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CONSTRUCTING HYPERBOLIC SYSTEMS IN THE ASHTEKAR FORMULATION OF GENERAL RELATIVITY

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Hyperbolic formulations of the equations of motion are essential technique for proving the well-posedness of the Cauchy problem of a system, and are also helpful for implementing stable long time evolution in numerical applications. We, here, present three kinds of hyperbolic systems in the Ashtekar formulation of general relativity for Lorentzian vacuum spacetime. We exhibit several (I) weakly hyperbolic, (II) diagonalizable hyperbolic, and (III) symmetric hyperbolic systems, with each their eigenvalues. We demonstrate that Ashtekar's original equations form a weakly hyperbolic system. We discuss how gauge conditions and reality conditions are constrained during each step toward constructing a symmetric hyperbolic system.

1. Introduction

Developing hyperbolic formulations of the Einstein equation is growing into an important research areas in general relativity.[1] These formulations are used in the analytic proof of the existence, uniqueness and stability (well-posedness) of the solutions of the Einstein equation.[2] So far, several first order hyperbolic formulations have been proposed; some of them are flux conservative,[3] some of them are symmetrizable or symmetric hyperbolic systems.[4–11] The recent interest in hyperbolic formulations arises from their application to numerical relativity. One of the most useful features is the existence of characteristic speeds in hyperbolic systems. We expect more stable evolutions and expect implements boundary conditions in their numerical simulation. Some numerical tests have been reported along this direction.[12–14]

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Ashtekar's formulation of general relativity[15] has many advantages. By using his special pair of variables, the constraint equations which appear in the theory become low-order polynomials, and the theory has the correct form for gauge theoretical interpretation. These features suggest the possibility for developing a nonperturbative quantum description of gravity. Classical applications of the Ashtekar's formulation have also been discussed by several authors. For example, we[16] discussed the reality conditions for the metric and triad and proposed a new set of variables for Lorentzian dynamics. We[17] also showed an example of passing a degenerate point in 3-space by locally relaxing the reality condition. Although there is always a problem of reality conditions in applying Ashtekar formulation to dynamics, we think that this new approach is quite attractive, and broadens our possibilities to attack dynamical issues.

A symmetric hyperbolic formulation of Ashtekar's variables was first developed by Iriondo, Leguizamón and Reula (ILR).[18] They use *anti*-Hermiticity of the principal symbol for defining their *symmetric* system. Unfortunately, in their first short paper,[18] they did not discuss the consistency of their system with the reality conditions, which are crucial in the study of the Lorentzian dynamics using the Ashtekar variables. We considered this point in Ref. 19, and found that there are strict reality constraints (alternatively they can be interpreted as gauge conditions). Note that we primarily use the Hermiticity of the characteristic matrix to define a symmetric hyperbolic system, which we think the more conventional notation. The difference between these definitions of symmetric hyperbolicity is commented in Appendix C.

The dynamical equations in the Ashtekar formulation of general relativity are themselves quite close to providing a hyperbolic formulation. As we will show in Sec. 4, the original set of equations of motion is a first-order (weakly) hyperbolic system. One of the purposes of this paper is to develop several hyperbolic systems based on the Ashtekar formulation for Lorentzian vacuum spacetime, and discuss how gauge conditions and reality conditions are to be implemented. We categorize hyperbolic systems into three classes: (I) weakly hyperbolic (system has all real eigenvalues), (II) diagonalizable hyperbolic (characteristic matrix is diagonalizable), and (III) symmetric hyperbolic system. These three classes have the relation (III) $\in$ (II) $\in$ (I), and are defined in detail in Sec. 2. As far as we know, only a symmetric hyperbolic systems provide a fully well-posed initial value formulation of partial differential equations systems. However, there are two reasons to consider the two other classes of hyperbolic systems, (I) and (II). First, as we found in our previous short paper,[19] the symmetric hyperbolic system we obtained using Ashtekar's variables has strict restrictions on the gauge conditions, while the original Ashtekar equations constitute a weakly hyperbolic system. We are interested in these differences, and show how additional constraints appear during the steps toward constructing a symmetric hyperbolic system. Second, many numerical experiments show that there are several advantages if we apply a certain form of hyperbolic formulation. Therefore, we think that presenting these three hyperbolic systems is valuable to stimulate the studies in this field. To aid in possibly applying these systems in numerical applications, we present characteristic speeds of each system we construct.

ILR, in their second paper,[20] expand their previous discussion[18] concerning reality conditions during evolution. They demand that the metric is real-valued (metric reality condition), and use the freedom of internal rotation during the time evolution to set up their soldering form so that it forms an anti-Hermitian principle symbol, which is their basis to characterize the system symmetric. However, we adopt the view that re-defining inner product of the fundamental variables introduces additional complications. In our procedure, we first fix the inner product to construct a symmetric hyperbolic system. As we will describe in Sec. 5, our symmetric hyperbolic system then requires a reality condition on the triad (triad reality condition), and in order to be consistent with its secondary condition we need to impose further gauge conditions. The lack of these constraints in ILR, we believe, comes from their incomplete treatment of a new gauge freedom, so-called *triad lapse* $\mathcal{A}^a_0$ (discussed in Sec. 3), for dynamical evolutions in the Ashtekar formulation. In Appendix C, we show that ILR's proposal to use internal rotation to re-set triad reality does not work if we adopt our conventional definition of hyperbolicity.

The layout of this paper is as follows: In Sec. 2, we define the three kinds of hyperbolic systems which are considered in this paper. In Sec. 3, we briefly review Ashtekar's formulation and the way of handling reality conditions. The following Secs. 4 and 5 are devoted to constructing hyperbolic systems. Summary and discussion are in Sec. 6. Appendix A supplements our proof of the uniqueness of our symmetric hyperbolic system. Appendices B and C are comments on ILR's treatment of the reality conditions.

2. Three Definitions of Hyperbolic Systems

We start by defining the hyperbolic systems which are used in this paper.

Definition 2.1. *We assume a certain set of (complex) variables* u_α $(\alpha = 1, \ldots, n)$ *forms a first-order (quasi-linear) partial differential equation system,*

$$\partial_t u_\alpha = J^{l\beta}{}_\alpha(u)\partial_l u_\beta + K_\alpha(u)\,, \tag{2.1}$$

where J *(the characteristic matrix) and* K *are functions of* u *but do not include any derivatives of* u*. We say that the system* (2.1) *is:*

(I) weakly hyperbolic, *if all the eigenvalues of the characteristic matrix are real.*[21]
(II) diagonalizable hyperbolic, *if the characteristic matrix is diagonalizable and has all real eigenvalues.*[22]
(III) symmetric hyperbolic, *if the characteristic matrix is a Hermitian matrix.*[7,23]

Here we state each definition more concretely. We treat $J^{l\beta}{}_{\alpha}$ as a $n \times n$ matrix when the l-index is fixed. The following properties of these matrices are for every basis of l-index.

We say λ^l is an eigenvalue of $J^{l\beta}{}_{\alpha}$ when the characteristic equation, $\det(J^{l\beta}{}_{\alpha} - \lambda^l \delta^{\beta}{}_{\alpha}) = 0$, is satisfied. The eigenvectors, p^{α}, are given by solving $J^{l\alpha}{}_{\beta} p^{l\beta} = \lambda^l p^{l\alpha}$.

The weakly hyperbolic system, (I), is obtained when J^l has *real spectrum* for every l, that is, when this characteristic equation can be divided by n real first-degree factors. For any single equation system, the Cauchy problem under weak hyperbolicity is not, in general, C^{∞} well-posed, while it is solvable in the class of the real analytic functions and in some suitable Gevrey classes, provided that the coefficients of the principal part are sufficiently smooth.

The diagonalizable hyperbolic system, (II), is obtained when J is *real diagonalizable*, that is, when there exists complex regular matrix P^l such that $((P^l)^{-1})^{\alpha}{}_{\gamma} J^{l\gamma}{}_{\delta} P^{l\delta}{}_{\beta}$ is real diagonal matrix for every l. We can construct characteristic curves if the system is in this class. This system is often used as a model in the studies of well-posedness in coupled linear hyperbolic system. (This is the same as *strongly hyperbolic* system as defined by some authors,[24,25] but we use the word *diagonalizable* since there exist other definitions for *strongly hyperbolic* systems.[27])

In order to define the symmetric hyperbolic system, (III), we need to declare an inner product $\langle u|u \rangle$ to judge whether $J^{l\beta}{}_{\alpha}$ is Hermitian. In other words, we are required to define the way of lowering the index α of u^{α}. We say $J^{l\beta}{}_{\alpha}$ is Hermitian with respect to this index rule, when $J^{l}{}_{\beta\alpha} = \bar{J}^{l}{}_{\alpha\beta}$ for every l, where the overhead bar denotes complex conjugate.

Any Hermitian matrix is real diagonalizable, so that (III) $\in$ (II) $\in$ (I). There are other definitions of hyperbolicity; such as *strictly hyperbolic* or *effectively hyperbolic*, if all eigenvalues of the characteristic matrix are real and distinct (and nonzero for the latter). These definitions are stronger than (II), but exhibit no inclusion relation with (III). In this paper, however, we only consider (I)–(III) above.

The symmetric system gives us the energy integral inequalities, which are the primary tools for studying well-posedness of the system. As was discussed by Geroch,[26] most physical systems can be expressed as symmetric hyperbolic systems.

3. Ashtekar Formulation

3.1. *Variables and equations*

The key feature of Ashtekar's formulation of general relativity[15] is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write the metric $g_{\mu\nu}$ using the tetrad E^I_μ, with E^I_μ satisfying the gauge condition $E^0_a = 0$. Define its inverse, E^μ_I, by $g_{\mu\nu} = E^I_\mu E^J_\nu \eta_{IJ}$ and $E^\mu_I := E^J_\nu g^{\mu\nu} \eta_{IJ}$, where we use μ, $\nu = 0, \dots, 3$ and $i, j = 1, \dots, 3$ as spacetime indices, while I, $J = (0), \dots, (3)$ and a, $b = (1), \dots, (3)$ are SO(1,3), SO(3) indices respectively. We raise and lower $\mu, \nu, \dots$ by $g^{\mu\nu}$ and $g_{\mu\nu}$ (the Lorentzian metric); $I, J, \dots$ by $\eta^{IJ} = \mathrm{diag}(-1, 1, 1, 1)$ and η_{IJ};

$i, j, \ldots$ by γ^{ij} and γ_{ij} (the 3-metric); $a, b, \ldots$ by δ^{ab} and δ_{ab}. We also use volume forms ϵ_{abc}: $\epsilon_{abc}\epsilon^{abc} = 3!$. We define SO$(3, C)$ self-dual and antiself-dual connections

$$ {}^{\pm}\mathcal{A}^a_\mu := \omega^{0a}_\mu \mp \left(\frac{i}{2}\right) \epsilon^a{}_{bc}\omega^{bc}_\mu \,, \tag{3.1} $$

where ω^{IJ}_μ is a spin connection 1-form (Ricci connection), $\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu E^J_\nu$. Ashtekar's plan is to use only a self-dual part of the connection ${}^{+}\mathcal{A}^a_\mu$ and to use its spatial part ${}^{+}\mathcal{A}^a_i$ as a dynamical variable. Hereafter, we simply denote ${}^{+}\mathcal{A}^a_\mu$ as $\mathcal{A}^a_\mu$.

The lapse function, N, and shift vector, N^i, both of which we treat as real-valued functions, are expressed as $E^\mu_0 = (1/N, -N^i/N)$. This allows us to think of E^μ_0 as a normal vector field to Σ spanned by the condition $t = x^0 = \text{const.}$, which plays the same role as that of Arnowitt–Deser–Misner (ADM) formulation. Ashtekar treated the set $(\tilde{E}^i_a,\ \mathcal{A}^a_i)$ as basic dynamical variables, where $\tilde{E}^i_a$ is an inverse of the densitized triad defined by

$$ \tilde{E}^i_a := e E^i_a \,, \tag{3.2} $$

where $e := \det E^a_i$ is a density. This pair forms the canonical set.

In the case of pure gravitational spacetime, the Hilbert action takes the form

$$ S = \int d^4x \left[\left(\partial_t \mathcal{A}^a_i\right)\tilde{E}^i_a + \left(\frac{i}{2}\right) \underset{\sim}{N} \tilde{E}^i_a \tilde{E}^j_b F^c_{ij} \epsilon^{ab}{}_c - e^2 \Lambda \underset{\sim}{N} - N^i F^a_{ij}\tilde{E}^j_a + \mathcal{A}^a_0 \mathcal{D}_i \tilde{E}^i_a \right], \tag{3.3} $$

where $\underset{\sim}{N} := e^{-1}N$, $F^a_{\mu\nu} := (d\mathcal{A}^a)_{\mu\nu} - (i/2)\epsilon^a{}_{bc}(\mathcal{A}^b \wedge \mathcal{A}^c)_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu - i\epsilon^a{}_{bc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu$ is the curvature 2-form, Λ is the cosmological constant, $\mathcal{D}_i \tilde{E}^j_a := \partial_i \tilde{E}^j_a - i\epsilon_{ab}{}^c \mathcal{A}^b_i \tilde{E}^j_c$, and $e^2 = \det \tilde{E}^i_a = (\det E^a_i)^2$ is defined to be $\det \tilde{E}^i_a = (1/6)\epsilon^{abc}\underset{\sim}{\epsilon}_{ijk}\tilde{E}^i_a \tilde{E}^j_b \tilde{E}^k_c$, where $\epsilon_{ijk} := \epsilon_{abc}E^a_i E^b_j E^c_k$ and $\underset{\sim}{\epsilon}_{ijk} := e^{-1}\epsilon_{ijk}$ [When $(i, j, k) = (1, 2, 3)$, we have $\epsilon_{ijk} = e$, $\underset{\sim}{\epsilon}_{ijk} = 1$, $\epsilon^{ijk} = e^{-1}$, and $\tilde{\epsilon}^{ijk} = 1$.].

Varying the action with respect to the non-dynamical variables $\underset{\sim}{N}$, N^i and $\mathcal{A}^a_0$ yields the constraint equations,

$$ \mathcal{C}_H := \left(\frac{i}{2}\right) \epsilon^{ab}{}_c \tilde{E}^i_a \tilde{E}^j_b F^c_{ij} - \Lambda \det \tilde{E} \approx 0 \,, \tag{3.4} $$

$$ \mathcal{C}_{Mi} := -F^a_{ij}\tilde{E}^j_a \approx 0 \,, \tag{3.5} $$

$$ \mathcal{C}_{Ga} := \mathcal{D}_i \tilde{E}^i_a \approx 0 \,. \tag{3.6} $$

The equations of motion for the dynamical variables ($\tilde{E}^i_a$ and $\mathcal{A}^a_i$) are

$$ \partial_t \tilde{E}^i_a = -i\mathcal{D}_j\left(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b\right) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c \,, \tag{3.7} $$

$$ \partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i \,, \tag{3.8} $$

where $\mathcal{D}_j X^{ji}_a := \partial_j X^{ji}_a - i\epsilon_{ab}{}^c \mathcal{A}^b_j X^{ji}_c$, for $X^{ij}_a + X^{ji}_a = 0$.

In order to construct metric variables from the variables $(\tilde{E}^i_a, \mathcal{A}^a_i, \underset{\sim}{N}, N^i)$, we first prepare tetrad E^μ_I as $E^\mu_0 = \left(1/e\underset{\sim}{N}, -N^i/e\underset{\sim}{N}\right)$ and $E^\mu_a = (0, \tilde{E}^i_a/e)$. Using them, we obtain metric $g^{\mu\nu}$ such that

$$g^{\mu\nu} := E^\mu_I E^\nu_J \eta^{IJ} \,. \tag{3.9}$$

3.2. *Reality conditions*

The metric (3.9), in general, is not real-valued in the Ashtekar formulation. To ensure that the metric is real-valued, we need to impose real lapse and shift vectors together with two conditions (the metric reality condition);

$$\mathrm{Im}(\tilde{E}^i_a \tilde{E}^{ja}) = 0 \,, \tag{3.10}$$

$$\mathrm{Re}\left(\epsilon^{abc}\tilde{E}^k_a \tilde{E}^{(i}_b \mathcal{D}_k \tilde{E}^{j)}_c\right) = 0 \,, \tag{3.11}$$

where the latter comes from the secondary condition of reality of the metric $\mathrm{Im}\{\partial_t(\tilde{E}^i_a \tilde{E}^{ja})\} = 0$,[28] and we assume $\det \tilde{E} > 0$ (see Ref. 16). These metric reality conditions, (3.10) and (3.11), are automatically preserved during the evolution if the variables satisfy the conditions on the initial data.[28,16]

For later convenience, we also prepare stronger reality conditions. These conditions are

$$\mathrm{Im}(\tilde{E}^i_a) = 0 \,, \quad \text{and} \tag{3.12}$$

$$\mathrm{Im}(\partial_t \tilde{E}^i_a) = 0 \,, \tag{3.13}$$

and we call them the "primary triad reality condition" and the "secondary triad reality condition," respectively. Using the equations of motion of $\tilde{E}^i_a$, the gauge constraint (3.4)–(3.6), the metric reality conditions (3.10), (3.11) and the primary condition (3.12), we see that (3.13) is equivalent to[16]

$$\mathrm{Re}(\mathcal{A}^a_0) = \partial_i(\underset{\sim}{N})\tilde{E}^{ia} + \frac{1}{2e}E^b_i \underset{\sim}{N} \tilde{E}^{ja} \partial_j \tilde{E}^i_b + N^i \,\mathrm{Re}(\mathcal{A}^a_i) \,, \tag{3.14}$$

or with undensitized variables,

$$\mathrm{Re}(\mathcal{A}^a_0) = \partial_i(N)E^{ia} + N^i \,\mathrm{Re}(\mathcal{A}^a_i) \,. \tag{3.15}$$

From this expression we see that the secondary triad reality condition restricts the three components of "triad lapse" vector $\mathcal{A}^a_0$. Therefore (3.14) is not a restriction on the dynamical variables ($\tilde{E}^i_a$ and $\mathcal{A}^a_i$) but on the slicing, which we should impose on each hypersurface. Thus the secondary triad reality condition does not restrict the dynamical variables any further than the secondary metric condition does.

Throughout this paper, we basically impose metric reality condition. We assume that initial data of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ for evolution are solved so as to satisfy all three constraint equations and metric reality condition (3.10) and (3.11). Practically, this is obtained, for example, by solving ADM constraints and by transforming a set of initial data to Ashtekar's notation.

3.3. *Characteristic matrix*

We shall see how the definitions of hyperbolic systems in Sec. 2 can be applied for Ashtekar's equations of motion (3.7) and (3.8). Since both dynamical variables, $\tilde{E}^i_a$ and $\mathcal{A}^a_i$, have 9 components each (spatial index: $i = 1, 2, 3$ and SO(3) index: $a = (1), (2), (3)$), the combined set of variables, $u^\alpha = (\tilde{E}^i_a, \mathcal{A}^a_i)$, has 18 components. Ashtekar's formulation itself is in the first-order (quasilinear) form in the sense of (2.1), but is not in a symmetric hyperbolic form.

We start by writing the principal part of the Ashtekar's evolution equations as

$$\partial_t \begin{bmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \end{bmatrix} \cong \begin{bmatrix} A^l{}_a{}^{bi}{}_j & B^l{}_{ab}{}^{ij} \\ C^{lab}{}_{ij} & D^{la}{}_{bi}{}^j \end{bmatrix} \partial_l \begin{bmatrix} \tilde{E}^j_b \\ \mathcal{A}^b_j \end{bmatrix}, \tag{3.16}$$

where $\cong$ means that we have extracted only the terms which appear in the principal part of the system. We name these components as A, B, C and D for later convenience.

The characteristic equation becomes

$$\det \begin{pmatrix} A^l{}_a{}^{bi}{}_j - \lambda^l \delta^b_a \delta^i_j & B^l{}_{ab}{}^{ij} \\ C^{lab}{}_{ij} & D^{la}{}_{bi}{}^j - \lambda^l \delta^a_b \delta^j_i \end{pmatrix} = 0\,. \tag{3.17}$$

If $B^l{}_{ab}{}^{ij}$ and $C^{lab}{}_{ij}$ vanish, then the characteristic matrix is diagonalizable if A and D are diagonalizable, since the spectrum of the characteristic matrix is composed of those of A and D. The eigenvectors for every l-index, $\left(p^{li}{}_a, q^{la}{}_i\right)$, are given by

$$\begin{pmatrix} A^l{}_a{}^{bi}{}_j & B^l{}_{ab}{}^{ij} \\ C^{lab}{}_{ij} & D^{la}{}_{bi}{}^j \end{pmatrix} \begin{pmatrix} p^{lj}{}_b \\ q^{lb}{}_j \end{pmatrix} = \lambda^l \begin{pmatrix} p^{li}{}_a \\ q^{la}{}_i \end{pmatrix} \quad \text{for every } l\,. \tag{3.18}$$

The lowering rule for the α of u^α follows those of the spacetime or internal indices. The corresponding inner product takes the form $\langle u|u\rangle := u_\alpha \bar{u}^\alpha$. According to this rule, we say the characteristic matrix is a Hermitian when

$$0 = A^{labij} - \bar{A}^{lbaji}\,, \tag{3.19}$$

$$0 = D^{labij} - \bar{D}^{lbaji}\,, \tag{3.20}$$

$$0 = B^{labij} - \bar{C}^{lbaji}\,. \tag{3.21}$$

4. Constructing Hyperbolic Systems with Original Equations of Motion

In this section, we consider which form of hyperbolicity applies to the original equations of motion, (3.7) and (3.8), under the metric reality condition (Sec. 4.1) or under the triad reality condition (Sec. 4.2).

4.1. *Under metric reality condition (system Ia and IIa)*

As the first approach, we take the equations of motion (3.7) and (3.8) with metric reality conditions (3.10) and (3.11). The principal term of (3.7) and (3.8) become

$$
\begin{aligned}
\partial_t \tilde{E}^i_a &= -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j\big(N^{[j}\tilde{E}^{i]}_a\big) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c \\
&\cong -i\epsilon^{cb}{}_a \underset{\sim}{N}(\partial_j \tilde{E}^j_c)\tilde{E}^i_b - i\epsilon^{cb}{}_a \underset{\sim}{N}\tilde{E}^j_c(\partial_j \tilde{E}^i_b) + \mathcal{D}_j(N^j \tilde{E}^i_a) - \mathcal{D}_j(N^i \tilde{E}^j_a) \\
&\cong \big[-i\epsilon^{bc}{}_a \underset{\sim}{N}\delta^l_j \tilde{E}^i_c - i\epsilon^{cb}{}_a \underset{\sim}{N}\tilde{E}^l_c \delta^i_j + N^l \delta^i_j \delta^b_a - N^i \delta^l_j \delta^b_a\big](\partial_l \tilde{E}^j_b)\,, \\
\partial_t \mathcal{A}^a_i &= -i\epsilon^{ab}{}_c \underset{\sim}{N}\tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N}\tilde{E}^a_i \\
&\cong -i\epsilon^{ab}{}_c \underset{\sim}{N}\tilde{E}^j_b(\partial_i \mathcal{A}^c_j - \partial_j \mathcal{A}^c_i) + N^j(\partial_j \mathcal{A}^a_i - \partial_i \mathcal{A}^a_j) \\
&\cong \big[+i\epsilon^a{}_b{}^c \underset{\sim}{N}\tilde{E}^j_c \delta^l_i - i\epsilon^a{}_b{}^c \underset{\sim}{N}\tilde{E}^l_c \delta^j_i + N^l \delta^a_b \delta^j_i - N^j \delta^a_b \delta^l_i\big](\partial_l \mathcal{A}^b_j)\,.
\end{aligned}
$$

The principal terms in the notation of (3.16) become

$$A^{labij} = -i\underset{\sim}{N}\epsilon^{abc}\tilde{E}^i_c \gamma^{lj} + i\underset{\sim}{N}\epsilon^{abc}\tilde{E}^l_c \gamma^{ij} + N^l \delta^{ab}\gamma^{ij} - N^i \delta^{ab}\gamma^{lj}\,, \tag{4.1}$$

$$B^{labij} = C^{labij} = 0\,, \tag{4.2}$$

$$D^{labij} = +i\underset{\sim}{N}\epsilon^{abc}\tilde{E}^j_c \gamma^{li} - i\underset{\sim}{N}\epsilon^{abc}\tilde{E}^l_c \gamma^{ij} + N^l \delta^{ab}\gamma^{ij} - N^j \delta^{ab}\gamma^{li}\,. \tag{4.3}$$

We get the 18 eigenvalues of the characteristic matrix, all of which are independent of the choice of triad:

$$0 \ (\text{multiplicity} = 6)\,, \quad N^l \ (4)\,, \quad N^l \pm N\sqrt{\gamma^{ll}} \ (4 \text{ each})\,,$$

where we do *not* take the sum in γ^{ll} (and we maintain this notation hereafter for eigenvalues and related discussions). Therefore we can say that this system is weakly hyperbolic, of type (I).

We note that this system is not type (II) in general, because this is not diagonalizable, for example, when $N^l = 0$. We classify this system as type (I), and call this *system Ia*, hereafter.

The necessary and sufficient conditions to make this system diagonalizable, type (II), are that the gauge conditions

$$N^l \neq 0 \text{ nor } \pm N\sqrt{\gamma^{ll}}\,, \quad \text{and} \quad \gamma^{ll} > 0\,, \tag{4.4}$$

where the last one is the positive definiteness of γ^{ll}. This can be proved as follows. Suppose that (4.4) is satisfied. Then 0, N^l, $N^l \pm \sqrt{\gamma^{ll}}$ are four distinct eigenvalues and we see $\mathrm{rank}(J^l) = 12$, $\mathrm{rank}(J^l - N^l I) = 14$, $\mathrm{rank}(J^l - (N^l \pm N\sqrt{\gamma^{ll}})I) = 14$. Therefore the characteristic matrix is diagonalizable. Conversely suppose that $N^l = 0$ or $N^l = \pm N\sqrt{\gamma^{ll}}$, then we see the characteristic matrix is not diagonalizable in each case.

The components of the characteristic matrix are the same as *system Ia*, so all eigenvalues are equivalent with *system Ia*. We can also show that this system is not Hermitian hyperbolic. Therefore we classify the system [*Ia* + (4.4)] to real diagonalizable hyperbolic, type (II), and call this set as *system IIa*. However, we will show in the next section that real diagonalizable hyperbolic system can also be constructed with less strict gauge conditions by modifying right-hand-side of equations of motion (*system IIb*).

4.2. *Under triad reality condition (system Ib)*

Next, we consider systems of the original equations of motion, (3.7) and (3.8), with the triad reality condition. Since this reality condition requires the additional (3.14) or (3.15) as the secondary condition (that is, to preserve the reality of triad during time evolution), in order to be consistent with this requirement and to avoid the system becoming second order in fundamental variables, we need to set $\partial_i N = 0$. This fixes the real part of the *triad lapse* gauge as $\mathrm{Re}(\mathcal{A}^a_0) = \mathrm{Re}(\mathcal{A}^a_i N^i)$. We naturally define its imaginary part as $\mathrm{Im}(\mathcal{A}^a_0) = \mathrm{Im}(\mathcal{A}^a_i N^i)$. Thus the triad lapse is fixed as $\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$. This gauge restriction does not affect principal part of the evolution equation for $\tilde{E}^i_a$, but requires us to add the term

$$\mathcal{D}_i \mathcal{A}^a_0 \cong \partial_i \mathcal{A}^a_0 = \partial_i(\mathcal{A}^a_j N^j) \cong N^j(\partial_i \mathcal{A}^a_j) = N^j \delta^a_b \delta^l_i (\partial_l \mathcal{A}^b_j)$$

to the right-hand-side of the equation of $\mathcal{A}^a_i$. That is, we need to add $N^j \delta^a_b \delta^l_i$ to ${D^{la}}_{bi}{}^j$ in (4.3),

$$D^{labij} = i\epsilon^{abc} \underset{\sim}{N} \tilde{E}^j_c \gamma^{li} - i\epsilon^{abc} \underset{\sim}{N} \tilde{E}^l_c \gamma^{ji} + N^l \delta^{ab} \gamma^{ji} \,.$$

The other components of the characteristic matrix remain the same [(4.1) and (4.2)]. We find that the set of eigenvalues of this system is

$$0 \ (\text{multiplicity} = 3)\,, \quad N^l \ (7)\,, \quad N^l \pm N\sqrt{\gamma^{ll}} \ (4 \text{ each})\,.$$

Therefore the system is again, type (I). This system is not real diagonalizable because D^l is not. So we classify this system as type (I) and call this set as *system Ib*. We note that this system is not real diagonalizable for any choice of gauge. Therefore we cannot construct a system of type (II) using the same technique of constructing *system IIa*. However, as we will show in the next section, the system becomes diagonalizable (and symmetric) hyperbolic under the triad reality condition if we modify the equations of motion.

5. Constructing a Symmetric Hyperbolic System

From the analysis of the previous section, we found that the original set of equations of motion in the Ashtekar formulation constitute a weakly hyperbolic system, type (I), or a diagonalizable hyperbolic system, type (II), under appropriate gauge conditions, but we also found that we could not obtain a symmetric hyperbolic system, type (III). In this section, we show that type (III) is obtained if we modify the equations of motion. We begin by describing our approach without considering reality conditions, but we will soon show that the triad reality condition is required for making the characteristic matrix Hermitian.

We first prepare the constraints (3.4)–(3.6) as

$$\begin{aligned}\mathcal{C}_H &\cong i\epsilon^{ab}{}_c\tilde{E}^i_a\tilde{E}^j_b\partial_i\mathcal{A}^c_j = i\epsilon^{dc}{}_b\tilde{E}^l_d\tilde{E}^j_c(\partial_l\mathcal{A}^b_j)\\ &= -i\epsilon_b{}^{cd}\tilde{E}^j_c\tilde{E}^l_d(\partial_l\mathcal{A}^b_j)\,, \qquad (5.1)\\ \mathcal{C}_{Mk} &= -F^a_{kj}\tilde{E}^j_a \cong -(\partial_k\mathcal{A}^a_j - \partial_j\mathcal{A}^a_k)\tilde{E}^j_a\\ &= \big[-\delta^l_k\tilde{E}^j_b + \delta^j_k\tilde{E}^l_b\big](\partial_l\mathcal{A}^b_j)\,, \qquad (5.2)\\ \mathcal{C}_{Ga} &= \mathcal{D}_i\tilde{E}^i_a \cong \partial_l\tilde{E}^l_a\,. \qquad (5.3)\end{aligned}$$

We apply the same technique as used by ILR to modify the equation of motion of $\tilde{E}^i_a$ and $\mathcal{A}^a_i$; by adding the constraints which weakly produce $\mathcal{C}_H \approx 0$, $\mathcal{C}_{Mk} \approx 0$, and $\mathcal{C}_{Ga} \approx 0$. (Indeed, this technique has also been used for constructing symmetric hyperbolic systems for the original Einstein equations.[9,11]) We also assume the triad lapse $\mathcal{A}^a_0$ is

$$\partial_i\mathcal{A}^a_0 \cong T^{lab}{}_{ij}\partial_l\tilde{E}^j_b + S^{la}{}_{bi}{}^j\partial_l\mathcal{A}^b_j\,, \qquad (5.4)$$

where T and S are parameters which do not include derivatives of the fundamental variables. This assumption is general for our purpose of studying the principal part of the system.

One natural way to construct a symmetric hyperbolic system is to keep $B = C = 0$ and modify the A and D terms in (3.16), so that we modify (3.7) using $\mathcal{C}_G$, and modify (3.8) using $\mathcal{C}_H$ and $\mathcal{C}_M$. That is, we add the following terms to the equations of motion:

$$\begin{aligned}&\text{modifying term for } \partial_t\tilde{E}^i_a\\ &\quad = P^i{}_{ab}\mathcal{C}^b_G \cong P^i{}_a{}^b\partial_j\tilde{E}^j_b = (P^i{}_a{}^b\delta^l_j)(\partial_l\tilde{E}^j_b)\,, \qquad (5.5)\\ &\text{modifying term for } \partial_t\mathcal{A}^a_i\\ &\quad = \mathcal{D}_i\mathcal{A}^a_0 + Q^a_i\mathcal{C}_H + R_i{}^{ja}\mathcal{C}_{Mj}\\ &\quad \cong T^{lab}{}_{ij}\partial_l\tilde{E}^j_b + S^{la}{}_{bi}{}^j\partial_l\mathcal{A}^b_j - iQ^a_i\epsilon_b{}^{cd}\tilde{E}^j_c\tilde{E}^l_d(\partial_l\mathcal{A}^b_j)\\ &\qquad + R_i{}^{ka}[-\delta^l_k\tilde{E}^j_b + \delta^j_k\tilde{E}^l_b]\partial_l\mathcal{A}^b_j\\ &\quad \cong \big[S^{la}{}_{bi}{}^j - iQ^a_i\epsilon_b{}^{cd}\tilde{E}^j_c\tilde{E}^l_d - R_i{}^{la}\tilde{E}^j_b + R_i{}^{ja}\tilde{E}^l_b\big](\partial_l\mathcal{A}^b_j) + T^{lab}{}_{ij}\partial_l\tilde{E}^j_b\,, \qquad (5.6)\end{aligned}$$

where P, Q, and R are parameters and will be fixed later. In Appendix A.2, we show that the modifications to the off-diagonal blocks B and C, i.e. modifying (3.7) using $\mathcal{C}_H$ and $\mathcal{C}_M$ and modify (3.8) using $\mathcal{C}_G$, will not affect the final conclusion at all. Note that we truncated $\mathcal{A}^a_0$ in (5.5), while it remains in (5.6), since only the derivative of $\mathcal{A}_0^a$ effects the principal part of the system. The terms in (3.16) become

$$
\begin{aligned}
A^{labij} &= -i\epsilon^{bca}\underset{\sim}{N}\gamma^{lj}\tilde{E}_c^i - i\epsilon^{cba}\underset{\sim}{N}\tilde{E}_c^l\gamma^{ij} \\
&\quad + N^l\gamma^{ij}\delta^{ab} - N^i\gamma^{lj}\delta^{ab} + P^{iab}\gamma^{lj}\,, \qquad (5.7)\\
B^{labij} &= 0\,, \qquad (5.8)\\
C^{labij} &= T^{labij}\,, \qquad (5.9)\\
D^{labij} &= i\epsilon^{abc}\underset{\sim}{N}\tilde{E}_c^j\gamma^{li} - i\epsilon^{abc}\underset{\sim}{N}\tilde{E}_c^l\gamma^{ji} \\
&\quad + N^l\delta^{ab}\gamma^{ji} - N^j\delta^{ab}\gamma^{li} + S^{labij} \\
&\quad - iQ^{ai}\epsilon^{bcd}\tilde{E}_c^j\tilde{E}_d^l - R^{ila}\tilde{E}^{jb} + R^{ija}\tilde{E}^{lb}\,. \qquad (5.10)
\end{aligned}
$$

The condition (3.21) immediately shows $T^{labij} = 0$. The condition (3.19) is written as

$$
\begin{aligned}
0 = &-i\epsilon^{abc}\underset{\sim}{N}\gamma^{lj}\tilde{E}_c^i + i\epsilon^{abc}\underset{\sim}{N}\gamma^{li}\bar{\tilde{E}}_c^j - 2\epsilon^{abc}\underset{\sim}{N}\gamma^{ij}\,\mathrm{Im}(\tilde{E}_c^l) \\
&- N^i\gamma^{lj}\delta^{ab} + N^j\gamma^{li}\delta^{ab} + P^{iab}\gamma^{lj} - \bar{P}^{jba}\gamma^{li} := \dagger^{labij}\,. \qquad (5.11)
\end{aligned}
$$

By contracting $\dagger^l{}_{ab}{}^{ij}$, we get $\mathrm{Re}(\epsilon_{abc}\dagger^{labik}\gamma_{li} - 2\epsilon_{abc}\dagger^{kabij}\gamma_{ij}) = 20\underset{\sim}{N}\,\mathrm{Im}(\tilde{E}_c^k)$. This suggests that we should impose $\mathrm{Im}(\tilde{E}_c^l) = 0$, in order to get $\dagger^l{}_{ab}{}^{ij} = 0$. This means that the triad reality condition is required for making the characteristic matrix Hermitian.

5.1. *Under triad reality condition (system IIIa)*

In this subsection, we assume the triad reality condition hereafter. In order to be consistent with the secondary triad reality condition (3.15) during time evolution, and in order to avoid the system becoming second order, we need to specify the lapse function as $\partial_i N = 0$. This lapse condition reduces to

$$\mathrm{Re}(\mathcal{A}_0^a) = N^i\,\mathrm{Re}(\mathcal{A}_i^a)\,, \qquad (5.12)$$

$$\partial_i\,\mathrm{Re}(\mathcal{A}_0^a) \cong N^j\partial_i\,\mathrm{Re}(\mathcal{A}_j^a)\,. \qquad (5.13)$$

By comparing these with the real and imaginary components of (5.4), i.e.,

$$\partial_i\,\mathrm{Re}(\mathcal{A}_0^a) \cong \mathrm{Re}(S^{la}{}_{bi}{}^j)\partial_l\,\mathrm{Re}(\mathcal{A}_j^b) - \mathrm{Im}(S^{la}{}_{bi}{}^j)\partial_l\,\mathrm{Im}(\mathcal{A}_j^b)\,, \qquad (5.14)$$

$$\partial_i\,\mathrm{Im}(\mathcal{A}_0^a) \cong \mathrm{Im}(S^{la}{}_{bi}{}^j)\partial_l\,\mathrm{Re}(\mathcal{A}_j^b) + \mathrm{Re}(S^{la}{}_{bi}{}^j)\partial_l\,\mathrm{Im}(\mathcal{A}_j^b)\,, \qquad (5.15)$$

we obtain

$$\mathrm{Re}(S^{la}{}_{bi}{}^j) = N^j\delta_i^l\delta_b^a \quad \text{and} \quad \mathrm{Im}(S^{la}{}_{bi}{}^j) = 0\,.$$

Thus S is determined as

$$S^{la}{}_{bi}{}^j = N^j\delta_i^l\,\delta_b^a\,. \qquad (5.16)$$

This value of S, and $T = 0$, determine the form of the triad lapse as

$$\mathcal{A}_0^a = \mathcal{A}_i^a N^i + \text{nondynamical terms}\,. \tag{5.17}$$

ILR do not discuss consistency of their system with the reality condition (especially with the secondary reality condition). However, since ILR assume $\mathcal{A}_0^a = \mathcal{A}_i^a N^i$, we think that ILR also need to impose a similar restricted lapse condition in order to preserve reality of their system.

By decomposing †, that is (3.19), into its real and imaginary parts, we get

$$0 = -N^i\gamma^{lj}\delta^{ab} + N^j\gamma^{li}\delta^{ba} + \gamma^{lj}\,\mathrm{Re}(P)^{iab} - \gamma^{li}\,\mathrm{Re}(P)^{jba}\,,$$
$$0 = -\epsilon^{bca}\,\underset{\sim}{N}\gamma^{lj}\tilde{E}^i_c - \epsilon^{acb}\,\underset{\sim}{N}\gamma^{li}\tilde{E}^j_c + \gamma^{lj}\,\mathrm{Im}(P)^{iab} + \gamma^{li}\,\mathrm{Im}(P)^{jba}\,.$$

By multiplying γ_{li} to these and taking symmetric and antisymmetric components on the indices ab, we have

$$0 = 2N^j\delta^{(ba)} + \mathrm{Re}(P)^{j(ab)} - 3\,\mathrm{Re}(P)^{j(ba)} = 2N^j\delta^{ba} - 2\,\mathrm{Re}(P)^{j(ab)}\,,$$
$$0 = 2N^j\delta^{[ba]} + \mathrm{Re}(P)^{j[ab]} - 3\,\mathrm{Re}(P)^{j[ba]} = 4\,\mathrm{Re}(P)^{j[ab]}\,,$$
$$0 = \mathrm{Im}(P)^{j(ab)} + 3\,\mathrm{Im}(P)^{j(ba)} = 4\,\mathrm{Im}(P)^{j(ab)}\,,$$
$$0 = -2\epsilon^{acb}\,\underset{\sim}{N}\tilde{E}^j_c + \mathrm{Im}(P)^{j[ab]} + 3\,\mathrm{Im}(P)^{j[ba]} = -2\epsilon^{acb}\,\underset{\sim}{N}\tilde{E}^j_c - 2\,\mathrm{Im}(P)^{j[ab]}\,.$$

These imply

$$P^{iab} = N^i\delta^{ab} + i\,\underset{\sim}{N}\epsilon^{abc}\tilde{E}^i_c\,. \tag{5.18}$$

Our task is finished when we specify the parameters Q and R. By substituting (5.16) into (5.10), the condition (3.20) becomes

$$\begin{aligned}0 = {} & i\epsilon^{abc}\,\underset{\sim}{N}\tilde{E}^j_c\gamma^{li} + i\epsilon^{bac}\,\underset{\sim}{N}\tilde{E}^i_c\gamma^{lj} - iQ^{ai}\epsilon^{bcd}\tilde{E}^j_c\tilde{E}^l_d - i\bar{Q}^{bj}\epsilon^{acd}\tilde{E}^i_c\tilde{E}^l_d \\ & - R^{ila}\tilde{E}^{jb} + R^{ija}\tilde{E}^{lb} + \bar{R}^{jlb}\tilde{E}^{ia} - \bar{R}^{jib}\tilde{E}^{la}\,.\end{aligned} \tag{5.19}$$

We found that a combination of the choice

$$Q^{ai} = e^{-2}\,\underset{\sim}{N}\tilde{E}^{ia}\,, \quad \text{and} \quad R^{ila} = ie^{-2}\,\underset{\sim}{N}\epsilon^{acd}\tilde{E}^i_d\tilde{E}^l_c\,, \tag{5.20}$$

satisfies the condition (5.19). We show in Appendix A.1 that this pair of Q and R satisfies (5.19) and that this choice is unique.

The final equations of motion are

$$A^{labij} = i\epsilon^{abc}\,\underset{\sim}{N}\tilde{E}^l_c\gamma^{ij} + N^l\gamma^{ij}\delta^{ab}\,, \tag{5.21}$$

$$B^{labij} = C^{labij} = 0\,, \tag{5.22}$$

$$\begin{aligned}D^{labij} = {} & i\,\underset{\sim}{N}(\epsilon^{abc}\tilde{E}^j_c\gamma^{li} - \epsilon^{abc}\tilde{E}^l_c\gamma^{ji} \\ & - e^{-2}\tilde{E}^{ia}\epsilon^{bcd}\tilde{E}^j_c\tilde{E}^l_d - e^{-2}\epsilon^{acd}\tilde{E}^i_d\tilde{E}^l_c\tilde{E}^{jb} \\ & + e^{-2}\epsilon^{acd}\tilde{E}^i_d\tilde{E}^j_c\tilde{E}^{lb}) + N^l\delta^{ab}\gamma^{ij}\,.\end{aligned} \tag{5.23}$$

To summarize, we obtain a symmetric hyperbolic system, type (III) by modifying the equations of motion, restricting the gauge to: $\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$, $\partial_i N = 0$, and assuming the triad reality condition. We name this set *system IIIa*. The eigenvalues of this system are

$$N^l \text{ (multiplicity} = 6), \quad N^l \pm \sqrt{\gamma^{ll}} N \text{ (5 each)} \quad \text{and} \quad N^l \pm 3\sqrt{\gamma^{ll}} N \text{ (1 each)}. \tag{5.24}$$

These speeds are again independent of the choice of (real) triad.

5.2. *Under metric reality condition (system IIb)*

Using this technique, we can also construct another example of diagonalizable hyperbolic system. Since the parameters S and T specify *triad lapse*, a gauge variable for time evolutions, it is possible to change our interpretation that we take the evolution of the system within the *metric* reality condition. Of course, the characteristic matrix is no longer Hermitian. From the fact that we do not use the triad reality condition in the process of modifying the characteristic matrix using parameters (5.18) and (5.20) nor in the process of deriving the eigenvalues, this system has the same components in its characteristic matrix and has the same eigenvalues. The process of examining diagonalizability is independent of the reality conditions. Therefore this system is classified as a diagonalizable hyperbolic system, type (II).

To summarize, we gain another diagonalizable hyperbolic system by modifying the equations of motion using terms from constraint equations, with characteristic matrix (5.21)–(5.23) under metric reality condition. The eigenvalues are (5.24), and this system is restricted only by a condition on triad lapse, $\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$, and not on lapse and shift vector like *system IIa*. We call this system *system IIb*.

6. Discussion

We have constructed several hyperbolic systems based on the Ashtekar formulation of general relativity, together with discussions of the required gauge conditions and reality conditions. We summarize their features in Table 1.

Table 1. List of obtained hyperbolic systems. The system *I*, *II* and *III* denote weakly hyperbolic, diagonalizable hyperbolic and symmetric hyperbolic systems, respectively.

System	Equations of motion	Reality condition	Gauge conditions required	First order	All real eigenvals	Diagonalizable	Sym. matrix
Ia	original	metric	—	yes	yes	no	no
Ib	original	triad	$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$, $\partial_i N = 0$	yes	yes	no	no
IIa	original	metric	$N^l \neq 0$, $\pm N\sqrt{\gamma^{ll}}$ $(\gamma^{ll} \neq 0)$	yes	yes	yes	no
IIb	modified	metric	$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$	yes	yes	yes	no
IIIa	modified	triad	$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$, $\partial_i N = 0$	yes	yes	yes	yes

The original dynamical equations in the Ashtekar formulation are classified as a weakly hyperbolic system. If we further assume a set of gauge conditions or reality conditions or both, then the system can be either a diagonalizable or a symmetric hyperbolic system. We think such a restriction process helps in understanding the structure of this dynamical system, and also that of the original Einstein equations. From the point of view of numerical applications, weakly and diagonalizable hyperbolic systems are still good candidates to describe the spacetime dynamics since they have much more gauge freedom than the obtained symmetric hyperbolic system.

The symmetric hyperbolic system we obtained, is constructed by modifying the right-hand-side of the dynamical equations using appropriate combinations of the constraint equations. This is a modification of somewhat popular technique used also by Iriondo, Leguizamón and Reula. We exhibited the process of determining coefficients, showing how uniquely they are determined (cf. Appendix A). In result, this symmetric hyperbolic formulation requires a triad reality condition, which we suspect that Iriondo *et al* implicitly assumed in their system. As we demonstrated in Sec. 5, in order to keep the system first order, and to be consistent with the secondary triad reality condition, the lapse function is strongly restricted in form; it must be constant. The shift vectors and triad lapse $\mathcal{A}^a_0$ should have the relation (5.17). This can be interpreted as the shift being free and the triad lapse determined. This gauge restriction sounds tight, but this arises from our general assumption of (5.4). ILR propose to use the internal rotation to reduce this reality constraint, however this proposal does not work in our notation (see Appendices B and C).

There might be a possibility to improve the situation by renormalizing the shift and triad lapse terms into the left-hand-side of the equations of motion like the case of general relativity.[9] Or this might be because our system is constituted by Ashtekar's original variables. We are now trying to relax this gauge restriction and/or to simplify the characteristic speeds by other gauge choices and also by introducing new dynamical variables. This effort will be reported elsewhere.

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Appendix A. Detail Processes of Deriving the Symmetric Hyperbolic System IIIa

In this Appendix, we show several detail calculations for obtaining the symmetric hyperbolic system *IIIa*.

Appendix A.1. *Determining Q and R*

We show here that the choice Q and R of (5.20) satisfies (5.19). That is, the final D in (5.23) satisfies (3.20), and that this choice Q and R is unique.

First we show that D in (5.23) satisfies Hermiticity, (3.20). From the direct calculation, we get

$$\begin{aligned} D^{labij} - \bar{D}^{lbaji} &= i\underset{\sim}{N}\big(\epsilon^{abc}\tilde{E}^j_c\gamma^{li} - \epsilon^{abc}\tilde{E}^l_c\gamma^{ji} - e^{-2}\epsilon^{bcd}\tilde{E}^{ia}\tilde{E}^j_c\tilde{E}^l_d \\ &\quad + e^{-2}\epsilon^{acd}\tilde{E}^{jb}\tilde{E}^i_c\tilde{E}^l_d - e^{-2}\epsilon^{acd}\tilde{E}^{lb}\tilde{E}^i_c\tilde{E}^j_d\big) \\ &\quad + i\underset{\sim}{N}\big(\epsilon^{bac}\tilde{E}^i_c\gamma^{lj} - \epsilon^{bac}\tilde{E}^l_c\gamma^{ij} - e^{-2}\epsilon^{acd}\tilde{E}^{jb}\tilde{E}^i_c\tilde{E}^l_d \\ &\quad + e^{-2}\epsilon^{bcd}\tilde{E}^{ia}\tilde{E}^j_c\tilde{E}^l_d - e^{-2}\epsilon^{bcd}\tilde{E}^{la}\tilde{E}^j_c\tilde{E}^i_d\big) \\ &= i\underset{\sim}{N}\big(\epsilon^{abc}\tilde{E}^j_c\gamma^{li} - \epsilon^{abc}\tilde{E}^i_c\gamma^{lj} - e^{-2}\epsilon^{acd}\tilde{E}^{lb}\tilde{E}^i_c\tilde{E}^j_d \\ &\quad - e^{-2}\epsilon^{bcd}\tilde{E}^{la}\tilde{E}^j_c\tilde{E}^i_d\big) =: i\underset{\sim}{N}\ \dagger^{labij}\ . \end{aligned}$$

(This $\dagger^{labij}$ definition is used only within this Appendix A.1.) Hermiticity, $\dagger^{labij} = 0$, can be shown from the fact

$$\begin{aligned} 2\dagger^{l(ab)ij} &= -e^{-2}\epsilon^{acd}\tilde{E}^{lb}\tilde{E}^i_c\tilde{E}^j_d - e^{-2}\epsilon^{acd}\tilde{E}^{lb}\tilde{E}^j_c\tilde{E}^i_d \\ &\quad - e^{-2}\epsilon^{bcd}\tilde{E}^{la}\tilde{E}^j_c\tilde{E}^i_d - e^{-2}\epsilon^{bcd}\tilde{E}^{la}\tilde{E}^i_c\tilde{E}^j_d = 0\,, \end{aligned}$$

and its antisymmetric part $\dagger^{l[ab]ij} = 0$, which is derived from

$$\begin{aligned} \epsilon_{abe}\dagger^{labij} &= \epsilon_{abe}\epsilon^{abc}\tilde{E}^j_c\gamma^{li} - \epsilon_{abe}\epsilon^{abc}\tilde{E}^i_c\gamma^{lj} \\ &\quad - e^{-2}\epsilon_{abe}\epsilon^{acd}\tilde{E}^{lb}\tilde{E}^i_c\tilde{E}^j_d - e^{-2}\epsilon_{bea}\epsilon^{bcd}\tilde{E}^{la}\tilde{E}^j_c\tilde{E}^i_d \\ &= 2\tilde{E}^j_e\gamma^{li} - 2\tilde{E}^i_e\gamma^{lj} - e^{-2}\tilde{E}^{lb}\tilde{E}^i_b\tilde{E}^j_e + e^{-2}\tilde{E}^{lb}\tilde{E}^i_e\tilde{E}^j_b \\ &\quad - e^{-2}\tilde{E}^{la}\tilde{E}^j_e\tilde{E}^i_a + e^{-2}\tilde{E}^{la}\tilde{E}^j_a\tilde{E}^i_e \\ &= 2\tilde{E}^j_e\gamma^{li} - 2\tilde{E}^i_e\gamma^{lj} - \tilde{E}^j_e\gamma^{il} + \tilde{E}^i_e\gamma^{lj} - \tilde{E}^j_e\gamma^{li} + \tilde{E}^i_e\gamma^{lj} = 0\,. \end{aligned}$$

Next we show that the choice Q and R of (5.20) is unique in order to satisfy (5.19). Suppose we have two pairs of (Q, R), say (Q_1, R_1) and (Q_2, R_2), as solutions of (5.19). Then the pair $(Q_1 - Q_2, R_1 - R_2)$ should satisfy a truncated part of (5.19),

$$\begin{aligned} \ddagger^{labij} &:= -iQ^{ai}\epsilon^{bcd}\tilde{E}^j_c\tilde{E}^l_d - i\bar{Q}^{bj}\epsilon^{acd}\tilde{E}^i_c\tilde{E}^l_d - R^{ila}\tilde{E}^{jb} \\ &\quad + R^{ija}\tilde{E}^{lb} + \bar{R}^{jlb}\tilde{E}^{ia} - \bar{R}^{jib}\tilde{E}^{la} = 0\,. \end{aligned} \tag{A.1}$$

Now we show that the equation $\ddagger^{labij} = 0$ has only the trivial solution $Q = R = 0$. By preparing

$$\begin{aligned} \ddagger^{labij}\gamma_{li} &= -iQ^{ai}\epsilon^{bcd}\tilde{E}^j_c\tilde{E}_{di} - R^{ila}\gamma_{li}\tilde{E}^{jb} + R^{ija}\tilde{E}^b_i\,, \\ \ddagger^{labij}\gamma_{li}\tilde{E}_{jb} &= -3e^2R^{ila}\gamma_{li} + e^2R^{ija}\gamma_{ij} = -2e^2R^{ija}\gamma_{ij}\,, \end{aligned} \tag{A.2}$$

we get $R^{ija}\gamma_{ij} = 0$. By substituting this into (A.2), we can express R by Q as

$$R^{ija} = ie^{-2}Q^{ak}\epsilon^{bcd}\tilde{E}^j_c\tilde{E}_{kd}\tilde{E}^i_b = iQ^{ak}\tilde{\epsilon}^{ij}{}_k \,. \tag{A.3}$$

Therefore (A.1) becomes

$$\ddagger^{labij} = -iQ^{ai}\epsilon^{bcd}\tilde{E}^j_c\tilde{E}^l_d - i\bar{Q}^{bj}\epsilon^{acd}\tilde{E}^i_c\tilde{E}^l_d - iQ^{ak}\tilde{\epsilon}^{il}{}_k\tilde{E}^{jb} + iQ^{ak}\tilde{\epsilon}^{ij}{}_k\tilde{E}^{lb} - i\bar{Q}^{bk}\tilde{\epsilon}^{jl}{}_k\tilde{E}^{ia} + i\bar{Q}^{bk}\tilde{\epsilon}^{ji}{}_k\tilde{E}^{la} \,.$$

From this equation, we get the following contracted relations:

$$\begin{aligned} e^{-2}\ddagger^{labij} &= -iQ^{ai}\epsilon^{bjl} - i\bar{Q}^{bj}\epsilon^{ail} - iQ^{ak}\epsilon^{il}{}_kE^{jb} \\ &\quad + iQ^{ak}\epsilon^{ij}{}_kE^{lb} - i\bar{Q}^{bk}\epsilon^{jl}{}_kE^{ia} + i\bar{Q}^{bk}\epsilon^{ji}{}_kE^{la} \,, \\ e^{-2}\ddagger^{labij}\,E_{ia} &= -iQ^a{}_a\epsilon^{bjl} + 2iQ^{ak}\epsilon_a{}^{[j}{}_kE^{l]b} - 2i\bar{Q}^{bk}\epsilon^{jl}{}_k \,, \\ e^{-2}\ddagger^{labij}\,E_{ia}\epsilon_{ljc} &= 2iQ^a{}_a\delta^b_c + 2iQ^{bc} - 2iQ^{cb} + 4i\bar{Q}^{bc} \,, \\ e^{-2}\ddagger^{labij}\,E_{ia}\epsilon_{ljc}\delta^c_b &= 6iQ^a{}_a + 4i\bar{Q}^a{}_a = 10i\,\mathrm{Re}(Q^a{}_a) - 2\,\mathrm{Im}(Q^a{}_a) \,, \end{aligned} \tag{A.4}$$

where $Q^{ab} := Q^{ai}E^b_i$ and $\epsilon^{bjl} := \epsilon^{ijl}E^b_i$. From the last one, we get $Q^a{}_a = 0$. By substituting this into (A.4), we get

$$e^{-2}\ddagger^{labij}\,E_{ia}\epsilon_{lj}{}^c = 2iQ^{bc} - 2iQ^{cb} + 4i\bar{Q}^{bc} = 4iQ^{[bc]} + 4i\bar{Q}^{bc} \,. \tag{A.5}$$

The symmetric part of (A.5) indicates $Q^{(bc)} = 0$, and

$$e^{-2}\ddagger^{labij}\,E_{ja}E_{lb} = 2iQ^{bc}\epsilon_{bc}{}^i - 3i\bar{Q}^{bc}\epsilon_{bc}{}^i = -\mathrm{Re}(Q^{bc}\epsilon_{bc}{}^i) + 5i\,\mathrm{Im}(Q^{bc}\epsilon_{bc}{}^i)$$

gives us $Q^{bc}\epsilon_{bc}{}^i = Q^{[bc]} = 0$. Therefore $Q^{bc} = Q^{ai} = 0$ is determined uniquely. From (A.3), we also get $R^{ija} = 0$.

Appendix A.2. *Modifications to off-diagonal blocks*

On the starting point of the modifications to the equations of motions (5.5) and (5.6), we assumed that off-diagonal terms keep vanishing. In this subsection, we show that the modifications to the off-diagonal blocks B and C in the matrix notation of (3.16), i.e. modifying (3.7) using $\mathcal{C}_H$ and $\mathcal{C}_M$ and modify (3.8) using $\mathcal{C}_G$, does not affect the final conclusion at all.

Suppose we have a symmetric hyperbolic system (5.21)–(5.23), and suppose we additionally modify the equations of motion (3.7) and (3.8) as

$$\begin{aligned} \text{modifying term for } \partial_t\tilde{E}^i_a &= G^i_a\mathcal{C}_H + H^{ij}_a\mathcal{C}_{Mj} \\ &\cong G^i_a(-i\epsilon_b{}^{cd}\tilde{E}^j_c\tilde{E}^l_d)(\partial_l\mathcal{A}^b_j) + H^{ik}_a(-\delta^l_k\tilde{E}^j_b + \delta^j_k\tilde{E}^l_b)(\partial_l\mathcal{A}^b_j) \\ &= (-iG^i_a\epsilon_b{}^{cd}\tilde{E}^j_c\tilde{E}^l_d - H^{il}_a\tilde{E}^j_b + H^{ij}_a\tilde{E}^l_b)(\partial_l\mathcal{A}^b_j) \,, \end{aligned} \tag{A.6}$$

$$\text{modifying term for } \partial_t\mathcal{A}^a_i = I^{ab}{}_i\mathcal{C}_{Gb} \cong (I^{ab}{}_i\delta^l_j)(\partial_l\tilde{E}^j_b) \,, \tag{A.7}$$

where G^i_a, H^{ij}_a and $I^{ab}{}_i$ are parameters to be determined. In the matrix notation, these can be written as

$$B^l{}_{ab}{}^{ij} = -iG^i_a \epsilon_b{}^{cd} \tilde{E}^j_c \tilde{E}^l_d - H^{il}_a \tilde{E}^j_b + H^{ij}_a \tilde{E}^l_b \,, \tag{A.8}$$

$$C^{lab}{}_{ij} = I^{ab}{}_i \delta^l_j \,. \tag{A.9}$$

The Hermitian condition (3.21) becomes

$$0 = -iG^i_a \epsilon_b{}^{cd} \tilde{E}^j_c \tilde{E}^l_d - H^{il}_a \tilde{E}^j_b + H^{ij}_a \tilde{E}^l_b - \bar{I}_{ba}{}^j \gamma^{li} =: \dagger^l{}_{ab}{}^{ij} \,. \tag{A.10}$$

(We use this $\dagger^l{}_{ab}{}^{ij}$ definition only inside of Appendix A.2.)

If there exists a nontrivial combination of G^i_a, H^{ij}_a and $I^{ab}{}_i$ which satisfy this relation, then it will constitute alternative symmetric hyperbolic system. However, we see only the trivial solution is allowed for (A.10) as follows. From the relations of $\dagger^{kabij}\gamma_{ij} + \dagger^{labik}\gamma_{li} = -4\bar{I}^{bak}$, we obtain $I^{ab}{}_i = 0$. With this $I^{ab}{}_i = 0$, we obtain $\dagger^l{}_{ab}{}^{ij} \tilde{E}^b_j = -2e^2 H^{il}_a$, which determine $H^{ij}_a = 0$. Similarly, from $I^{ab}{}_i = 0$ and $H^{ij}_a = 0$, we get $\dagger^l{}_a{}^{bij} \underset{\sim}{\epsilon}_{jlk} \tilde{E}^k_b = -6ie^2 G^i_a$, which determine $G^i_a = 0$.

Appendix B. Internal Rotation and Ashtekar Equations

In this Appendix, we consider the effect of a SO(3) rotation on the triad, which corresponds to a SU(2) rotation on the soldering form. The equations that we derive here will be applied in the discussion in Appendix C.

Appendix B.1. *Primary and secondary conditions of internal rotation*

The SO(3) internal transformation only affects inner space, and not the spacetime quantities. Let us write U for such a rotation. U should satisfy the condition

$$U^a{}_c U^{bc} = \delta^{ab} \,. \tag{B.1}$$

This comes from the transformation of δ^{ab} to $\delta^{*ab} := U^a{}_c U^b{}_d \delta^{cd}$, which should satisfy $\delta^{*ab} = \delta^{ab}$. The determinant $\det U$ must be ± 1, and we choose $\det U = 1$ for later convenience. The transformation $\delta^a{}_b \to \delta^{*a}{}_b$ is naturally defined by $\delta^{*a}{}_b := U^a{}_c U_b{}^d \delta^c{}_d$. From (B.1), we get the fundamental relations: $\delta^{*a}{}_b = \delta^a{}_b$, $\delta^*{}_{ab} = \delta_{ab}$, and $\epsilon^{*abc} = \epsilon^{abc}$.

Now we define the transformation of the triad E^i_a and of the inverse triad E^a_i as

$$E^{*i}_{\ a} := U_a{}^b E^i_b \,, \tag{B.2}$$

$$E^{*a}_{\ i} := U^a{}_b E^b_i \,. \tag{B.3}$$

The 3-metric, γ^{ij}, is preserved under this transformation, since $\gamma^{ij} = E^i_a E^{ja} = E^{*i}_{\ a} E^{*ja}$. We note that this secondary condition, $\partial_t \gamma^{ij} = \partial_t(E^i_a E^{ja}) = \partial_t(E^{*i}_{\ a} E^{*ja})$, will not give us further conditions. This is equivalent with the time derivative of (B.1).

Appendix B.2. *Internal rotation of Ashtekar variables*

Using $\det U = 1$, the transformation of the densitized triad becomes

$$\tilde{E}^{*i}_{\ a} = U_a{}^b \tilde{E}^i_b \,, \tag{B.4}$$

and straightforward calculation shows

$$\mathcal{A}^{*a}_{\ i} = U^a{}_b \mathcal{A}^b_i - \frac{i}{2}\epsilon^{ab}{}_c U_b{}^d (\partial_i U^c{}_d) \,, \tag{B.5}$$

where we also note that $\omega^{*0a}_{\ i} = U^a{}_b \omega^{0b}_i$, and $\omega^{*bc}_{\ i} = U^a{}_e(\epsilon^e{}_{bc}\omega^{bc}_i) - \epsilon^a{}_{bc}(\partial_i U^{bd})U^c{}_d$. We remark that the second term in (B.5) arises because $\mathcal{A}^a_i$ includes the spatial derivative of the triad. The relations of triad lapse and curvature 2-form become

$$\mathcal{A}^{*a}_{\ 0} = U^a{}_b \mathcal{A}^b_0 - \frac{i}{2}\epsilon^{ab}{}_c U_b{}^d (\partial_t U^c{}_d) \,, \tag{B.6}$$

$$F^{*a}_{\ ij} = U^a{}_b F^b_{ij} \,, \tag{B.7}$$

and constraints (3.4)–(3.6) are transformed into

$$\mathcal{C}^*{}_H = \mathcal{C}_H \,, \tag{B.8}$$

$$\mathcal{C}^*{}_{Mi} = \mathcal{C}_{Mi} \,, \tag{B.9}$$

$$\mathcal{C}^*{}_{Ga} = U_a{}^b \mathcal{C}_{Gb} \,. \tag{B.10}$$

The Hilbert action (3.3) will be preserved ($S^* = S$) under U, which is demonstrated by the "cancellation relation"

$$(\partial_t \mathcal{A}^{*a}_{\ i})\tilde{E}^{*i}_{\ a} + \mathcal{A}^{*a}_{\ 0}\mathcal{C}^*{}_{Ga} = (\partial_t \mathcal{A}^a_i)\tilde{E}^i_a + \mathcal{A}^a_0 \mathcal{C}_{Ga} \,. \tag{B.11}$$

Therefore the equations of motion for $\tilde{E}^{*i}_{\ a}$ and $\mathcal{A}^{*a}_{\ i}$ are equivalent with the original ones, (3.7) and (3.8), putting a $*$ on all terms.

The secondary metric reality condition (3.11), $W^{ij} := \mathrm{Re}(\epsilon^{abc}\tilde{E}^{*k}_{\ a}\tilde{E}^{*(i}_{\ b}\mathcal{D}^*{}_k\tilde{E}^{*j)}_{\ c})$, retains its form,

$$W^{*ij} = W^{ij} \,,$$

while the secondary triad reality condition (3.15), $Y^a := -\mathrm{Re}(\mathcal{A}^a_0) + \partial_i(N)E^{ia} + N^i\,\mathrm{Re}(\mathcal{A}^a_i)$, is transformed as

$$\begin{aligned} Y^{*a} = {} & \mathrm{Re}(U^a{}_b)Y^b - i\partial_i(N)\,\mathrm{Re}(U^a{}_b)\,\mathrm{Im}(E^i_b) \\ & + \mathrm{Im}(U^a{}_b)[\mathrm{Im}(\mathcal{A}^b_0) - \partial_i(N)\,\mathrm{Im}(E^{ib}) - N^i\,\mathrm{Im}(\mathcal{A}^b_i)] \\ & + \frac{1}{2}N^i\epsilon^a{}_{bc}\,\mathrm{Im}(U^{bd})\big(\partial_i\,\mathrm{Im}(U^c{}_d)\big) \,. \end{aligned} \tag{B.12}$$

This equation has many unexpected terms, even if we assume the triad reality, $\mathrm{Im}(E^i_a) = 0$, before the transformation.

To summarize, under triad transformations, $\mathcal{A}^a_i$, $\mathcal{A}^a_0$, and Y^a are not transformed covariantly, while the other variables are transformed covariantly.

Appendix B.3. *Make triad real using internal rotation*

Suppose all the variables satisfy the metric reality conditions, that is, $\tilde{E}^i_a$ satisfies $\mathrm{Im}(\tilde{E}^i_a \tilde{E}^{ja}) = 0$. Can we obtain the triad which satisfies the triad reality condition, $\mathrm{Im}(\tilde{E}^{*i}{}_a) = 0$, by an internal rotation?

The answer is affirmative. However, such a rotation U must satisfy

$$0 = \mathrm{Im}(\tilde{E}^{*i}{}_a) = \mathrm{Im}(U_a{}^b \tilde{E}^i_b) = \mathrm{Re}(U_a{}^b)\,\mathrm{Im}(\tilde{E}^i_b) + \mathrm{Im}(U_a{}^b)\,\mathrm{Re}(\tilde{E}^i_b)\,, \qquad \text{(B.13)}$$

and its secondary condition

$$0 = \mathrm{Im}(\partial_t \tilde{E}^{*i}{}_a) = \mathrm{Im}[(\partial_t U_a{}^b)\tilde{E}^i_b + U_a{}^b(\partial_t \tilde{E}^i_b)]\,. \qquad \text{(B.14)}$$

The application of this technique will be discussed in Sec. Appendix C.2. Before ending this section, we remark two points. First, $\mathcal{A}^a_i$ is not transformed covariantly by this rotation U. Second, when we consider the evolution of $\tilde{E}^{*i}{}_a$, the evolution should be consistent with the secondary triad reality condition (3.14).

Appendix C. Consideration of ILR's Treatment of Reality Conditions

The symmetric hyperbolic system (system *IIIa*) that we obtained in Sec. 5 is strictly restricted by the triad reality condition. ILR (in their second paper[20]) propose to use an internal rotation to de-constrain this situation. Here we comment on this possibility.

Appendix C.1. *Difference of definition of symmetric hyperbolic system*

First of all, we should point out again that there is a fundamental difference in the definition used to characterize the system as *symmetric*. As we discussed in Sec. 2, we define symmetry using the fact that the characteristic matrix is Hermitian, while ILR[18,20] define it when the principal symbol of the system $iB^l{}_j{}^a k_a$ ($iJ^{l\beta}{}_\alpha k_l$ in our notation) is anti-Hermitian.

We suspect that these two definitions are equivalent when the vector k_a (k_l in our notation) is arbitrary real. Actually, ILR have advanced a suggestion that our definition and their 'modern' version are equivalent. The judgement which is conventional or not, however, we would like to leave to the reader. Concerning our definition of symmetric hyperbolicity, we think that the readers can quite easily compare our system with other proposed symmetric hyperbolic systems in general relativity: all eigenvalues (in the system we presented) are all real-valued, while ILR's are all pure imaginary. (Even if the distinction of real and pure imaginary is ignored, the eigenvalues calculated by us (5.24) and by ILR are different.)

We note that, in addition, this fundamental difference will lead to different conclusions regarding the treatment of the reality condition (see the proceeding discussion).

Appendix C.2. *Can we obtain a symmetric hyperbolic system by internal rotation?*

What ILR proposed is the following: Suppose the system satisfies the reality condition on the metric, but not on the triad. By using the freedom of making an internal rotation, we can transform the soldering form to satisfy the triad reality condition, in such a way it forms symmetric hyperbolic system. (In their terminologies, they seek a "rotated" scalar product that is to find a more general symmetrizer.) Therefore we can remove the additional constraints of the triad reality.

This procedure, however, includes changing inner product of dynamical variables, which might cause the topology of well-posedness of the initial value formulation to change. Here, we examine whether such a re-definition of the inner product is acceptable in our definition of symmetric hyperbolicity.

Suppose we have a system which satisfies the constraints, and the metric reality condition, but not the triad reality conditions. As we commented in Sec. 3, metric reality will be preserved automatically by the dynamical equations (3.16) and (5.21)–(5.23). Now we apply a SO(3) rotation $E^i_a \to E^{*i}_{\ a} := U_a{}^b E^i_b$ to the system. We summarized the transformations of Ashtekar's variables and equations by U in Appendix B. In the new variables $(\tilde{E}^{*i}_{\ a}, \mathcal{A}^{*a}_{\ i})$, transformed via U, the equations of motions are written covariantly.

As discussed in Appendix B.3, it is possible to construct the real triad by using U. However, we always should verify the triad reality condition, both its primary condition (3.12), and its secondary condition (3.13). The latter is expressed as (B.12) or (B.14). If we interpret this secondary condition as a restriction on the gauge variables, lapse N, shift N^i, and triad lapse $\mathcal{A}^a_0$, then we only need to solve the primary condition in order to obtain triad reality on 3-hypersurface. This is indeed solvable. For example, ILR explain a way to get a real triad using orthonormality of the basis in their Appendix A in Ref. 20.

Next, let us see whether a symmetric hyperbolic system is obtained by the new pair of variables $(\tilde{E}^{*i}_{\ a}, \mathcal{A}^{*a}_{\ i})$. We define the equations of motion similarly as

$$\partial_t \begin{bmatrix} \tilde{E}^{*i}_{\ a} \\ \mathcal{A}^{*a}_{\ i} \end{bmatrix} = \begin{bmatrix} A^{*l}{}_a{}^{bi}{}_j & B^{*l}{}_{ab}{}^{ij} \\ C^{*lab}{}_{ij} & D^{*la}{}_{bi}{}^j \end{bmatrix} \partial_l \begin{bmatrix} \tilde{E}^{*j}_{\ b} \\ \mathcal{A}^{*b}_{\ j} \end{bmatrix} + \text{terms with no } \partial_l \tilde{E}^{*j}_{\ b} \text{ nor } \partial_l \mathcal{A}^{*a}_{\ i}\,. \qquad \text{(C.1)}$$

By applying the same modifications as those in Sec. 5, we get

$$A^{*labij} = i\epsilon^{abc} \underset{\sim}{N} \tilde{E}^{*l}{}_c \gamma^{ij} + N^l \gamma^{ij} \delta^{ab}\,, \qquad \text{(C.2)}$$

$$B^{*labij} = C^{*labij} = 0\,, \qquad \text{(C.3)}$$

$$\begin{aligned} D^{*labij} = {} & i\underset{\sim}{N}(\epsilon^{abc} \tilde{E}^{*j}{}_c \gamma^{li} - \epsilon^{abc} \tilde{E}^{*l}{}_c \gamma^{ji} \\ & - e^{-2} \tilde{E}^{*ia} \epsilon^{bcd} \tilde{E}^{*j}{}_c \tilde{E}^{*l}{}_d - e^{-2} \epsilon^{acd} \tilde{E}^{*i}{}_d \tilde{E}^{*l}{}_c \tilde{E}^{*jb} \\ & + e^{-2} \epsilon^{acd} \tilde{E}^{*i}{}_d \tilde{E}^{*j}{}_c \tilde{E}^{*lb}) + N^l \delta^{ab} \gamma^{ij}\,. \end{aligned} \qquad \text{(C.4)}$$

These equations are related to (5.21)–(5.23). We note that, in the modification here, we added the terms $(N^i\delta^{ab} + i\underset{\sim}{N}\epsilon^{abc}\tilde{E}^{*i}_{\ c})\mathcal{C}^*_{Gb}$ coming from the terms of the gauge constraint. This corresponds to the relation $A^{*labij} = U^a{}_c U^b{}_d A^{lcdij}$.

Equations (C.2)–(C.4) forms a Hermitian matrix in the principal part of (C.1), but it contradicts the consistent evolution with triad reality. That is, for example, the left-hand-side of dynamical equation $\partial_t \tilde{E}^{*i}_{\ a} = \cdots$ [upper half of (C.1)] is real-valued since we impose $\mathrm{Im}(\tilde{E}^{*i}_{\ a}) = 0$, while in the right-hand-side includes complex value in the nonprincipal part. To explain this in another words, the system (C.1)–(C.4) will not preserve the triad reality. Therefore we again need to control gauge variables through the secondary triad reality condition, and this discussion again returns the same gauge restrictions with those in Sec. 5.

We also point out that the inner product of the fundamental variables in our notation does not form Hermitian like in the case of ILR. The inner product before the rotation U can be written

$$\langle(\tilde{E}^i_a, \mathcal{A}^a_i)|(\tilde{E}^i_a, \mathcal{A}^a_i)\rangle := \delta^{ab}\gamma_{ij}\tilde{E}^i_a \bar{\tilde{E}}^j_b + \delta_{ab}\gamma^{ij}\mathcal{A}^a_i \bar{\mathcal{A}}^b_j\,, \tag{C.5}$$

which is common to ours and ILR's, while after the rotation the inner product becomes

$$\begin{aligned}
&\langle(\tilde{E}^{*i}_{\ a}, \mathcal{A}^{*a}_{\ i})|(\tilde{E}^{*i}_{\ a}, \mathcal{A}^{*a}_{\ i})\rangle \\
&\quad = U_c{}^a \bar{U}^{cb}\tilde{E}^i_a \bar{\tilde{E}}^j_b + U_c{}^a \bar{U}^{cb}\gamma^{ij}\mathcal{A}^a_i \bar{\mathcal{A}}^b_j \\
&\qquad - \frac{i}{2}\gamma^{ij}\left(\epsilon_{agf}\bar{U}^{gh}(\partial_j \bar{U}^f{}_h)U^a{}_c\mathcal{A}^c_i + \epsilon^a{}_{ec}U^{ed}(\partial_i U^c{}_d)\bar{U}_{af}\bar{\mathcal{A}}^f_j\right) \\
&\qquad - \frac{1}{4}U_e{}^d(\partial_i U_{cd})\bar{U}^{eh}(\partial_j \bar{U}^c{}_h) + \frac{1}{4}U_e{}^d(\partial_i U_{cd})\bar{U}^{ch}(\partial_j \bar{U}^e{}_h)\,,
\end{aligned} \tag{C.6}$$

which is not Hermitian, and can not be used as the inner product of the original variable $(\tilde{E}^i_a, \mathcal{A}^a_i)$ as in the ILR's proposal.

As the final remark, we would like to comment that both the variables to evolve by the equations, and the variables used to confirm the Hermiticity of the system should be common throughout all evolutions. Otherwise, we cannot apply the energy inequality for the evolution of that system. From this point of view, we think it necessary to consider the secondary triad reality condition throughout evolution of this system.

To summarize, we tried to follow ILR's procedure to remove the restriction of the triad reality condition in our system, which casts on our definition of symmetric hyperbolicity, and which is based on the fixed inner product as of its Hermitian form. We, however, see that ILR's procedure does not work in our system since it requires the restriction of the secondary reality conditions of the triad. Therefore we conclude that we cannot de-constrain restrictions any further.

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