i \tag{3.33}$$

3. Apply the inverse conformal transformation and get the metric and matter components $\gamma_{ij},\ K_{ij},\ \rho,\ J^i$ in the physical frame:

$$\begin{aligned}
\gamma_{ij} &= \psi^{4/(N-2)}\hat{\gamma}_{ij},\\
K_{ij} &= \psi^{\ell}[\hat{A}^{TT}_{ij} + (\hat{\mathbb{l}}W)_{ij}] + \frac{1}{N}\psi^{\ell-4/(N-2)}\hat{\gamma}_{ij}\mathrm{tr}K,\\
\rho &= \psi^{-p}\hat{\rho},\\
J^i &= \psi^{-q}\hat{J}^i
\end{aligned}$$

References

[1] N.ÓMurchadha and J.W.York Jr., Phys. Rev. **D 10**, 428 (1974)

[2] T. Torii and H. Shinkai, Phys. Rev. **D 78**, 084037 (2008).

3.2 どのようにゲージを設定するか

The standard 3 + 1 formulation allows us to choose gauge conditions (slicing conditions) for every time step. The fundamental guidelines for fixing the lapse function α and the shift vector β_i:

- to avoid the foliation hitting the physical and coordinate singularity in its evolution.
- to make system suitable for physical situation.
- to make the evolution system as simple as possible.
- to enable the gravitational wave extraction easy.

I list several essential slicing conditions below. The notations hereafter follows those of §2.1 (ADM formulation).

3.2.1 Lapse conditions

geodesic slice	$\alpha = 1$	GOOD BAD	simple, easy to understand no singularity avoidance	
harmonic slice	$\nabla_a \nabla^a x^b = 0$	GOOD GOOD BAD	simplify eqs., easy to compare analytical investigations no singularity avoidance or coordinate pathologies	[2]-[7]
maximal slice	$K = 0$	GOOD BAD	singularity avoidance have to solve an elliptic eq.	[1],[8]-[15]
maximal slice (K-driver)	$\partial_t K = -c^2 K$	G&B GOOD	same with maximal slice, easy to maintain $K = 0$	[12]
constant mean curvature	$K = const.$	G&B GOOD	same with maximal slice, suitable for cosmological situation	[16]-[18]
polar slicing	$K_\theta^\theta + K_\varphi^\varphi = 0$, or $K = K_r^r$	GOOD BAD	singularity avoidance in isotropic coord. trouble in Schwarzschild coord.	[19]-[21]
algebraic	$\alpha \sim \sqrt{\gamma}$, $\alpha \sim 1 + \log \gamma$	GOOD BAD	easy to implement not avoiding singularity	

Maximal slicing

This is always the first one to be mentioned as a singularity avoiding gauge condition. The name of 'maximal' comes from the fact that the deviation of the 3-volume $V = \int \sqrt{\gamma} d^3x$ along to the normal line becomes maximal when we set $K = 0$. This is simply written as

$$K = 0 \qquad \text{on} \quad \Sigma(t). \tag{3.34}$$

Pioneering idea can be seen in Lichnerowicz [8], and it was extended by York [1]. This condition is supposed to be applied in simulations that a singularity will appear during evolutions such as gravitational collapses. The actual equation for determining the lapse function α can be obtained from $\partial_t K = \partial_t (K_{ij}\gamma^{ij}) = 0$. By substituting the evolution equations, we get

$$D^i D_i \alpha = \{ {}^{(3)}R + K^2 + 4\pi G(S - 3\rho_H) - 3\Lambda\}\alpha, \tag{3.35}$$

or by using the Hamiltonian constraint further,

$$D^i D_i \alpha = \{K_{ij}K^{ij} + 4\pi G(S + \rho_H) - \Lambda\}\alpha. \tag{3.36}$$

This is an elliptic equation. When the curvature is strong (i.e. close to the appearance of a singularity), the RHS of equation become larger, hence the lapse becomes smaller. Therefore the foliation near the singularity evolves slowly.

For Schwarzschild black-hole space-time, Estabrook *et al.* [10] showed that the maximal slicing condition allows the 3-surface to reach into $r = 1.5M$ in the limit $t \to \infty$, that is inside of the event horizon, $r = 2M$. However, it is also reported that the difference of α-evolution causes the grid-stretching problem.

3.2.2 Shift conditions

geodesic slice	$\beta^i = 0$	`GOOD`	simple, easy to understand	
		`BAD`	too simple	
minimal distortion	$\min \Sigma^{ij}\Sigma_{ij}$	`GOOD`	geometrical meaning	[1]
		`BAD`	elliptic eqs., hard to solve	
minimal strain	$\min \Theta^{ij}\Theta_{ij}$	`G&B`	same with minimal distortion	[1]

Minimal distortion condition, minimal strain condition
Any singularity avoiding slice conditions causes the grid stretching problem. Smarr and York [1] proposed the condition which minimize the distortion in a global sense.

Let us define the expansion tensor $\Theta_{\mu\nu}$ and the distortion tensor Σ_{ij}. Let the normal direction to the surface n^μ, and the coordinate-constant congruence $t_\mu = \alpha n_\mu + \beta_\mu$. By projecting t^μ onto the hypersurface using the projection operator $\perp^a_b = \delta^a_b + n^a n_b$,

$$\Theta_{\mu\nu} = \perp \nabla_{(\nu} t_{\mu)} = -\alpha K_{\mu\nu} + \frac{1}{2} D_{(\mu}\beta_{\nu)} \tag{3.37}$$

We then extract this traceless part and define,

$$\Sigma_{ij} = \Theta_{ij} - \frac{1}{3}\Theta\gamma_{ij} = -2\alpha \left(K_{ij} - \frac{1}{3}\gamma_{ij} K \right) + \frac{1}{2}\left(D_{(i}\beta_{j)} - \frac{1}{3} D^k \beta_k \right). \tag{3.38}$$

The minimal distortion condition is to choose β^i which minimize the action

$$\delta S[\beta] = \delta\{\frac{1}{2}\int \Sigma_{ij}\Sigma^{ij} d^3x\} = 0. \tag{3.39}$$

This condition can be written as $D^j \Sigma_{ij} = 0$, or

$$D^j D_j \beta_i + D^j D_i \beta_j - \frac{2}{3} D_i D_j \beta^j = D^j \left[2\alpha \left(K_{ij} - \frac{1}{3} trK \gamma_{ij} \right) \right], \tag{3.40}$$

$$\text{or} \quad \Delta\beta_i + \frac{1}{3} D_i (D^j \beta_j) + R^j_i \beta_j = D^j \left[2\alpha \left(K_{ij} - \frac{1}{3} trK \gamma_{ij} \right) \right], \tag{3.41}$$

where $\Delta = D^i D_i$.

Similarly, we can define the minimal strain condition by minimizing $\Theta^{ij}\Theta_{ij}$.

The both requires non-linear elliptic equations and hard to solve. Several group solves "pseudo"-minimal distortion condition by replacing the covariant derivatives to the partial derivatives [22]. This simplification also works for inspiral binary neutron star evolution.

References

[1] J.W. York,Jr., “Kinematics and Dynamics of General Relativity”, in *Sources of Gravitational Radiation*, ed. by L.Smarr, (Cambridge, 1979) ;
L.Smarr and J.W.York,Jr., Phys. Rev. D **17**, 2529 (1978)

[2] C. Bona and J. Massó, Phys. Rev. D **38**, 3419 (1988).

[3] C. Bona and J. Massó, Phys. Rev. Lett. **68**, 1097 (1992).

[4] G.B. Cook and M.A. Scheel, Phys. Rev. D **56**, 4775 (1997).

[5] M. Alcubierre, Phys. Rev. D **55**, 5981 (1997).

[6] M. Alcubierre and J. Massó, Phys. Rev. D **57**, R4511 (1998).

[7] M. Alcubierre, gr-qc/0210050.

[8] A. Lichnerowicz, J. Math. Pures Appl. **23**, 37 (1944).

[9] D. Bernstein, D.W. Hobill, and L. Smarr, in *Frontiers in Numerical Relativity*, edited by C.R. Evans, L.S. Finn, and D.W. Hobill (Cambridge Univ. Press, 1989)

[10] F. Estabrook, H. Wahlquist, S. Christensen, B. DeWitt, L. Smarr, and E. Tsiang, Phys. Rev. D. **7**, 2814 (1973).

[11] M.J. Duncan, Phys. Rev. D **31**, 1267 (1985).

[12] J. Balakrishna, G. Daues, E. Seidel, W-M. Suen, M. Tobias and E. Wang, Class. Quant. Grav. **13**, L135 (1996).

[13] A. Geyer and H. Herold , Phys. Rev. D. **52**, 6182 (1995).

[14] A. Geyer and H. Herold , Gen. Rel. Grav. **29**, 1257 (1997).

[15] L. I. Petrich, S.L. Shapiro, and S.A. Teukolsky, Phys. Rev. D **31**, 2459 (1985).

[16] D.M. Eardley, L. Smarr, Phys. Rev. D **19**, 2239 (1979)

[17] D.R. Brill, J.M. Cavallo, J.A. Isenberg, J. Math. Phys. **21**, 2789 (1980)

[18] K. Nakao, T. Nakamura, K. Oohara, K. Maeda, Phys. Rev. D **44**, 1326 (1991)

[19] J.M. Bardeen and T. Piran, Phys. Rep. **96**, 205 (1983).

[20] R. A. d'Inverno, Class. Quantum Grav. **12**, L75 (1995).

[21] L. I. Petrich, S.L. Shapiro, and S.A. Teukolsky, Phys. Rev. D **33**, 2100 (1986).

[22] M. Shibata and K. Uryu, Prog. Theor. Phys. **107**, 265 (2002).

3.3 Ashtekar 形式を用いた数値相対論

3.3.1 History

Ashtekar's formulation of general relativity[11] has many attractive features comparing to the conventional ADM formulation. Therefore, an application to numerical simulations was suggested [2] soon after Ashtekar completed his formulation, but had not yet been completed more than a decade. Historically, an application to numerical relativity of the connection formulation was also suggested [3, 4] using Capovilla-Dell-Jacobson's version of the connection variables [5], which produce an direct relation to Newman-Penrose's Ψs.

The first full numerical application was reported by Shinkai and Yoneda [58, 71]. They developed a plane symmetric evolution code, and showed comparisons of numerical stability due to the different hyperbolicity in the context of formulation problem (§4, in this lecture note). They also showed that their new formulation called λ-system, which makes the evolution system asymptotically constrained, works as desired.

In this subsection, we only look at how they realized numerical experiments from the viewpoint of methodology.

3.3.2 Numerical treatments by Shinkai-Yoneda

Shinkai and Yoneda coded up the program so as to compare the evolutions of spacetime with three different sets of dynamical equations (Ashtekar's original, and two modified sets) but with the common conditions: the same initial data, the same boundary conditions, the same slicing condition and the same evolution scheme.

They considered the plane symmetric vacuum spacetime without cosmological constant. This spacetime has the true freedom of gravitational waves of two polarized (+ and $\times$) modes. They applied the periodic boundary conditions to remove any difficulties caused by numerical treatment of the boundary conditions. The initial data are given by solving constraint equations in ADM variables, using the standard conformal approach by York and O'Murchadha (Box 3.2 in this lecture note).

When we use Ashtekar's variables for evolution, we transform the ADM initial data in terms of Ashtekar's variables. The results are analyzed by monitoring the violation of the constraint equations which are expressed using the same (or transformed if necessary) variables.

Reformulation of the Ashtekar evolution equations

They constructed three variations of Ashtekar's evolution system (see Table 3.1 for summary).

(a) The original set of dynamical equations (2.75) and (2.76) [the *original* equations] already forms a weakly hyperbolic system [12]. So that we regard the mathematical structure of the original equations as one step advanced from the standard ADM.

	system	variables	Eqs of motion	remark
I	Ashtekar (weakly hyp.)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(2.75), (2.76) (original)	"original" eqs.
II	Ashtekar (strongly hyp.)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(3.42), (3.43) (with $\kappa = 1$)	(3.45) required
III	Ashtekar (symmetric hyp.)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(3.42), (3.43) (with $\kappa = 1$)	(3.44) required
adj	Ashtekar (adjusted)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(3.42), (3.43) (with $\kappa \neq 1$)	
λ	Ashtekar-λ-system	$(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a)$		controls $\mathcal{C}_H, \mathcal{C}_{Mi}, \mathcal{C}_{Ga}$

Table 3.1: List of Ashtekar evolution systems that applied in [58].

(b) Further, we can construct higher levels of hyperbolic systems by restricting the gauge condition and/or by adding constraint terms, $\mathcal{C}_H^{\rm ASH}, \mathcal{C}_{Mi}^{\rm ASH}$ and $\mathcal{C}_{Ga}^{\rm ASH}$, to the original equations.

- by requiring additional gauge conditions *or* adding constraints to the dynamical equations, we can obtain a strongly hyperbolic system [12],
- by requiring additional gauge conditions *and* adding constraints to the dynamical equations, we can obtain a symmetric hyperbolic system [70, 12].

(c) Based on the above symmetric hyperbolic system, we can construct an Ashtekar version of the λ-system [20] which is robust against perturbative errors for both constraints and reality conditions [57].

In order to obtain a symmetric hyperbolic system, we add constraint terms to the right-hand-side of (2.75) and (2.76). The adjusted dynamical equations,

$$\begin{aligned}
\partial_t \tilde{E}^i_a &= -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c + \kappa_1 P^i{}_{ab}\mathcal{C}_G^{{\rm ASH}b}, &(3.42)\\
&\quad \text{where} \qquad P^i{}_{ab} \equiv N^i \delta_{ab} + i\underset{\sim}{N}\epsilon_{ab}{}^c \tilde{E}^i_c,\\
\partial_t \mathcal{A}^a_i &= -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \kappa_2 Q^a_i \mathcal{C}_H^{\rm ASH} + \kappa_3 R_i{}^{ja}\mathcal{C}_{Mj}^{\rm ASH}, &(3.43)\\
&\quad \text{where} \qquad Q^a_i \equiv e^{-2}\underset{\sim}{N}\tilde{E}^a_i, \qquad R_i{}^{ja} \equiv ie^{-2}\underset{\sim}{N}\epsilon^{ac}{}_b \tilde{E}^b_i \tilde{E}^j_c
\end{aligned}$$

form a symmetric hyperbolicity if we further require $\kappa_1 = \kappa_2 = \kappa_3 = 1$ and the gauge conditions,

$$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i, \qquad \partial_i N = 0. \tag{3.44}$$

We remark that the adjusted coefficients, $P^i{}_{ab}, Q^a_i, R_i{}^{ja}$, for constructing the symmetric hyperbolic system are uniquely determined, and there are no other additional terms (say, no $\mathcal{C}_H^{\rm ASH}, \mathcal{C}_M^{\rm ASH}$ for $\partial_t \tilde{E}^i_a$, no $\mathcal{C}_G^{\rm ASH}$ for $\partial_t \mathcal{A}^a_i$) [12]. The gauge conditions, (3.44), are consequences of the consistency with (triad) reality conditions.

We can also construct a strongly (or diagonalizable) hyperbolic system by restricting to a gauge $N^l \neq 0, \pm N\sqrt{\gamma^{ll}}$ (where γ^{ll} is the three-metric and we do not sum indices here) for the original equations (2.75), (2.76). Or we can also construct from the adjusted equations, (3.42) and (3.43), together with the gauge condition

$$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i. \tag{3.45}$$

As for the strongly hyperbolic system, we hereafter take the latter expression.

metric and the initial data construction

We consider the plane symmetric metric,

$$ds^2 = (-N^2 + N_x N^x)dt^2 + 2N_x dx dt + \gamma_{xx}dx^2 + \gamma_{yy}dy^2 + \gamma_{zz}dz^2 + 2\gamma_{yz}dydz \tag{3.46}$$

where the components are the function of $N(x,t), N_x(x,t), \gamma_{xx}(x,t), \gamma_{yy}(x,t), \gamma_{zz}(x,t), \gamma_{yz}(x,t)$. N and N^x are called the lapse function and the shift vector.

We prepare our initial data by solving the ADM constraint equations, (2.14) and (**??**), using the conformal approach (Box 3.2). Since we consider only the vacuum spacetime, the input quantities are the initial guess of the 3-metric $\hat{\gamma}_{ij}$, the trace part of the extrinsic curvature tr$K$, and the transverse traceless part of the extrinsic curvature $\hat{A}_{TT}$. For simplicity, we impose $\hat{A}_{TT} = 0$ and tr$K = K_0$ (constant). The Hamiltonian constraint, then, becomes an equation for the conformal factor, ψ:

$$8\hat{\Delta}\psi := 8\frac{1}{\sqrt{\hat{\gamma}}}\partial_i(\hat{\gamma}^{ij}\sqrt{\hat{\gamma}}\partial_j\psi) = \hat{R}\psi + \frac{2}{3}(K_0)^2\psi^5, \tag{3.47}$$

where $\hat{\gamma} = \det \hat{\gamma}_{ij}$. The momentum constraint is automatically satisfied by assumption. The initial dynamical quantities γ_{ij}, K_{ij} are given by the conformal transformation,

$$\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}, \qquad K_{ij} = \frac{1}{3}\psi^4 \hat{\gamma}_{ij} K_0. \tag{3.48}$$

We solve (3.47) under the periodic boundary conditions using the incomplete Cholesky conjugate gradient (ICCG) method.

We can set two different modes of gravitational waves. One is the +-mode waves, which is given by setting a conformal guess metric as (in a matrix form)

$$\hat{\gamma}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ sym. & 1 + a\exp(-b(x-c)^2) & 0 \\ sym. & sym. & 1 - a\exp(-b(x-c)^2) \end{pmatrix} \tag{3.49}$$

where a, b, c are parameters. The other is the $\times$-mode waves, given by

$$\hat{\gamma}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ sym. & 1 & a\exp(-b(x-c)^2) \\ sym. & sym. & 1 \end{pmatrix} \tag{3.50}$$

where a, b, c are parameters again. Both cases, we expect non-linear behavior when wave's curvature becomes quite large compared to the background. In the collision of a +-mode wave and a $\times$-mode wave, we also expect to see the mode-mixing phenomena which is known as gravitational Faraday effect. These effects are confirmed in our numerical simulations.

Transformation of variables: From ADM to Ashtekar

We need to transform the dynamical variables on the initial data when we evolve them in the connection variables. We list the procedure to obtain $(\tilde{E}^i_a, \mathcal{A}^a_i)$ from (γ_{ij}, K_{ij}). This procedure is used also when we evaluate the constraints, $\mathcal{C}^{\rm ASH}_H, \mathcal{C}^{\rm ASH}_{Mi}, \mathcal{C}^{\rm ASH}_{Ga}$ for the data evolved using ADM variables.

From the three-metric γ_{ij} to $\tilde{E}^i_a$:

1. Define the triad E^a_i corresponding to the three-metric γ_{ij}. We take

$$E^a_i = \begin{bmatrix} E^1_x & E^1_y & E^1_z \\ E^2_x & E^2_y & E^2_z \\ E^3_x & E^3_y & E^3_z \end{bmatrix} = \begin{bmatrix} \sqrt{\gamma_{xx}} & 0 & 0 \\ 0 & e_{22} & e_{23} \\ 0 & e_{32} & e_{33} \end{bmatrix}. \tag{3.51}$$

   and set simply $e_{23} = e_{32}$. The relation between the metric and the triad becomes

$$e_{22}^2 + e_{33}^2 = \gamma_{yy}, \qquad e_{23}^2 + e_{33}^2 = \gamma_{zz}, \qquad (e_{22} + e_{33})e_{23} = \gamma_{yz}. \tag{3.52}$$

   For the case of +-mode waves, we define naturally, $e_{22} = \sqrt{\gamma_{yy}}, e_{33} = \sqrt{\gamma_{zz}}, e_{23} = 0$. For $\times$-mode waves, we also take a natural set of definitions, $e_{22} = e_{33} = [(\gamma_{yy} + (\gamma_{yy}^2 - \gamma_{yz}^2)^{1/2})/2]^{1/2}$ and $e_{23} = \gamma_{yz}/2e_{22}$ which are given by solving $e_{22}^2 + e_{33}^2 = \gamma_{yy}$ and $2e_{22}e_{23} = \gamma_{yz}$.

2. obtain the inverse triad E^i_a from triad E^a_i.

3. calculate the density, e, as $e = \det E^a_i$.

4. obtain the densitized triad, $\tilde{E}^i_a = eE^i_a$.

From three-metric (γ_{ij}, K_{ij}) to $\mathcal{A}^a_i$:

1. prepare the triad E^a_i and its inverse E^i_a.

2. calculate the connection 1-form $\omega_i^{bc} = E^{b\mu}\nabla_i E^c_\mu$. This is expressed only using partial derivatives as [9]
$$\omega_i^{bc} = E^{jb}\partial_{[i}E^c_{j]} - E_{id}E^{kb}E^{jc}\partial_{[k}E^d_{j]} + E^{jc}\partial_{[j}E^b_{i]}. \tag{3.53}$$
3. $\mathcal{A}^a_i = -K_{ij}E^{ja} - \frac{i}{2}\epsilon^a{}_{bc}\omega_i^{bc}$.

Transformation of variables: From Ashtekar to ADM
In contrast to the previous transformation, we also need to obtain (γ_{ij}, K_{ij}) from $(\tilde{E}^i_a, \mathcal{A}^a_i)$ when we evaluate the metric output or ADM constraints when we evolve the spacetime using connection variables. This process is only required at an evaluation times, not required at every time step (unless we use the gauge condition which is primarily defined using ADM quantities).

From densitized inverse triad $\tilde{E}^i_a$ to three-metric γ_{ij}:

1. calculate the density e as $e = (\det \tilde{E}^i_a)^{1/2}$.
2. get the three inverse metric as $\gamma^{ij} = \tilde{E}^i_a\tilde{E}^j_a/e^2$.
3. obtain γ_{ij}.

From $(\tilde{E}^i_a, \mathcal{A}^a_i)$ to the extrinsic curvature K_{ij}:

1. prepare the un-densitized inverse triad, $E^i_a = \tilde{E}^i_a/e$.
2. prepare triad E^a_i.
3. calculate the connection 1-form $\epsilon^a{}_{bc}\omega_i^{bc}$.
4. calculate Z^a_i, which is defined as [10] $Z^a_i := -\mathcal{A}^a_i + \frac{i}{2}\epsilon^a{}_{bc}\omega_i^{bc} (= K_{ij}E^{ja})$, and get $K_{ij} = Z^a_i E_{ja}$.

Gauge conditions
Their choice of the slicing (gauge) condition was the simplest one.

(1) the simplest geodesic slicing condition for the lapse function,

(2) the simplest zero shift vector $N^x = 0$, and

(3) the natural choice of triad lapse function $\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$ [$= 0$ if $N^x = 0$, which is suggested from (3.44) or (3.45)].

However, in the Ashtekar formalism, the densitized lapse function $\underset{\sim}{N}$ is the fundamental gauge quantity (rather than N). Therefore we try two conditions for the lapse,

(1a) the standard geodesic slicing condition $N = 1$, which will be transformed to $\underset{\sim}{N} = 1/e$ when we apply this condition in Ashtekar's evolution system, and

[9] This is from the definitions, $\omega_i^{bc} := E^{jb}\nabla_i E^c_j$ and $\omega^{abc} := E^{ja}\omega_j^{bc}$, and a relation
$$3\omega^{[abc]} - 2\omega^{[bc]a} = \omega^{a[bc]} + \omega^{b[ca]} + \omega^{c[ab]} - \omega^{abc} + \omega^{cba} = \omega^{abc}.$$
Using the densitized triad, eq. (3.53) can be also expressed as
$$\omega_i^{bc} = \frac{2}{e^2}\tilde{E}^{jb}(\partial_{[i}\tilde{E}^c_{j]}) + \frac{1}{e^4}\tilde{E}^{jb}\tilde{E}^c_i\tilde{E}^a_k(\partial_j\tilde{E}^k_a) + \frac{1}{4e^4}\tilde{E}_{ia}\tilde{E}^{kb}\tilde{E}^j_c(\partial_j\tilde{E}^a_k), \quad \text{taking } [bc].$$

[10] This is from the original definition of $\mathcal{A}^a_i$, $\mathcal{A}^a_i := \omega_i^{0a} - (i/2)\epsilon^a{}_{bc}\,\omega_i^{bc}$.

(1b) the densitized geodesic slicing condition $\underset{\sim}{N} = 1$, which will be transformed to $N = e$ when we evolve the system using ADM equations.

In practice, such a transformation using the density e will not guarantee that the Courant condition holds if we fix the time evolution step Δt [11]. Therefore we need to rescale the transformed lapse [$\underset{\sim}{N}$ in (1a), N in (1b)] so that it has a maximum value of unity, in order to keep our evolution system stable.

If we apply the standard geodesic slice, then we can compare the weakly hyperbolic system with the symmetric hyperbolic one. Similarly if we apply the densitized geodesic slice, then we can compare the (original) weakly hyperbolic system with the strongly hyperbolic one.

References

[1] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. **D36**, 1587 (1987).

[2] A. Ashtekar and J.D. Romano, “Key (3+1)-equations in terms of new variables for numerical relativity”, Syracuse University Report (1989).

[3] A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).

[4] D.C. Salisbury, L.C. Shepley, A. Adams, D. Mann, L.Turvan and B. Turner, Class. Quantum Grav. **11**, 2789 (1994).

[5] R. Capovilla, T. Jacobson, and J. Dell, Phys. Rev. Lett. **63**, 2325 (1989); Class. Quantum Grav. **8**, 59 (1991).

[6] H. Shinkai and G. Yoneda, Class. Quant. Grav. **17**, 4799 (2000).

[7] G. Yoneda and H. Shinkai, Class. Quant. Grav. **18**, 441 (2001).

[8] A. Ashtekar, J.D. Romano and R. S. Tate, Phys. Rev. **D40**, 2572 (1989).

[9] G. Yoneda and H. Shinkai, Class. Quantum Grav. **13**, 783 (1996).

[10] H. Shinkai and G. Yoneda, Phys. Rev. D **60**, 101502 (1999).

[11] G. Yoneda and H. Shinkai, Phys. Rev. Lett. **82**, 263 (1999).

[12] G. Yoneda and H. Shinkai, Int. J. Mod. Phys. D. **9**, 13 (2000).

[13] O. Brodbeck, S. Frittelli, P. Hübner, and O.A. Reula, J. Math. Phys. **40**, 909 (1999).

[11] We here remind the reader of the stability condition, $N\Delta t \leq \Delta x$ for a standard forward-time centered-space (FTCS) scheme for a simple wave equation, in a $(\Delta t, \Delta x)$-spaced numerical grid. Note that this condition will be changed due to the choice of the evolution scheme and the equations of the system.