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Constraint propagation in the family of ADM systems

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The current important issue in numerical relativity is to determine which formulation of the Einstein equations provides us with stable and accurate simulations. Based on our previous work on ''asymptotically constrained'' systems, we here present constraint propagation equations and their eigenvalues for the Arnowitt-Deser-Misner (ADM) evolution equations with additional constraint terms (adjusted terms) on the right-hand side. We conjecture that the system is robust against violation of constraints if the amplification factors (eigenvalues of the Fourier component of the constraint propagation equations) are negative or purely imaginary. We show that such a system can be obtained by choosing multipliers of the adjusted terms. Our discussion covers Detweiler's proposal and Frittelli's analysis, and we also mention the so-called conformal-traceless ADM systems.

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I. INTRODUCTION

The effort to solve the Einstein equations numerically—so-called numerical relativity—is now providing an interesting bridge between mathematical relativists and numerical relativists. Most of the simulations have been performed using the Arnowitt-Deser-Misner (ADM) formulation [1] or a modified version. However, the ADM formulation has not been proven to be a well-posed system, since its evolution equations do not present a hyperbolic form in its original or standard formulation.

Most simulations are performed using ''free evolution'' procedures: (1) solve the Hamiltonian and momentum constraints to prepare the initial data, (2) integrate the evolution equations by fixing gauge conditions, and (3) monitor the accuracy or stability by evaluating the constraints. Many trials have been made in the last few decades, but we have not yet obtained a perfect recipe for the long-term stable evolution of the Einstein equations. Here we consider the problem through the form of the equations.

One direction in the community is to rewrite the Einstein evolution equations in a hyperbolic form and to apply it to numerical simulations [2]. This is motivated by the fact that we can prove well-posedness for the evolution of several systems if they have a certain kind of hyperbolic feature. The authors recently derived [3,4] three levels of a hyperbolic system of the Einstein equations using Ashtekar's connection variables [5],[1] and compared them numerically [6]. We found that (a) the three levels of hyperbolicity can be obtained by adding constraint terms and/or imposing gauge conditions, (b) there is no drastic difference in the accuracy of numerical evolutions in these three, and (c) the symmetric hyperbolic system is not always the best for reducing numerical errors. Similar results in regard to (a) and (b) are reported by Hern [7] based on the Frittelli-Reula formulation [8].

What are, then, the criteria for predicting the stable evolutions of a system? Inspired by the ''λ-system'' proposal [9], we have considered a so-called ''asymptotically constrained'' system, that is, a system robust against the violation of constraints [10]. The fundamental idea of the ''λ system'' is to introduce artificial flow onto the constraint surface. However, we also found that such a feature can be obtained simply by adding constraint terms to the evolution equations which we named ''adjusted systems'' [11]. We explained the reason why this works by analyzing the evolution equations of the constraints (the propagation of the constraints). We proposed that the stablity of the system can be predicted by analyzing the eigenvalues (amplification factors) of the constraint propagation equations (we describe this in detail in Sec. II). We confirmed that our proposal works in both the Maxwell and Ashtekar systems [11].

The purpose of this article is to apply our proposal to the

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[1] We derived weakly, strongly (= diagonalizable), and symmetric hyperbolic systems. The mathematical inclusion relation is

$$\text{weakly hyperbolic} \ni \text{strongly hyperbolic}$$
$$\ni \text{symmetric hyperbolic.}$$

See details in [4].

ADM system(s). Especially, we consider the ‘‘adjusting process’’ [adding constraints on the right-hand side (RHS) of evolution equations] and the resultant changes to the eigenvalues of the constraint propagation systems. This adjusting process can be seen in many constructions of hyperbolic systems in the references. In fact, the *standard ADM* for numerical relativists is the version which was introduced by York [12], where the *original ADM* system [1] has already been adjusted using the Hamiltonian constraint (see more detail in Sec. III). The advantage of the *standard ADM* system is reported by Frittelli [13] from the point of the hyperbolicity and the characteristic propagation speed of the constraints. Our discussion extends her analysis to the amplification factors.

One early effort of the adjusting mechanism was presented by Detweiler [14]. Our study also includes his system, and shows that this system actually works as desired for a certain choice of parameter (Sec. IV). We also study the same procedure for the ‘‘conformal-traceless’’ ADM (CT-ADM) formulations [15,16] which is recently the most popular system in numerical simulations (Sec. V).

The analysis in the text is for perturbational violation on a flat background. Further applications are available, but we will discuss them in future reports. In the Appendix, we also give numerical demonstrations of the adjusted ADM systems discussed in the text.

II. CONSTRAINT PROPAGATION AND ‘‘ADJUSTED SYSTEM’’

We begin by reviewing the background of ‘‘adjusted systems’’ and our conjecture.

The notion of the evolution equations of the constraints is often discussed from the point of whether they form a first class system or not. Fortunately, the constraints on the (original or standard) ADM formulation are known to form a first-class system. Because of this fact, numerical relativists only need to monitor violation of the Hamiltonian and momentum constraints during free evolution of the initial data.

Our essential idea here is to feed this procedure back into the evolution equations. That is, we adjust the system's evolution equations by characterizing the constraint propagation in advance. Let us describe the procedure in a general form. Suppose we have a set of dynamical variables, $u^a(x^i,t)$, and its evolution equations

$$\partial_t u^a = f(u^a, \partial_i u^a, \ldots), \qquad (2.1)$$

which should satisfy a set of constraints $C^\rho(u^a,\partial_i u^a,\ldots)\approx 0$. The evolution equation for C^a can be written as

$$\partial_t C^\rho = g(C^\rho, \partial_i C^\rho, \ldots). \qquad (2.2)$$

We can perform two main types of analysis on Eq. (2.2).

(1) If Eq. (2.2) is in a first-order form (that is, only includes first-order spatial derivatives), then the level of hyperbolicity and the characteristic speeds (eigenvalues λ^l of the principal matrix) will definitely determine the stability of the system. We expect mathematically rigorous well-posed features for strongly or symmetric hyperbolic systems, and the characteristic speeds suggest to us satisfactory criteria for stable evolutions if they are real and under the propagation speed of the original variables u^a and/or within the causal region of the numerical integration scheme applied.

(2) On the other hand, the Fourier transformed equation (2.2),

$$\partial_t \hat{C}^\rho = \hat{g}(\hat{C}^\rho), \qquad (2.3)$$

where $C^\rho(x,t)=\int \hat{C}^\rho(k,t)\exp(ik\cdot x)d^3k$ also characterizes the evolution of the constraints independently of its hyperbolicity. As we have proposed and confirmed in [11], the set of eigenvalues Λ^i of the coefficient matrix in Eq. (2.3) provides a kind of *amplification factor* of the constraint propagation and predicts the increase or decrease of the violation of the constraints if it exists. More precisely, we showed in [11] that if the eigenvalues of Eq. (2.3) (a) have a *negative* real-part or (b) are *nonzero* (*purely imaginary*) eigenvalues, then we see more stable evolutions than a system which does not.

This is because the constraints are damped if the eigenvalues are negative and are propagating away if the eigenvalues are purely imaginary. We found heuristically that the system becomes more stable (accurate) when the amplification factors, Λ's, satisfy as much the above criteria and/or as large magnitude of Λ's away from zeros. (Examples in [11] are of the plane wave propagation in the Maxwell system and the Ashtekar system.) We remark that this eigenvalue analysis requires that we fix a particular background metric for the situation we consider, since the amplification factor depends on the dynamical variables u^a.

The above features of the constraint propagation, Eq. (2.2), will change when we modify the original evolution equations. Suppose we adjust the RHS of Eq. (2.1) by adding the constraints,

$$\partial_t u^a = f(u^a, \partial_i u^a, \ldots) + F(C^\rho, \partial_i C^\rho, \ldots), \qquad (2.4)$$

then Eq. (2.2) will also be modified as

$$\partial_t C^\rho = g(C^\rho, \partial_i C^\rho, \ldots) + G(C^\rho, \partial_i C^\rho, \ldots). \qquad (2.5)$$

By taking the characteristic speed of Eq. (2.5) and the amplification factor of the Fourier transformed equation (2.5), the predicted stability of the system (2.4) becomes different to that of the original system, Eq. (2.2).

Our proposed ‘‘adjusted system’’ is obtained by finding a certain functional form of $F(C^\rho,\partial_i C^\rho,\ldots)$ in Eq. (2.4) so as to get a more stable prediction in the analysis of the eigenvalues λ^l and Λ^i. In the following discussion, we show two eigenvalues λ^l and Λ^i for each ADM system. We remark again that the term ‘‘characteristic speed’’ here is not for the dynamical equation (2.1), but for the constraint propagation equations (2.2).

III. STANDARD ADM SYSTEM

A. Standard ADM system and its constraint propagation

We start by analyzing the standard ADM system. By ‘‘standard ADM’’ we mean here the most widely adopted system, due to York [12], with evolution equations

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i , \tag{3.1}$$

$$\begin{aligned}\partial_t K_{ij} = {} & \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha \\ & + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} ,\end{aligned} \tag{3.2}$$

and constraint equations

$$\mathcal{H} \coloneqq R^{(3)} + K^2 - K_{ij} K^{ij}, \tag{3.3}$$

$$\mathcal{M}_i \coloneqq \nabla_j K^j{}_i - \nabla_i K, \tag{3.4}$$

where (γ_{ij}, K_{ij}) are the induced three-metric and the extrinsic curvature, (α, β_i) are the lapse function and the shift covector, ∇_i is the covariant derivative adapted to γ_{ij}, and $R^{(3)}_{ij}$ is the three Ricci tensor.

The constraint propagation equations, which are the time evolution equations of the Hamiltonian constraint (3.3) and the momentum constraints (3.4), can be written as

$$\begin{aligned}\partial_t \mathcal{H} = {} & \beta^j (\partial_j \mathcal{H}) - 2\alpha \gamma^{ij} (\partial_i \mathcal{M}_j) + 2\alpha K \mathcal{H} + \alpha (\partial_l \gamma_{mk}) \\ & \times (2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj}) \mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha) \mathcal{M}_i ,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\partial_t \mathcal{M}_i = {} & -(1/2)\alpha(\partial_i \mathcal{H}) + \beta^j (\partial_j \mathcal{M}_i) + \alpha K \mathcal{M}_i - (\partial_i \alpha)\mathcal{H} \\ & - \beta^k \gamma^{jl} (\partial_i \gamma_{lk}) \mathcal{M}_j + (\partial_i \beta_k) \gamma^{kj} \mathcal{M}_j .\end{aligned} \tag{3.6}$$

The simplest derivation of Eqs. (3.5) and (3.6) is by using the Bianchi identity, which can be seen in Frittelli [13]. [Note that C in [13] is half our $\mathcal{H}$, and we have corrected typos in Eq. (11) in [13].]

The characteristic part of Eqs. (3.5) and (3.6) can be extracted as

$$\partial_t \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} \simeq \begin{pmatrix} \beta^l & -2\alpha\gamma^{il} \\ -(1/2)\alpha\delta^l_i & \beta^l \delta^j{}_i \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_j \end{pmatrix} \eqqcolon P^l \partial_l \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_j \end{pmatrix}, \tag{3.7}$$

which indicates that the characteristic speeds (eigenvalues of the characteristic matrix, P^l) are

$$\lambda^l = (\beta^l, \beta^l, \beta^l \pm \alpha\sqrt{\gamma^{ll}}) \quad (\text{no sum over } l). \tag{3.8}$$

Since $\mathrm{rank}(P^l - \beta^l) = 2$, the matrix P^l is diagonalizable, but not the symmetric.

Simply by inserting (1/2) in front of $\mathcal{H}$ above, we obtain

$$\partial_t \begin{pmatrix} \mathcal{H}/2 \\ \mathcal{M}_i \end{pmatrix} \simeq \begin{pmatrix} \beta^l & -\alpha\gamma^{il} \\ -\alpha\delta^l_i & \beta^l \delta^j{}_i \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{H}/2 \\ \mathcal{M}_j \end{pmatrix}; \tag{3.9}$$

the characteristic matrix becomes symmetric (with the same eigenvalues). This is a feature of the standard ADM system that was pointed out by Frittelli. (Actually $\mathcal{H}/2$ is the form originally given by the Lagrangian formulation.)

B. Amplification factors on the Minkowskii background

As a first example, we consider the perturbation of Minkowskii spacetime: $\alpha = 1$, $\beta^i = 0$, $\gamma_{ij} = \delta_{ij}$. By taking the linear order contribution, Eqs. (3.5) and (3.6) are reduced to

$$\partial_t \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_i \end{pmatrix} = \begin{pmatrix} 0 & -2ik_j \\ -(1/2)ik_i & 0 \end{pmatrix} \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_j \end{pmatrix} \tag{3.10}$$

in Fourier components. The eigenvalues of the coefficient matrix of Eq. (3.10), which we call *amplification factors*, become

$$\Lambda^l = (0, 0, \pm i\sqrt{k^2}), \tag{3.11}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. These factors will be compared with others later, but we note that the real parts of all the amplification factors are zero.

IV. ADJUSTED ADM SYSTEMS

A. Adjustments

Generally, we can write the adjustment terms to Eqs. (3.1) and (3.2) using Eqs. (3.3) and (3.4) by the following combinations (using up to the first derivative of constraints):

adjustment term of $\partial_t \gamma_{ij}$:

$$+P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(D_k \mathcal{H}) + q^{kl}_{ij}(D_k \mathcal{M}_l), \tag{4.1}$$

adjustment term of $\partial_t K_{ij}$:

$$+R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(D_k \mathcal{H}) + s^{kl}{}_{ij}(D_k \mathcal{M}_l), \tag{4.2}$$

where P, Q, R, S and p, q, r, s are multipliers (do not confuse R_{ij} with the three Ricci curvature that we write as $R^{(3)}_{ij}$). Since this expression is too general, we mention some restricted cases below.

We remark that our starting system, Eqs. (3.1) and (3.2), is the *standard ADM* system for numerical relativists introduced by York [12]. This expression can be obtained from the originally formulated canonical expression by ADM [1], but in that process the Hamiltonian constraint equation is used to eliminate the three-dimensional Ricci scalar. Therefore the *standard ADM* system is already adjusted from the *original ADM* system. We start our comparison with this point.

B. Original ADM vs standard ADM

Frittelli's adjustment analysis [13] can be written in terms of Eqs. (4.1) and (4.2), as

$$R_{ij} = (1/4)(\mu - 1)\alpha\gamma_{ij}, \tag{4.3}$$

where μ is a constant and set other multipliers in Eqs. (4.1) and (4.2) to zero. Here $\mu = 1$ corresponds to the standard ADM system (no adjustment, since $R_{ij} = 0$) and $\mu = 0$ to the original ADM system (without any adjustment to the canonical formulation by the ADM system).

Keeping the multiplier (4.3) in mind, we here discuss the case of nonzero R_{ij}, S^k_{ij} (all other multipliers being zero). The constraint propagation equations become

$$\partial_t \mathcal{H} = \beta^j(\partial_j \mathcal{H}) - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) + 2\alpha K \mathcal{H} + \alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i + 2KR\mathcal{H} - 2K^{ij}R_{ij}\mathcal{H}$$
$$+ 2K\gamma^{ij}S^k_{ij}\mathcal{M}_k - 2K^{ij}S^k{}_{ij}\mathcal{M}_k, \tag{4.4}$$

$$\partial_t \mathcal{M}_i = -(1/2)\alpha(\partial_i \mathcal{H}) + \beta^j(\partial_j \mathcal{M}_i) + \alpha K \mathcal{M}_i - (\partial_i \alpha)\mathcal{H} - \beta^k\gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j + \gamma^{kj}(\partial_j R_{ki})\mathcal{H} - \gamma^{jk}(\partial_i R_{jk})\mathcal{H}$$
$$+ R^j_i(\partial_j \mathcal{H}) - R_{jk}\gamma^{jk}(\partial_i \mathcal{H}) + \gamma^{lj}(\partial_j S^k_{li})\mathcal{M}_k - \gamma^{jl}(\partial_i S^k_{jl})\mathcal{M}_k + S^{kj}_i(\partial_j \mathcal{M}_k) - \gamma^{jl}S^k_{jl}(\partial_i \mathcal{M}_k) + (\partial_j \gamma^{kj})R_{ki}\mathcal{H}$$
$$+ \Gamma^j_{jk}R^k_i\mathcal{H} - \Gamma^k_{ji}R^j_k\mathcal{H} - (\partial_i \gamma^{jk})R_{jk}\mathcal{H} + (\partial_j \gamma^{lj})S^k_{li}\mathcal{M}_k + \Gamma^j_{jl}S^{kl}_i\mathcal{M}_k - \Gamma^l_{ji}S^{kj}_l\mathcal{M}_k - (\partial_i \gamma^{jl})S^k_{jl}\mathcal{M}_k; \tag{4.5}$$

that is, Eqs. (4.4) and (4.5) form a first-order system. The principal part can be written as

$$\partial_t \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} \simeq \begin{pmatrix} \beta^l & -2\alpha\gamma^{jl} \\ -(1/2)\alpha\delta^l_i + R^l_i - \delta^l_i R_{km}\gamma^{km} & \beta^l\delta^j_i + S^{jl}_i - \gamma^{mk}\delta^l_i S^j_{mk} \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_j \end{pmatrix}. \tag{4.6}$$

The general discussion of the hyperbolicity and characteristic speed of the system (4.6) is hard, so hereafter we restrict ourselves to the case

$$R_{ij} = \kappa_1 \alpha \gamma_{ij}, \quad S^k_{ij} = \kappa_2 \beta^k \gamma_{ij}, \tag{4.7}$$

where we recover Eq. (4.3) by choosing $\kappa_1 = (\mu - 1)/4$ and $\kappa_2 = 0$. The eigenvalues of Eq. (4.6) then become

$$\lambda^l = (\beta^l, \beta^l, (1-\kappa_2)\beta^l$$
$$\pm \sqrt{\alpha^2\gamma^{ll}(1+4\kappa_1) + (\kappa_2\beta^l)^2}) \quad \text{(no sum over } l) \tag{4.8}$$

and the hyperbolicity of Eq. (4.6) can be classified as (i) symmetric hyperbolic when $\kappa_1 = 3/2$ and $\kappa_2 = 0$, (ii) strongly hyperbolic when $\alpha^2\gamma^{ll}(1+4\kappa_1) + \kappa_2^2(\beta^l)^2 > 0$ where $\kappa_1 \neq -1/4$, and (iii) weakly hyperbolic when $\alpha^2\gamma^{ll}(1+4\kappa_1) + \kappa_2^2(\beta^l)^2 \geqslant 0$.

For the case of Eq. (4.7) on a Minkowskii background metric, the linear order terms of the constraint propagation equations become

$$\partial_l \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_i \end{pmatrix} = \begin{pmatrix} 0 & -2ik_j \\ -(1/2)(1+4\kappa_1)ik_i & 0 \end{pmatrix} \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_j \end{pmatrix} \tag{4.9}$$

whose Fourier transform gives the eigenvalues

$$\Lambda^l = (0, 0, \pm\sqrt{-k^2(1+4\kappa_1)}). \tag{4.10}$$

That is (two zeros, two purely imaginary) for the standard ADM system and (four zeros) for the original ADM system. Therefore, according to our conjecture, the standard ADM system is expected to have better stability than the original ADM system.

C. Detweiler's system

1. Detweiler's system and its constraint amplification

Detweiler's modification to the ADM system [14] can be realized through one choice of the multipliers in Eqs. (4.1) and (4.2). He found that with a particular combination the evolution of the energy norm of the constraints, $\mathcal{H}^2 + \mathcal{M}^2$, can be negative definite when we apply the maximal slicing condition $K = 0$. (We will comment more on his criteria in Sec. IV C 2.) His adjustment can be written in our notation in Eqs. (4.1) and (4.2) as

$$P_{ij} = -L\alpha^3\gamma_{ij}, \tag{4.11}$$

$$R_{ij} = L\alpha^3(K_{ij} - (1/3)K\gamma_{ij}), \tag{4.12}$$

$$S^k_{ij} = L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}], \tag{4.13}$$

$$s^{kl}_{ij} = L\alpha^3[\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}], \tag{4.14}$$

everything else zero, where L is a constant. Detweiler's adjustment, Eqs. (4.12)–(4.14), does not put the constraint propagation equation in first-order form, so we cannot discuss hyperbolicity or the characteristic speed of the constraints.

For the Minkowskii background spacetime, the adjusted constraint propagation equations with above choice of multiplier become

$$\partial_l \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_i \end{pmatrix} = \begin{pmatrix} -2Lk^2 & -2ik_j \\ -(1/2)ik_i & -(L/2)k^2\delta^j_i - (L/6)k_ik_j \end{pmatrix}$$
$$\times \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_j \end{pmatrix}. \tag{4.15}$$

The eigenvalues of the Fourier transform are

$$\Lambda^l = (-(L/2)k^2, -(L/2)k^2, -(4L/3)k^2$$
$$\pm \sqrt{k^2[-1+(4/9)L^2k^2]}). \tag{4.16}$$

This indicates negative real eigenvalues if we choose small positive L.

We confirmed numerically, using perturbation on Minkowskii spacetime, that Detweiler's system presents better accuracy than the standard ADM system, but only for small positive L. See the Appendix.

2. Differences with Detweiler's requirement

We comment here on the differences between Detweiler's criteria for stable evolution and ours.

Detweiler calculated the $L2$ norm of the constraints C_ρ over the three-hypersurface and imposed the negative definiteness of its evolution:

$$\text{Detweiler's criteria} \Leftrightarrow \partial_t \int C_\rho C^\rho dV < 0, \quad \forall \text{ nonzero } C_\rho, \tag{4.17}$$

where $C_\rho C^\rho =: G^{\rho\sigma} C_\rho C_\sigma$ and $G_{\rho\sigma} = \mathrm{diag}[1, \gamma_{ij}]$ for the pair of $C_\rho = (\mathcal{H}, \mathcal{M}_i)$.

Assuming the constraint propagation to be $\partial_t \hat{C}_\rho = A^\sigma_\rho \hat{C}_\sigma$ in the Fourier components, the time derivative of the $L2$ norm can be written as

$$\partial_t(\hat{C}_\rho \hat{C}^\rho) = (A^{\rho\sigma} + \overline{A}^{\sigma\rho} + \partial_t \overline{G}^{\rho\sigma}) \hat{C}_\rho \overline{\hat{C}}_\sigma. \tag{4.18}$$

Together with the fact that the $L2$ norm is preserved by Fourier transform, we can say, for the case of a *static* background metric,

$$\text{Detweiler's criteria} \Leftrightarrow \text{eigenvalues of } (A + A^\dagger) \text{ are all negative } \forall\ k. \tag{4.19}$$

On the other hand,

$$\text{our criteria} \Leftrightarrow \text{eigenvalues of } A \text{ are all negative } \forall\ k. \tag{4.20}$$

Therefore for the case of a static background, Detweiler's criterion is stronger than ours. For example, the matrix

$$A = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix} \quad \text{where } a \text{ is constant}, \tag{4.21}$$

for the evolution system $(\hat{C}_1, \hat{C}_2)$, satisfies our criterion but not Detweiler's when $|a| \geqslant \sqrt{2}$. This matrix, however, gives asymptotical decay for $(\hat{C}_1, \hat{C}_2)$. Therefore we may say that Detweiler requires monotonic decay of the constraints, while we assume only asymptotical decay.

We remark that Detweiler's truncations on higher-order terms in the C norm corresponds to our perturbational analysis; both are based on the idea that the deviations from the constraint surface (the errors expressed as a nonzero constraint value) are initially small.

D. Another possible adjustment

1. Simplified Detweiler system

Similar to Detweiler's equation (4.11), we next consider only the adjustment

$$P_{ij} = \kappa_0 \alpha \gamma_{ij}, \tag{4.22}$$

all other multipliers being zero in Eqs. (4.1) and (4.2).

On the Minkowskii background, the Fourier components of the constraint propagation equation can be written as

$$\partial_t \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_i \end{pmatrix} = \begin{pmatrix} 2\kappa_0 k^2 & -2ik_j \\ -(1/2)ik_i & 0 \end{pmatrix} \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_j \end{pmatrix}, \tag{4.23}$$

and the eigenvalues of the coefficient matrix are

$$\Lambda^l = (0,0,\kappa_0 k^2 \pm \sqrt{k^2(-1 + \kappa_0^2 k^2)}). \tag{4.24}$$

That is, the amplification factors become (0, 0, two negative real values) for the choice of relatively small negative κ_0.

We also confirmed that this system works as desired. We give a numerical example in the Appendix.

2. Adjusting the Hamiltonian constraint system

Our final example is a combination of the one in Sec. IV B and that above, that is,

$$P_{ij} = \kappa_0 \alpha \gamma_{ij}, \tag{4.25}$$

$$R_{ij} = \kappa_1 \alpha \gamma_{ij}, \tag{4.26}$$

all other multipliers being zero in Eqs. (4.1) and (4.2). Similar to the previous one, the Fourier-transformed constraint propagation equation is

$$\partial_t \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_i \end{pmatrix} = \begin{pmatrix} 2\kappa_0 k^2 & -2ik_i \\ -(1/2)ik_i - 2\kappa_1 ik_i & 0 \end{pmatrix} \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_j \end{pmatrix}, \tag{4.27}$$

which gives the eigenvalues

$$\Lambda^l = (0,0,\kappa_0 k^2 \pm \sqrt{k^2(-1 + \kappa_0 k^2 - 4\kappa_1)}). \tag{4.28}$$

We can expect a similar asymptotical stable evolution by choosing κ_0 and κ_1, so as to make the eigenvalues (0, 0, two negative real values).

V. CONFORMAL-TRACELESS ADM SYSTEMS

The so-called ‘‘conformally decoupled traceless ADM formulation’’ was first developed by the Kyoto group [15]. After the rediscovery that this formulation is more stable than the standard ADM formula by Baumgarte and Shapiro [16], several groups began to use the CT-ADM formulation for their numerical codes and reported an advantage in stability [17,18]. Along with this conformal decomposition, several hyperbolic formulations have also been proposed [19–21], but they have not yet been applied to numerical simulations.

However, it is not yet clear why the CT-ADM formulation gives better stability than the ADM formulation. The Potsdam group [22] found that the eigenvalues of the CT-ADM *evolution equations* have fewer ‘‘zero eigenvalues’’ than those of the ADM system, and they conjectured that instability can be caused by ‘‘zero eigenvalues’’ that violate the ‘‘gauge mode.’’ Miller [23] applied von Neumann's stability analysis to plane wave propagation and reported that the CT-ADM formulation has a wider range of parameters that give us stable evolution. These studies provide support for the CT-ADM formulation in some sense, but on the other hand, it is also shown that an example of an ill-posed solution in the CT-ADM formulation exists (as well in the ADM formulation) [24].

Here, we apply our constraint propagation analysis to this CT-ADM system.

A. CT-ADM equations

Since one reported feature of the CT-ADM formulation is the use of the momentum constraint on the RHS of the evolution equations [22], we here present the set of CT-ADM equations carefully for such a replacement of the constraint terms.

The widely used notation [15,16] is to use the variables $(\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$ instead of the standard ADM variables (γ_{ij}, K_{ij}), where

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij}, \tag{5.1}$$

$$\tilde{A}_{ij} = e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \tag{5.2}$$

$$\tilde{\Gamma}^i = \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk}, \tag{5.3}$$

and we impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions. The set of evolution equations becomes

$$(\partial_t - \mathcal{L}_\beta)\phi = (-1/6)\alpha K, \tag{5.4}$$

$$(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij}, \tag{5.5}$$

$$\begin{aligned}(\partial_t - \mathcal{L}_\beta)K = {}& \alpha(1-\kappa_1)R^{(3)} + \alpha(1-\kappa_1)K^2 + \alpha\kappa_1\tilde{A}_{ij}\tilde{A}^{ij} \\ & + (1/3)\alpha\kappa_1 K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha),\end{aligned} \tag{5.6}$$

$$\begin{aligned}(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} = {}& -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} \\ & - e^{-4\phi}\alpha(1/3)\gamma_{ij}(1-\kappa_3)R^{(3)} \\ & + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j) + e^{-4\phi}\alpha(1/3)\gamma_{ij}\kappa_3 \\ & \times[-\tilde{A}_{kl}\tilde{A}^{kl} + (2/3)K^2],\end{aligned} \tag{5.7}$$

$$\begin{aligned}\partial_t\tilde{\Gamma}^i = {}& -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\kappa_2\alpha(\partial_j K)\tilde{\gamma}^{ij} \\ & + 12\kappa_2\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}^j_k(\partial_j\tilde{\gamma}^{ik}) \\ & - 2\kappa_2\alpha\tilde{\Gamma}^k_{lj}\tilde{A}^j_k\tilde{\gamma}^{il} - 2(1-\kappa_2)\alpha(\partial_j\tilde{A}_{kl})\tilde{\gamma}^{ik}\tilde{\gamma}^{jl} \\ & + 2\alpha(1-\kappa_2)\tilde{A}^i_j\tilde{\Gamma}^j - \partial_j\big(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) \\ & - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\big),\end{aligned} \tag{5.8}$$

where $\mathcal{L}_\beta$ is the Lie derivative along the shift vector β^i, and $R^{(3)}$ is the three-metric scalar curvature. Here we introduced parameters κ which show where we replace the terms with constraints. For example $(\kappa_1, \kappa_2, \kappa_3) = (0,0,0)$ is the case of no replacement [the standard ADM equations expressed using Eqs. (5.1)–(5.3)], while Baumgarte and Shapiro [16] use $(\kappa_1, \kappa_2, \kappa_3) = (1,1,0)$.

The constraint equations in the CT-ADM system can be expressed as

$$\begin{aligned}\mathcal{H} = {}& e^{-4\phi}\tilde{R}^{(3)} - 8e^{-4\phi}\tilde{\gamma}^{ij}(\partial_i\partial_j\phi) - 8e^{-4\phi}\tilde{\gamma}^{ij}(\partial_i\phi)(\partial_j\phi) \\ & + 8e^{-4\phi}(\partial_i\phi)\tilde{\Gamma}^i + (2/3)K^2 - \tilde{A}_{ij}\tilde{A}^{ij},\end{aligned} \tag{5.9}$$

$$\mathcal{M}_i = (\partial_j\tilde{A}_{ki})\tilde{\gamma}^{kj} - (2/3)(\partial_i K) - \tilde{A}_{ji}\tilde{\Gamma}^j + 6(\partial_j\phi)\tilde{A}^j_i - \tilde{\Gamma}^k_{ji}\tilde{A}^j_k, \tag{5.10}$$

$$\mathcal{G}^i = \tilde{\Gamma}^i + \partial_j\tilde{\gamma}^{ji}. \tag{5.11}$$

Here $\mathcal{H}, \mathcal{M}$ are the Hamiltonian and momentum constraints and the third one, $\mathcal{G}$, is a consistency relation due to the algebraic definition of Eq. (5.3).

B. Constraint propagation equations of the CT-ADM formulation

Similar to the ADM cases, we here show the propagation equations for Eqs. (5.9)–(5.11). The expressions are given using Eqs. (3.5) and (3.6), but we have to be careful to keep using the new variable Γ_i wherever it appears. Following [16], we express $\tilde{R}^{(3)}_{ij}$ as

$$\tilde{R}^{(3)}_{ij} = -(1/2)\tilde{\gamma}^{lm}(\partial_l\partial_m\tilde{\gamma}_{ij}) + (1/2)\tilde{\gamma}_{ki}\partial_j\tilde{\Gamma}^k + (1/2)\tilde{\gamma}_{kj}\partial_i\tilde{\Gamma}^k + (1/2)\tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + \tilde{\gamma}^{lm}\tilde{\Gamma}^k_{li}\tilde{\Gamma}_{jkm} + \tilde{\gamma}^{lm}\tilde{\Gamma}^k_{lj}\tilde{\Gamma}_{ikm} + \tilde{\gamma}^{lm}\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}. \tag{5.12}$$

The constraint propagation equations, then, are obtained by straightforward calculations as

$$\begin{aligned}\partial_t\mathcal{H} = {}& \beta^j(\partial_j\mathcal{H}) - 2\alpha e^{-4\phi}\tilde{\gamma}^{ij}(\partial_i\mathcal{M}_j) + 2\alpha K\mathcal{H} - 2\alpha e^{-4\phi}(\partial_i\tilde{\gamma}^{ij})\mathcal{M}_j - 4\alpha e^{-4\phi}(\partial_i\phi)\tilde{\gamma}^{ij}\mathcal{M}_j - 4e^{-4\phi}\tilde{\gamma}^{ij}(\partial_j\alpha)\mathcal{M}_i \\ & + 2\kappa_2 e^{-4\phi}(\partial_i\alpha)\tilde{\gamma}^{ij}\mathcal{M}_j + 2\kappa_2 e^{-4\phi}\alpha(\partial_i\tilde{\gamma}^{ij})\mathcal{M}_j + 2\kappa_2 e^{-4\phi}\alpha\tilde{\gamma}^{ij}(\partial_i\mathcal{M}_j) + 16\kappa_2\alpha e^{-4\phi}(\partial_i\phi)\tilde{\gamma}^{ij}\mathcal{M}_j - (4/3)\kappa_1\alpha K\mathcal{H},\end{aligned} \tag{5.13}$$

$$\partial_t \mathcal{M}_i = -(1/2)\alpha(\partial_i \mathcal{H}) + \beta^j(\partial_j \mathcal{M}_i) + \alpha K \mathcal{M}_i - (\partial_i \alpha)\mathcal{H} - 4\beta^j(\partial_i \phi)\mathcal{M}_j + \beta^k \tilde{\gamma}^{jl}(\partial_i \tilde{\gamma}_{lk})\mathcal{M}_j + (\partial_i \beta_k)e^{-4\phi}\tilde{\gamma}^{kj}\mathcal{M}_j$$
$$+(1/3)(2\kappa_1+\kappa_3)(\partial_i \alpha)\mathcal{H} + (1/3)(2\kappa_1+\kappa_3)\alpha(\partial_i \mathcal{H}) - 2\kappa_2 \alpha \tilde{A}_i^j \mathcal{M}_j - (1/3)\kappa_3 \alpha \mathcal{G}^j \tilde{\gamma}_{ji}\mathcal{H} + 2\kappa_3 \alpha(\partial_i \phi)\mathcal{H}, \tag{5.14}$$

$$\partial_t \mathcal{G}^i = 2\tilde{A}^i_j \mathcal{G}^j + 2\kappa_2 \alpha \tilde{\gamma}^{ij}\mathcal{M}_j . \tag{5.15}$$

These form a first-order system, and the characteristic part can be extracted as

$$\partial_t \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_i \\ \mathcal{G}^i \end{pmatrix} \cong \begin{pmatrix} \beta^l & 2(-1+\kappa_2)\alpha\gamma^{lj} & 0 \\ [(2/3)\kappa_1+(1/3)\kappa_3-(1/2)]\alpha\delta^l_i & \beta^l \delta^j_i & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_j \\ \mathcal{G}^j \end{pmatrix}, \tag{5.16}$$

whose characteristic speeds are

$$\lambda^l = (0,0,0,\beta^l,\beta^l,\beta^l \pm \alpha\sqrt{\gamma^{ll}(1-\kappa_2)[1-(4/3)\kappa_1-(2/3)\kappa_3]}) \;\; (\text{no sum over } l). \tag{5.17}$$

By analyzing the reality of the eigenvalues, the diagonalizability of the characteristic matrix, and the possibility of a symmetric characteristic matrix, we can classify the hyperbolicity of the system (5.16) as

$$\text{weakly hyperbolic} \Leftrightarrow (1-\kappa_2)[1-(4/3)\kappa_1-(2/3)\kappa_3] \geqslant 0, \tag{5.18}$$

$$\text{strongly hyperbolic} \Leftrightarrow (1-\kappa_2) = [1-(4/3)\kappa_1-(2/3)\kappa_3] = 0,$$

$$\text{or } (1-\kappa_2)[1-(4/3)\kappa_1-(2/3)\kappa_3] > 0, \tag{5.19}$$

$$\text{symmetric hyperbolic} \Leftrightarrow (-1+\kappa_2) = [1-(4/3)\kappa_1-(2/3)\kappa_3]. \tag{5.20}$$

That is, for the nonadjusted system $(\kappa_1,\kappa_2,\kappa_3)=(0,0,0)$, constraint propagation forms a strongly hyperbolic system, while the Baumgarte-Shapiro form gives only weak hyperbolicity. (We note that the first-order version of the CT-ADM system by Frittelli and Reula [20] has also well-posed constraint propagation equations.)

C. Amplification factors on a Minkowskii background

For a Minkowskii background, the constraint propagation equations at linear order become

$$\partial_t \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_i \\ {}^{(1)}\hat{\mathcal{G}}^i \end{pmatrix} = \begin{pmatrix} 0 & 2(\kappa_2-1)ik_j & 0 \\ [(2/3)\kappa_1+(1/3)\kappa_3-(1/2)]ik_i & 0 & 0 \\ 0 & 2\kappa_2\delta^{ij} & 0 \end{pmatrix} \begin{pmatrix} {}^{(1)}\hat{\mathcal{H}} \\ {}^{(1)}\hat{\mathcal{M}}_j \\ {}^{(1)}\hat{\mathcal{G}}^j \end{pmatrix}. \tag{5.21}$$

The constraint amplification factor becomes

$$\Lambda^l = (0,0,0,0,0,\pm\sqrt{-k^2(1-\kappa_2)[1-(4/3)\kappa_1-(2/3)\kappa_3]}). \tag{5.22}$$

That is, Λ^l are either zero, purely imaginary, or ± real numbers. For the nonadjusted system they are zero and purely imaginary [that is, the same as Eq. (3.11)], while the Baumgarte-Shapiro form gives us all zero eigenvalues. Therefore, from our point of view, these two are not very different in their characterization of constraint propagation.

VI. CONCLUDING REMARKS

We have reviewed ADM systems from the point of view of the adjustment of the dynamical equations by constraint terms. We have shown that characteristic speeds and amplification factors of the constraint propagation change due to their adjustments. We compared the equations for the ADM, adjusted ADM, and conformal traceless ADM (CT-ADM) systems, and tried to find a system that is robust for violation of the constraints, which we can call an ‘‘asymptotically constrained’’ system.

We conjectured that if the amplification factors (eigenvalues of the coefficient matrix of the Fourier-transformed constraint propagation equations) are negative or purely imaginary, then the system has better asymptotically constrained features than a system where they are not. According to our conjecture, the standard ADM system is expected to have better stability than the original ADM system (no growing mode in amplification factors). Detweiler's modified ADM system, which is one particular choice of adjustment, definitely has good properties in that there are no growing modes

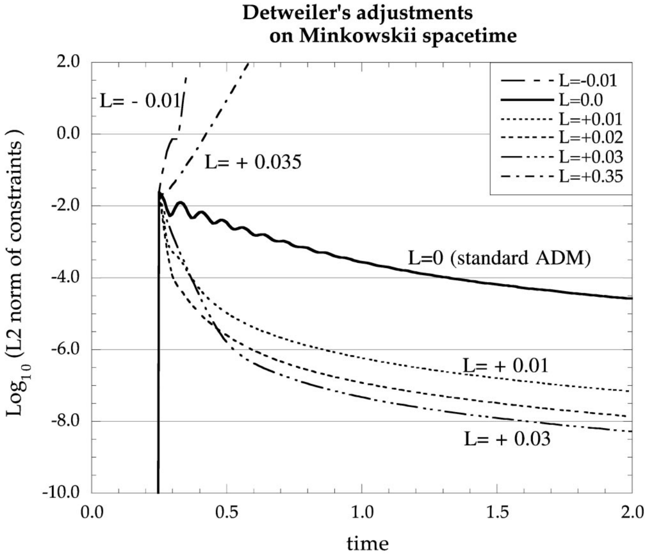

FIG. 1. Demonstration of the Detweiler's modified ADM system on Minkowskii background spacetime (the system of Sec. IV C). The $L2$ norm of the constraints is plotted as a function of time. Artificial error was added at $t=0.25$. Here L is the parameter used in Eqs. (4.12)–(4.14). We see that the evolution is asymptotically constrained for small $L>0$.

in the amplification factors. We also showed that this can be obtained by a simpler choice of adjustment multipliers.

We also studied the CT-ADM system which is popular with numerical relativists nowadays. However, from our point of view, we do not see any particular advantages for the CT-ADM system over the standard ADM system.

The reader might ask why we can break the time-reversal invariant feature of the evolution equations by a particular choice of adjusting multipliers against the fact that the ''Einstein equations'' are time-reversal invariant. This question can be answered by the following. If we take a time-reversal transformation $(\partial_t \to -\partial_t)$, the Hamiltonian constraint and the evolution equations of K_{ij} keep their signatures, while the momentum constraints and the evolution equations of γ_{ij} change their signatures. Therefore if we adjust the γ_{ij} equations using the Hamiltonian constraint and/or K_{ij} equations using the momentum constraints (supposing the multiplier has + parity), then we can break the time-reversal invariant feature of the ''ADM equations.'' In fact, the examples we obtained all obey this rule. The CT-ADM formulation keeps its signature against the adjustments we made, so that we cannot find any additional advantage from this analysis.

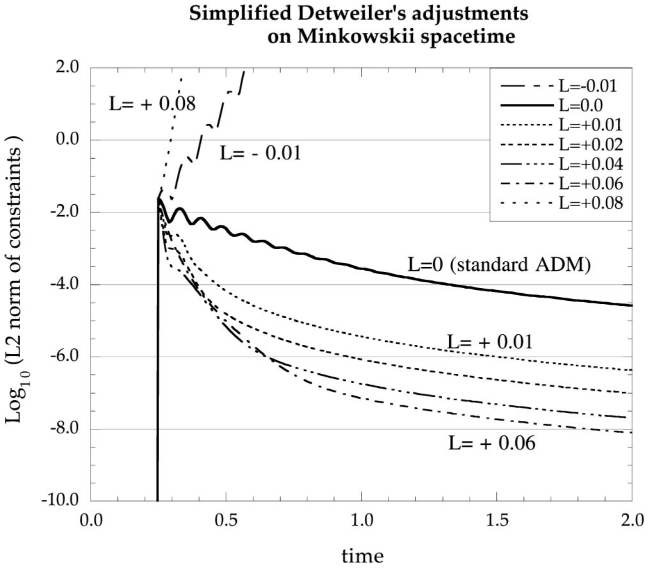

FIG. 2. Demonstration of the simplified Detweiler's modified ADM system on Minkowskii background spacetime (the system of Sec. IV D 1). For comparison with Fig. 1, we set $L=-\kappa_0$, where κ_0 is the parameter used in Eq. (4.22). We see the evolution is asymptotically constrained for small $L>0$.

Considering the constraint propagation equations is a kind of substitutional approach for numerical integrations of the dynamical equations. However, this might be one of the main directions for our future research, as Friedrich and Nagy [25] impose a zero speed of the constraint propagation as the first principle when they considered the initial boundary value problem of the Einstein equations [26].

We are now applying our discussion to more general spacetimes and trying to find guidelines for choosing appropriate gauge conditions from analysis of the constraint propagation equations. These efforts will be reported elsewhere [27].

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APPENDIX: NUMERICAL DEMONSTRATIONS OF ADJUSTED ADM SYSTEMS

We here show two numerical demonstrations of the adjusted ADM systems that were discussed in Sec. IV C (Detweiler's modified ADM system) and Sec. IV D 1 (simplified version).

Detweiler's adjustment, Eqs. (4.12)–(4.14), can be parametrized by a constant L, and our prediction from the amplification factor on the Minkowskii background is that this system will be asymptotically constrained for small positive L. Figure 1 is a demonstration of this system. We evolved Minkowskii spacetime numerically in a plane-symmetric spacetime and added artificial error in the middle of the evolution. Our numerical integration uses the Brailovskaya scheme, which was described in detail in our previous paper [6]. The code passes convergence tests and the plots are for 401 gridpoints in the range $x=[0,10]$, and we fix the time grid $\Delta t=0.2\Delta x$. The error was introduced as a pinpoint kick, in the form of $\Delta g_{yy}=10^{-3}$ at $x=5.0$ and $t=0.25$. We monitor how the $L2$ norm of the constraints $(\mathcal{H}^2+\mathcal{M}_x^2)$

behaves. From Fig. 1, we see that a small positive L reduces the $L2$ norm in time, which is the asymptotically constrained feature we expected. The case of slightly larger L will make the system unstable. This is the same feature we have seen in the numerical demonstration of the λ system or adjusted Maxwell system and Ashtekar system [11]; for that case the upper bound of the multiplier can be explained by violation of the Courant-Friedrich-Lewy condition, while in this system we cannot calculate the exact characteristics since the system is not first order.

Similarly, we plotted in Fig. 2 the case of a simplified version (the system of Sec. IV D 1). We see the desired feature again by changing the parameter κ_0 that appears in Eq. (4.22).

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