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LETTER TO THE EDITOR

Diagonalizability of constraint propagation matrices

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Abstract
In order to obtain stable and accurate general relativistic simulations, reformulations of the Einstein equations are necessary. In a series of our works, we have proposed using eigenvalue analysis of constraint propagation equations for evaluating violation behaviour of constraints. In this letter, we classify asymptotical behaviours of constraint violation into three types (asymptotically constrained, asymptotically bounded and diverge), and give their necessary and sufficient conditions. We find that degeneracy of eigenvalues sometimes leads constraint evolution to diverge (even if its real part is not positive) and conclude that it is quite useful to check the diagonalizability of constraint propagation matrices. The discussion is general and can be applied to any numerical treatments of constrained dynamics.

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1. Introduction

The so-called numerical relativity (computational simulations in general relativity) is a promising research field having implications for ongoing astrophysical observations such as gravitational wave astronomy [1]. Many simulations of binary compact objects have revealed that mathematically equivalent sets of evolution equations show different numerical stability in the free-evolution scheme.

There are many approaches to reformulate the Einstein equations for obtaining a long-term stable and accurate numerical evolution (e.g., see references in [2]). In a series of our works, we have proposed the construction of a system that has its constraint surface as an attractor. By applying eigenvalue analysis of constraint propagation equations, we showed that there *is* a constraint-violating mode in the standard Arnowitt–Deser–Misner (ADM) evolution system [3, 4] when it is applied to a single non-rotating black-hole spacetime [6]. We also found that

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such a constraint-violating mode can be compensated for if we adjust the evolution equations with a particular modification using constraint terms like the one proposed by Detweiler [5].

Our predictions are borne out in simple numerical experiments using the Maxwell, Ashtekar and ADM systems [6–9]. There are also several numerical experiments to confirm that our predictions are effective [10, 11]. However, we have not yet obtained definite guidelines for specifying the above adjusting terms and their multipliers.

In this letter, we show the essential steps in analysing constraint amplification factors (defined in section 2.2). In section 3, we show that degeneracy of eigenvalues sometimes leads constraint evolution to diverge. This observation suggests the importance of checking the diagonalizability of characteristic matrices and gives further insights for constructing an asymptotically constrained system.

2. A guideline to obtain a robust evolution system

2.1. Idea of adjusted system

We begin by reviewing our proposal for an 'adjusted system'.

Suppose we have a dynamical system of variables $u^a(x^i, t)$, which has evolution equations,

$$\partial_t u^a = f(u^a, \partial_i u^a, \ldots), \tag{1}$$

and the (first-class) constraints,

$$C^\alpha(u^a, \partial_i u^a, \ldots) \approx 0. \tag{2}$$

Note that we do not require (1) to form a first-order hyperbolic form. We propose to investigate the evolution equation of C^α (constraint propagation),

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \ldots), \tag{3}$$

for evaluating violation features of constraints.

The character of constraint propagation, (3), will vary when we modify the original evolution equations. Suppose we modify (adjust) (1) using constraints

$$\partial_t u^a = f(u^a, \partial_i u^a, \ldots) + F(C^\alpha, \partial_i C^\alpha, \ldots), \tag{4}$$

then (3) will also be modified as

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \ldots) + G(C^\alpha, \partial_i C^\alpha, \ldots). \tag{5}$$

Therefore, finding a proper adjustment $F(C^\alpha, \ldots)$ is a quite important problem.

Hyperbolicity analysis may be a way to evaluate constraint propagation, (3) and (5) [12]. However, this requires (3) to be a first-order system which is easy to break (see, e.g., Detweiler-type adjustment [5] in the ADM formulation [6]). Furthermore, hyperbolicity analysis only concerns the principal part of the equation, which may fail to analyse the detailed evaluation of the evolution.

Alternatively, we have proceeded with an eigenvalue analysis of the whole RHS in (3) and (5) after a suitable homogenization, which may compensate for the above difficulties of hyperbolicity analysis.

2.2. CP matrix and CAF

We propose to transform the constraint propagation equation, (3) and (5), into Fourier modes,

$$\partial_t \hat{C}^\alpha = \hat{g}(\hat{C}^\alpha) = M^\alpha{}_\beta \hat{C}^\beta, \qquad \text{where} \quad C(x,t)^\alpha = \int \hat{C}(k,t)^\alpha \exp(\mathrm{i}k\cdot x)\,\mathrm{d}^3k, \tag{6}$$

then to analyse the eigenvalues, say Λ_α, of the coefficient matrix, $M^\alpha{}_\beta$. We call Λ_α and $M^\alpha{}_\beta$ the constraint amplification factors (CAFs) and constraint propagation matrix (CP matrix), respectively.

So far we have proposed the following heuristic conjectures [6–9].

(A) If the CAF has a *negative real part* (the constraints are necessarily diminished), then we see more stable evolution than that in a system which has a positive CAF.

(B) If the CAF has a *non-zero imaginary part* (the constraints are propagating away), then we see more stable evolution than that in a system which has a zero CAF.

We observe that this eigenvalue analysis requires the fixing of a particular background spacetime, since the CAFs depend on the dynamical variables, u^a.

2.3. Classification of constraint propagations

The CAFs indicate the evolution of constraint violations (definitely its Fourier modes). It is natural to assume that a divergence of constraint norm is related to the numerical blow-ups. Therefore, we classify the fundamental evolution property of constraint propagation equation (6) as follows.

(C1) *Asymptotically constrained:* violation of constraints decays (converges to zero).
(C2) *Asymptotically bounded:* violation of constraints is bounded at a certain value.
(C3) *Diverge:* at least one constraint will diverge.

Note that (C1) $\subset$ (C2). We will derive the necessary and sufficient conditions for (C1) and (C2) in the following section.

3. Conditions for (C1) and (C2)

3.1. Preparation

Hereafter, we consider a set of evolution equations,

$$\partial_t C^i(t) = M^i{}_j C^j, \tag{7}$$

where C^i $(i = 1, \ldots, n)$ is a complex-valued vector, $M^i{}_j$ is an $n \times n$ complex-valued matrix and $C^i(t)$ is assumed to have finite-valued initial data $C^i(0)$.

Without loss of generality, the CP matrix M can be assumed to be a Jordan normal form, since within complex-valued operations all the matrices can be converted to this form. Suppose that M has r different eigenvalues $(\lambda_1, \ldots, \lambda_r)$, where $r \leqslant n$. Let the multiplicity of λ_k be n_k, where $\sum_{k=1}^r n_k = n$. M can be expressed as

$$M = J_1 \dot{+} \cdots \dot{+} J_r := \begin{pmatrix} J_1 & & O \\ & \ddots & \\ O & & J_r \end{pmatrix}, \tag{8}$$

where the cell size of J_k is $n_k \times n_k$. The Jordan matrix J_k is then expressed using a Jordan block $J_{k\ell}$,

$$J_k = J_{k1} \dot{+} \cdots \dot{+} J_{km}, \tag{9}$$

$$J_{k\ell} := \begin{pmatrix} \lambda_k & 1 & & O \\ & \ddots & \ddots & \\ O & & \lambda_k & 1 \end{pmatrix}. \tag{10}$$

Note that $J_{k\ell}$ is $n_{k\ell} \times n_{k\ell}$, $\sum_{\ell=1}^{m} n_{k\ell} = n_k$, $m = n - \mathrm{rank}(M - \lambda_k E)$ and $\max_\ell(n_{k\ell}) = \nu_k$. The minimum polynomial of M is written as

$$\mu_M(t) = (t - \lambda_1)^{\nu_1} \cdots (t - \lambda_r)^{\nu_r}. \tag{11}$$

If J_k is diagonal (i.e., $n_k = n - \mathrm{rank}(M - \lambda_k E)$), then $\nu_k = 1$ for that k. If M is diagonalizable (i.e., $n_k = n - \mathrm{rank}(M - \lambda_k E)\ \forall k$), then $\nu_k = 1$ for all k.

We then have the following statement.

Proposition 1. *The solution of*

$$\partial_t C_a = J_k C_a \tag{12}$$

can be expressed formally as

$$C_a(t) = \exp(\lambda_k t) \sum_{\ell=0}^{\nu_k - 1} \left(a_\ell^{(k)} t^\ell\right). \tag{13}$$

A proof is available by mathematical induction. Suppose that J_{k1} is $\nu_k \times \nu_k$ which is the maximal size J_k. By direct calculation, we have that $\partial_t C_a = J_{k1} C_a$ yields (13) with t-polynomial of degree $(\nu_k - 1)$. Then we see that (13) is satisfied in general.

From this proposition, the highest power N_k in t-polynomial in (13) is bounded by $0 \leqslant N_k \leqslant \nu_k - 1$. The matrix J_k in (12) can be directly extended to the full CP matrix, M, in (7). Therefore the highest power N in all constraints is bounded by

$$0 \leqslant N \leqslant \max_{1 \leqslant k \leqslant r} (n_k) - 1. \tag{14}$$

3.2. Asymptotically constrained CP

Propositions 2 and 3 give us the following theorem.

Theorem 1. *Asymptotically constrained evolution (violation of constraints converges to zero) is obtained if and only if all the real parts of the CAFs are negative.*

Proposition 2. *All the real parts of CAFs are negative ⇒ asymptotically constrained evolution.*

Proof. We use expression (13). If $\mathrm{Re}(\lambda_k) < 0\ \forall k$, then C_i will converge to zero at $t \to \infty$ no matter what the t-polynomial terms are. □

Proposition 3. *Asymptotically constrained evolution ⇒ all the real parts of the CAFs are negative.*

Proof. We show the contrapositive. Suppose there exists an eigenvalue λ_1 whose real part is non-negative. Then we get $\partial_t C_1 = \lambda_1 C_1$ whose solution is $C_1 = C_1(0) \exp(\lambda_1 t)$. C_1 does not converge to zero. □

3.3. Asymptotically bounded CP

Propositions 4 and 5 give us the following theorem.

Theorem 2. *Asymptotically bounded evolution (all the constraints are bounded at a certain value) is obtained if and only if all the real parts of CAFs are not positive and J_k is diagonal when* $\mathrm{Re}(\lambda_k) = 0$.

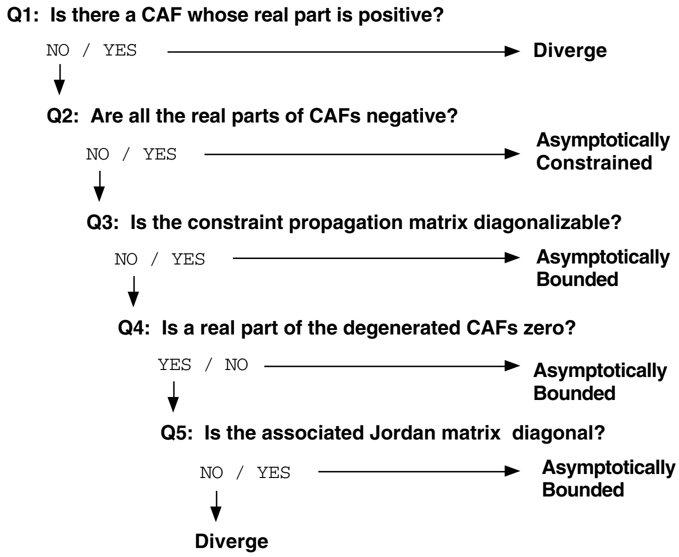

Figure 1. A flowchart to classify the fate of constraint propagation.

Corollary. *Asymptotically bounded evolution is obtained if the real parts of CAFs are not positive and the CP matrix $M^\alpha{}_\beta$ is diagonalizable.*

Proposition 4. *All the real parts of CAFs are not positive and J_k is diagonal when* $\mathrm{Re}(\lambda_k) = 0 \Rightarrow$ *asymptotically bounded evolution.*

Proof. We use expression (13). When $\mathrm{Re}(\lambda_k) < 0$, $\exp(\lambda_k t) \times$ (t-polynomials) will converge to zero no matter what the t-polynomial terms are. When $\mathrm{Re}(\lambda_k) = 0$, we see $\nu_k = 1$ from the assumption of diagonality of J_k. So we see that the t-polynomial terms are constant and $\exp(\lambda_k t)$ is bounded. □

Proposition 5. *Asymptotically bounded evolution $\Rightarrow$ all the real parts of the CAFs are not positive and J_k is diagonal when* $\mathrm{Re}(\lambda_k) = 0$.

Proof. We show the contrapositive. If there exists an eigenvalue whose real part is positive, then constraints will diverge no matter what the t-polynomial terms are. Therefore, we try to show that constraints will diverge when all the real parts of eigenvalues are non-positive, and there exists λ_k such that $\mathrm{Re}(\lambda_k) = 0$ and its Jordan matrix J_k is not diagonal.

Since Jordan matrix J_k is not diagonal, we see that the power of t-polynomial ν_k is greater than 1 in expression (13). Thus we have that (13) will diverge in $t \to \infty$. □

4. Concluding remarks

The two theorems will give us a guideline to analyse a constraint-violating mode of the system. The result not only supports our previous heuristic conjecture (A), but also suggests an ill-behaving case when CAFs are degenerated and their real part is zero, when the associated Jordan matrix is not diagonal. This indicates the importance of checking the diagonalizability of constraint propagation matrix M.

Along the line of our evaluation of constraint propagation equations (7), we propose a practical procedure for this classification in figure 1. We think that this diagram will provide systematic predictions for obtaining a robust evolution system in any constrained dynamics.

The present classification is only on the fixed background spacetime and only for $t \to \infty$. It is still not clear at what value the constraints are bounded if a limiting value exists. Thus further modifications are underway. We are also applying the present classification scheme to various adjusted systems of the Einstein equations (adjusted ADM, and further modified versions), together with numerical experiments. We hope to report on them in the near future.

The current constraint analysis only concentrates on the evolution equations and does not include the effect of the boundary treatments. Since the eigenvalues are evaluated locally, it will be possible to include the effect of numerical boundary conditions if they are expressed apparently as a part of the evolution equations. This is also the one direction to take our future research. Meanwhile, we would like to remark that one of our proposed adjustments in [9] contributes to enforce the computational ability of the black-hole excision boundary treatment [11].

By extending the notion of 'norm' or 'compactness' of constraint violations, it might be interesting to define a new measure which monitors a 'distance' between the constraint surface and an evolution sector in constraint dynamics.

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