Fate of the Traversible Wormholes

— Black-Hole Collapse or Inflationary Expansion —

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OUTLINE

- Traversible wormhole (Morris-Thorne wormhole, 1988)
- Black Hole - Wormhole synthesis (Hayward, 1999)
- "Dynamical Wormhole"
- A numerical approach, dual-null formulation
- A new type of critical behaviour??

HS and S.A. Hayward, Phys. Rev. D. **66** (2002) 044005

Morris-Thorne's "Traversable" wormhole

M.S. Morris and K.S. Thorne, Am. J. Phys. 56 (1988) 395
M.S. Morris, K.S. Thorne, and U. Yurtsever, PRL 61 (1988) 3182
H.G. Ellis, J. Math. Phys. 14 (1973) 104
(G. Clément, Am. J. Phys. 57 (1989) 967)

Desired properties of traversable WHs

1. Spherically symmetric and Static $\Rightarrow$ M. Visser, PRD 39(89) 3182 & NPB 328 (89) 203
2. Einstein gravity
3. Asymptotically flat
4. No horizon for travel through
5. Tidal gravitational forces should be small for traveler
6. Traveler should cross it in a finite and reasonably small proper time
7. Must have a physically reasonable stress-energy tensor
   $\Rightarrow$ Weak Energy Condition is violated at the WH throat.
   $\Rightarrow$ (Null EC is also violated in general cases.)
8. Should be perturbatively stable
9. Should be possible to assemble

1 Why Wormhole?

- They make great science fiction – short cuts between otherwise distant regions. Morris & Thorne 1988, Sagan "Contact" etc
- They increase our understanding of gravity when the usual energy conditions are not satisfied, due to quantum effects (Casimir effect, Hawking radiation) or alternative gravity theories, brane-world models etc.
- They are very similar to black holes –both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH).

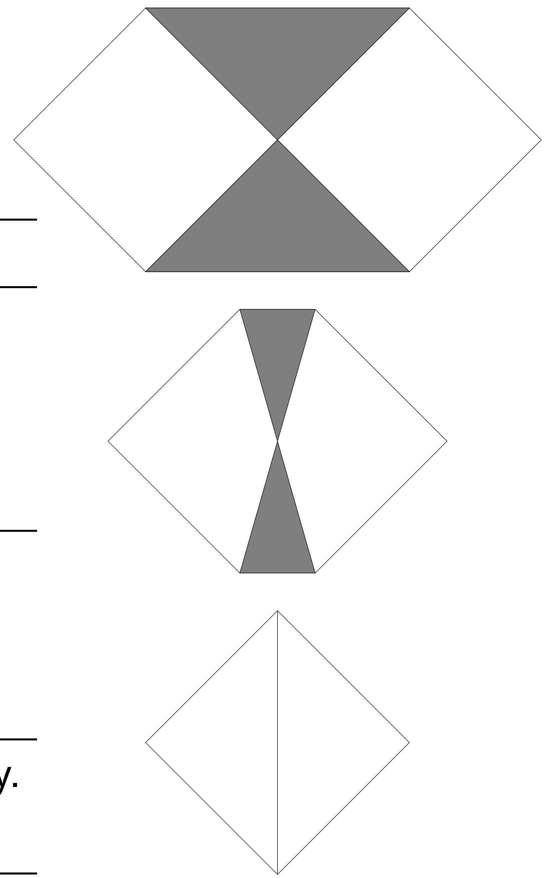
Wormhole $\equiv$ Hypersurface foliated by marginally trapped surfaces

- BH and WH are interconvertible?
  New duality?

BH and WH are interconvertible ? (New Duality?)

S.A. Hayward, Int. J. Mod. Phys. D 8 (1999) 373

- They are very similar – both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH)
- Only the causal nature of the THs differs, whether THs evolve in plus / minus density.



	Black Hole	Wormhole
Locally defined by	Achronal(spatial/null) outer TH ⇒ 1-way traversable	Temporal (timelike) outer THs ⇒ 2-way traversable
Einstein eqs.	Positive energy density normal matter (or vacuum)	Negative energy density "exotic" matter
Appearance	occur naturally	Unlikely to occur naturally. but constructible ???

2 Fate of Morris-Thorne (Ellis) wormhole?

- “Dynamical wormhole” defined by local trapping horizon
- spherically symmetric, both normal/ghost KG field
- apply dual-null formulation in order to seek horizons
- Numerical simulation

2.1 ghost/normal Klein-Gordon fields

Lagrangian:

$$\mathcal{L} = \sqrt{-g}\left[\frac{R}{16\pi} - \frac{1}{4\pi}\underbrace{\left(\frac{1}{2}(\nabla\psi)^2 + V_1(\psi)\right)}_{\text{normal}} + \frac{1}{4\pi}\underbrace{\left(\frac{1}{2}(\nabla\phi)^2 + V_2(\phi)\right)}_{\text{ghost}}\right]$$

The field equations

$$G_{\mu\nu} = 2\left[\psi_{,\mu}\psi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}(\nabla\psi)^2 + V_1(\psi)\right)\right] - 2\left[\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}(\nabla\phi)^2 + V_2(\phi)\right)\right]$$

$$\Box\psi = \frac{dV_1(\psi)}{d\psi}, \qquad \Box\phi = \frac{dV_2(\phi)}{d\phi}.$$ (Hereafter, we set $V_1(\psi) = 0, V_2(\phi) = 0$)

2.2 dual-null formulation, spherically symmetric spacetime

S A Hayward, CQG 10 (1993) 779, PRD 53 (1996) 1938, CQG 15 (1998) 3147

- The spherically symmetric line-element:

$$ds^2 = r^2 dS^2 - 2e^{-f} dx^+ dx^-,$$

where $r = r(x^+, x^-)$, $f = f(x^+, x^-)$, $\cdots$

- The Einstein equations:

$$\begin{aligned}
\partial_\pm\partial_\pm r + (\partial_\pm f)(\partial_\pm r) &= -r(\partial_\pm\psi)^2 + r(\partial_\pm\phi)^2, \\
r\partial_+\partial_- r + (\partial_+ r)(\partial_- r) + e^{-f}/2 &= 0, \\
r^2\partial_+\partial_- f + 2(\partial_+ r)(\partial_- r) + e^{-f} &= +2r^2(\partial_+\psi)(\partial_-\psi) - 2r^2(\partial_+\phi)(\partial_-\phi), \\
r\partial_+\partial_-\phi + (\partial_+ r)(\partial_-\phi) + (\partial_- r)(\partial_+\phi) &= 0, \\
r\partial_+\partial_-\psi + (\partial_+ r)(\partial_-\psi) + (\partial_- r)(\partial_+\psi) &= 0.
\end{aligned}$$

- To obtain a system accurate near $\Im^\pm$, we introduce the conformal factor $\boxed{\Omega = 1/r}$. We also define first-order variables, the conformally rescaled momenta

expansions	$\vartheta_\pm = 2\partial_\pm r = -2\Omega^{-2}\partial_\pm\Omega \qquad (\theta_\pm = 2r^{-1}\partial_\pm r)$	(1)
inaffinities	$\nu_\pm = \partial_\pm f$	(2)
momenta of ϕ	$\wp_\pm = r\partial_\pm\phi = \Omega^{-1}\partial_\pm\phi$	(3)
momenta of ψ	$\pi_\pm = r\partial_\pm\psi = \Omega^{-1}\partial_\pm\psi$	(4)

The set of equations (cont.):

$$\partial_\pm \vartheta_\pm = -\nu_\pm \vartheta_\pm - 2\Omega\pi_\pm^2 + 2\Omega\wp_\pm^2, \tag{5}$$

$$\partial_\pm \vartheta_\mp = -\Omega(\vartheta_+\vartheta_-/2 + e^{-f}), \tag{6}$$

$$\partial_\pm \nu_\mp = -\Omega^2(\vartheta_+\vartheta_-/2 + e^{-f} - 2\pi_+\pi_- + 2\wp_+\wp_-), \tag{7}$$

$$\partial_\pm \wp_\mp = -\Omega\vartheta_\mp\wp_\pm/2, \tag{8}$$

$$\partial_\pm \pi_\mp = -\Omega\vartheta_\mp\pi_\pm/2. \tag{9}$$

and remember the identity: $\partial_+\partial_- = \partial_-\partial_+$:

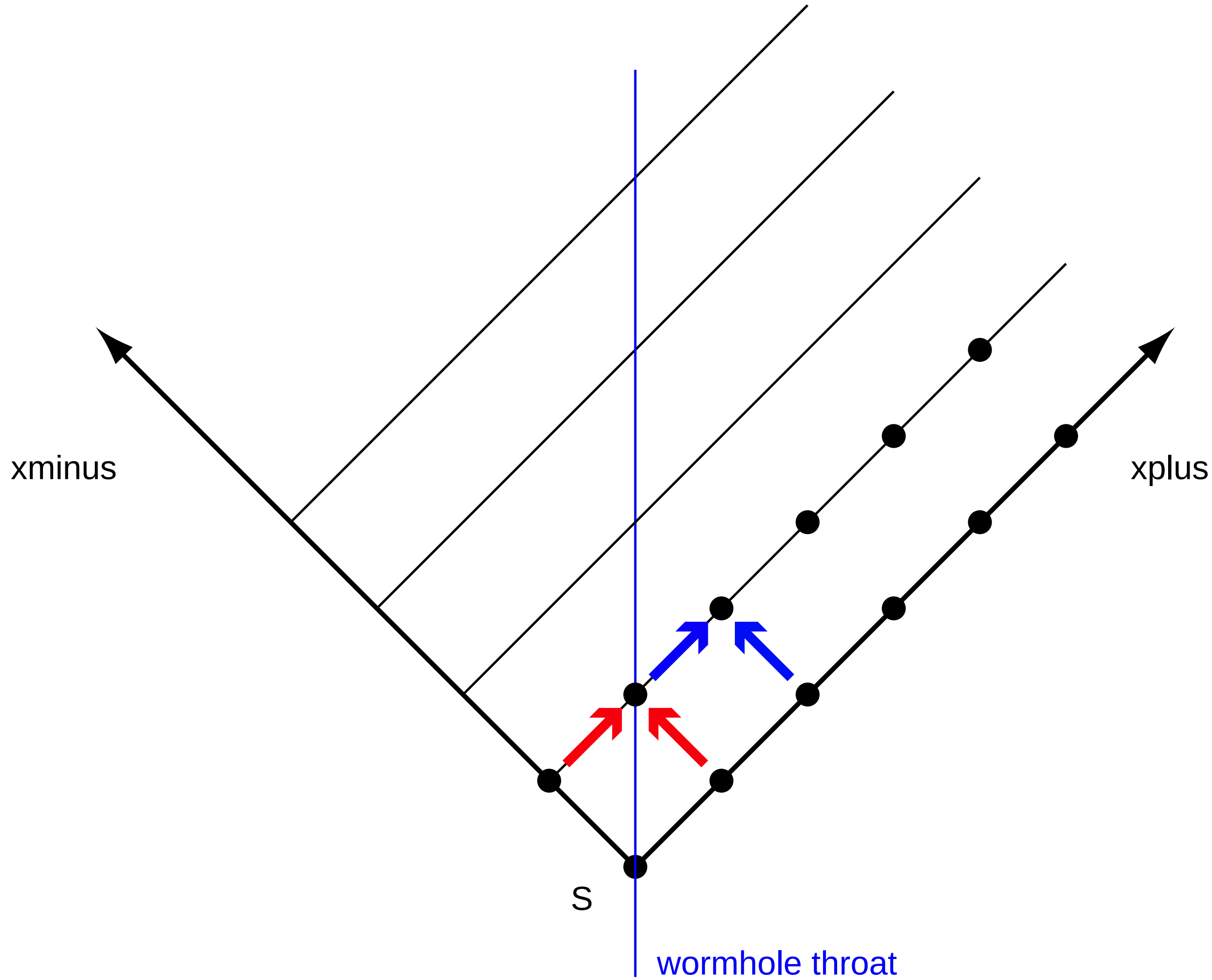
2.3 Initial data on $x^+ = 0$, $x^- = 0$ slices and on S

Generally, we have to set :

$(\Omega, f, \vartheta_\pm, \phi, \psi)$ on S: $x^+ = x^- = 0$

$(\nu_\pm, \wp_\pm, \pi_\pm)$ on $\Sigma_\pm$: $x^\mp = 0$, $x^\pm \geq 0$

Grid Structure for Numerical Evolution



2.4 Morris-Thorne (Ellis) wormhole as the initial data

	on Σ_+ ($x^- = 0$ surface)	on Σ_- ($x^+ = 0$ surface)
Ω	$1/\sqrt{a^2+z^2}$	$1/\sqrt{a^2+z^2}$
f	0	0
$\vartheta_\pm$	$\pm\sqrt{2}z/\sqrt{a^2+z^2}$	$\mp\sqrt{2}z/\sqrt{a^2+z^2}$
ν_+	0	
ν_-		0
ϕ	$\tan^{-1}(z/a)$	$-\tan^{-1}(z/a)$
$\wp_+$	$+a/\sqrt{2}\sqrt{a^2+z^2}$	
$\wp_-$		$-a/\sqrt{2}\sqrt{a^2+z^2}$
ψ	0	0
π_+	0	
π_-		0

where $z = (x^+ - x^-)/\sqrt{2}$.

We put the perturbation in $\wp_+$:
$\delta\wp_+ = c_a \exp(-c_b(z-c_c)^2)$
where c_a, c_b, c_c are parameters.

2.5 Gravitational mass-energy

- Localizing, the local gravitational mass-energy is given by the Misner-Sharp energy E,

$$E = (1/2)r[1 - g^{-1}(dr, dr)] = (1/2)r + e^f r\,(\partial_+ r)(\partial_- r) = \frac{1}{2\Omega}[1 + \frac{1}{2}e^f\,\vartheta_+\vartheta_-]$$

while the (localized Bondi) conformal flux vector components $\varphi^\pm$

$$\varphi^\pm = r^2 T^{\pm\pm}\partial_\pm r = r^2 e^{2f} T_{\mp\mp}\partial_\pm r = e^{2f}(\pi_\mp^2 - \wp_\mp^2)\vartheta_\pm/8\pi.$$

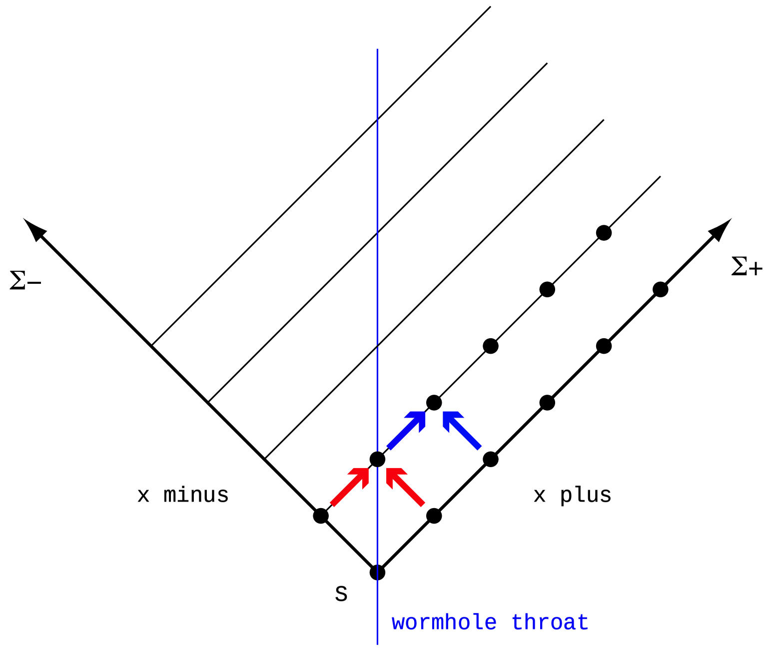
- They are related by the energy propagation equations or unified first law. $\partial_\pm E = 4\pi\varphi_\pm$,

$$E(x^+, x^-) = \frac{a}{2} + 4\pi\int_{(0,0)}^{(x^+,x^-)}(\varphi_+ dx^+ + \varphi_- dx^-),$$

where the integral is independent of path, by conservation of energy.

  - $\lim_{x^+\to\infty} E$ is the Bondi energy
  - $\lim_{x^+\to\infty}\varphi_-$ the Bondi flux for the right-hand universe.
  - For the static wormhole, the energy $E = a^2/2\sqrt{a^2+z^2}$ is everywhere positive, maximal at the throat and zero at infinity, $z \to \pm\infty$, i.e. the Bondi energy is zero.
  - Generally, the Bondi energy-loss property, that it should be non-increasing for matter satisfying the null energy condition, is reversed for the ghost field.

Numerical Grid / Convergence test



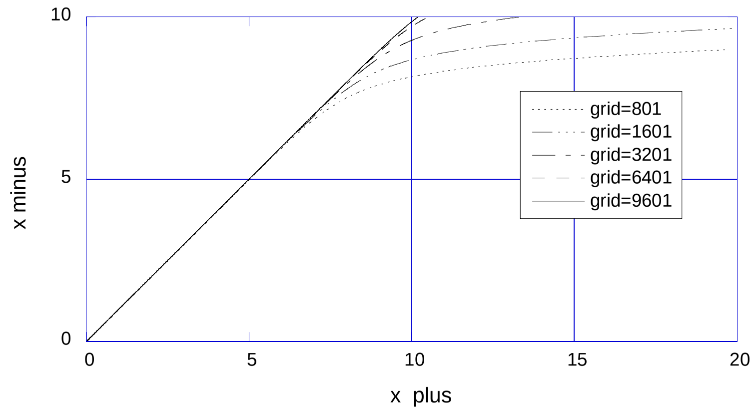


Figure 1: Numerical grid structure. Initial data are given on null hypersurfaces $\Sigma_\pm$ ($x^\mp = 0$, $x^\pm > 0$) and their intersection S.
Figure 2: Convergence behaviour of the code for exact static wormhole initial data. The location of the trapping horizon $\vartheta_- = 0$ is plotted for several resolutions labelled by the number of grid points for $x^+ = [0, 20]$. We see that numerical truncation error eventually destroys the static configuration.

Stationary Configurations

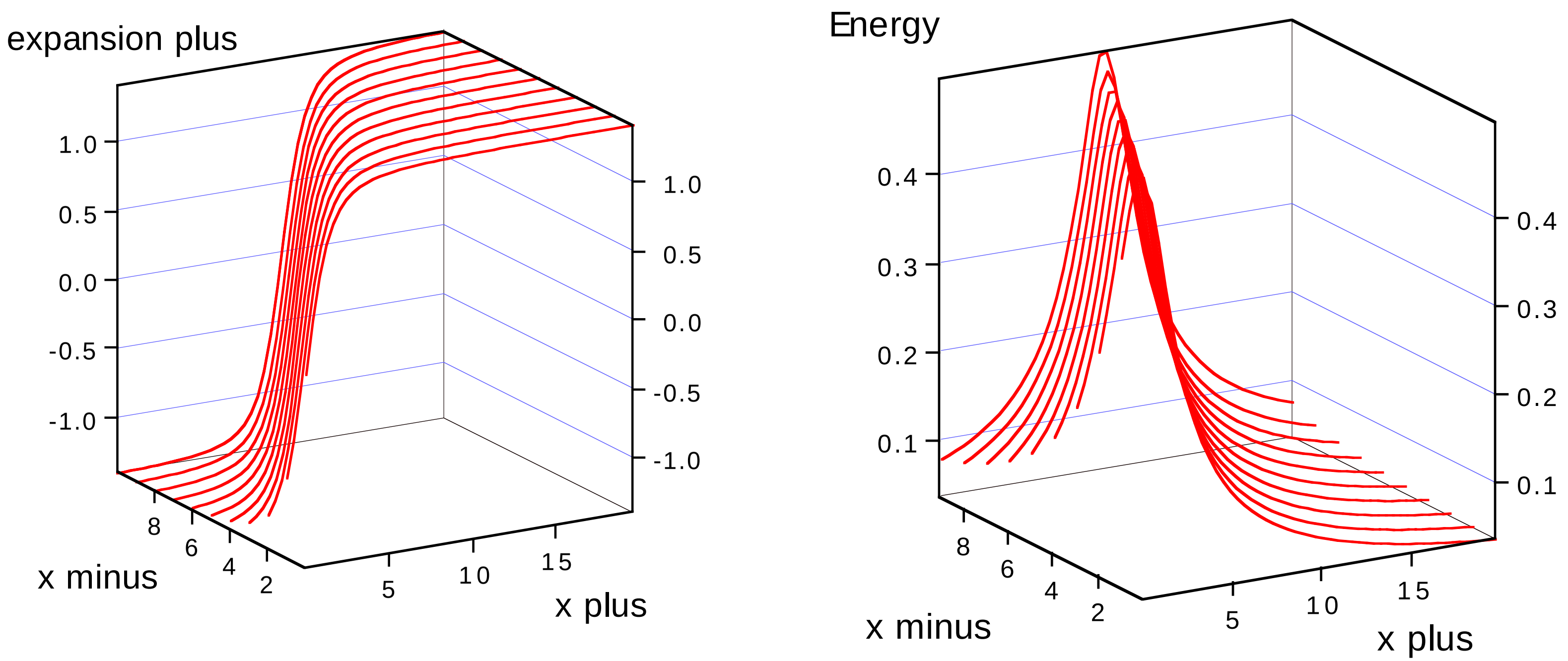


Figure 2: Static wormhole configuration obtained with the highest resolution calculation: (a) expansion ϑ_+ and (b) local gravitational mass-energy E are plotted as functions of (x^+, x^-). Note that the energy is positive and tends to zero at infinity.

Ghost pulse input – Bifurcation of the horizons

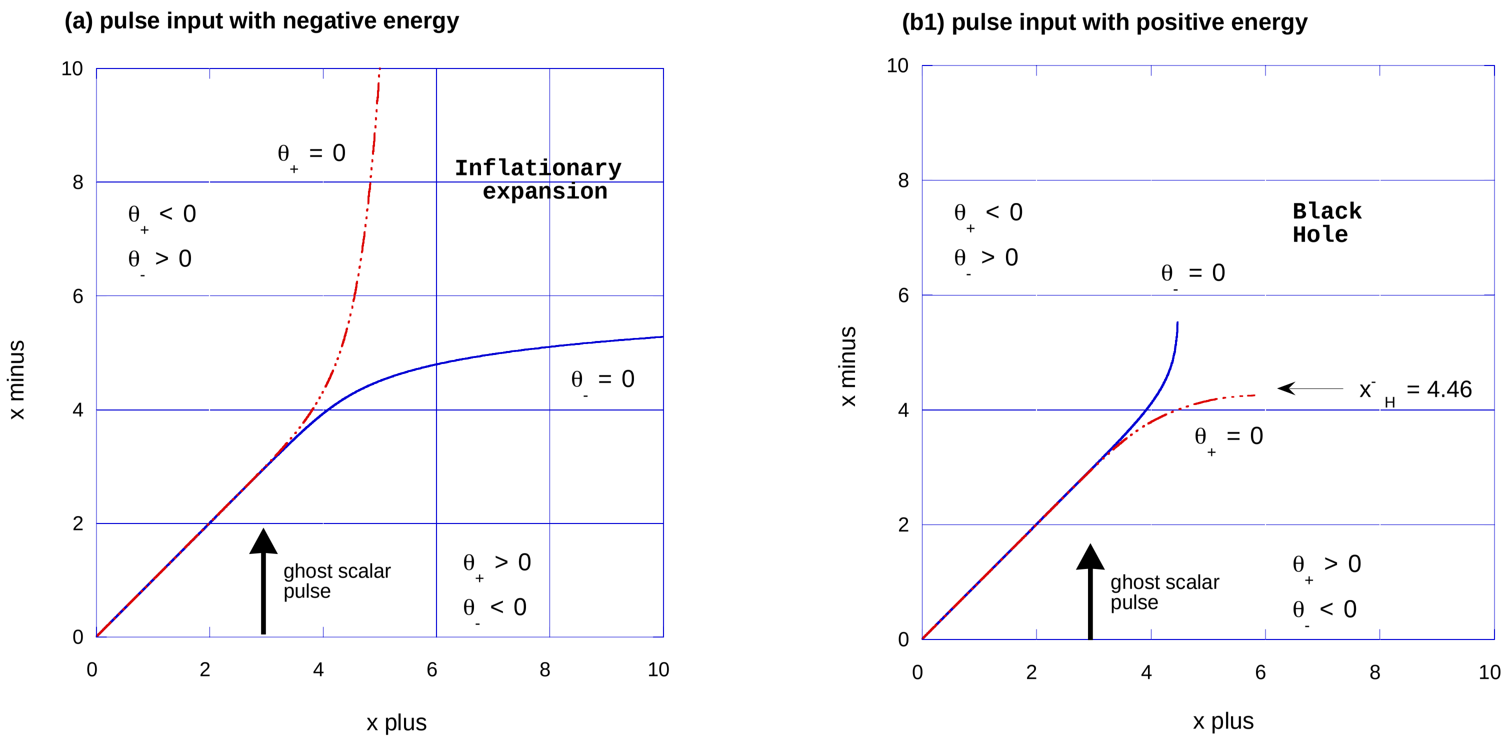
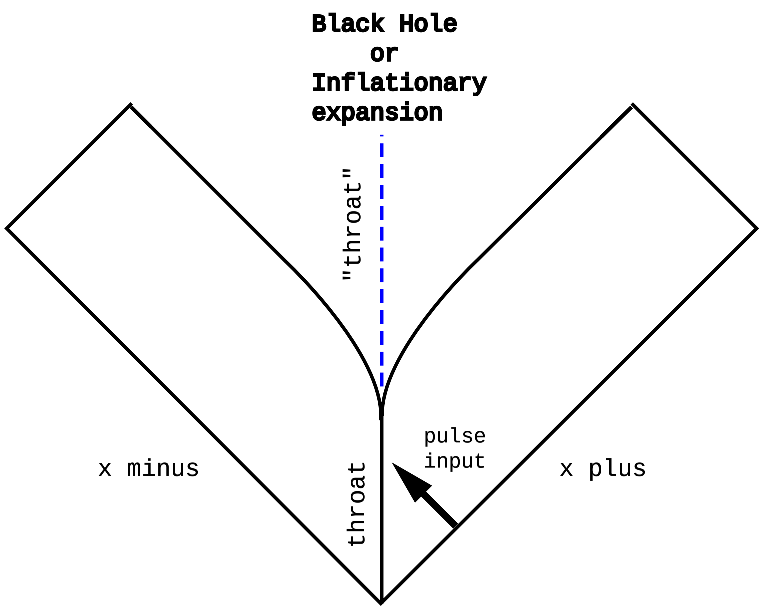


Figure 3: Horizon locations, $\vartheta_{\pm} = 0$, for perturbed wormhole. Fig.(a) is the case we supplement the ghost field, $c_a = 0.1$, and (b1) and (b2) are where we reduce the field, $c_a = -0.1$ and -0.01. Dashed lines and solid lines are $\vartheta_+ = 0$ and $\vartheta_- = 0$ respectively. In all cases, the pulse hits the wormhole throat at $(x^+, x^-) = (3, 3)$. A 45° counterclockwise rotation of the figure corresponds to a partial Penrose diagram.

Bifurcation of the horizons – go to a Black Hole or Inflationary expansion



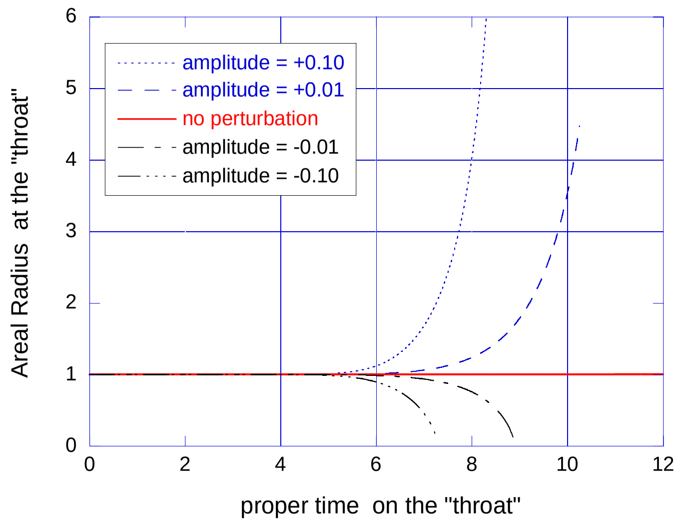


Figure 4: Partial Penrose diagram of the evolved space-time.
Figure 6: Areal radius r of the "throat" $x^+ = x^-$, plotted as a function of proper time. Additional negative energy causes inflationary expansion, while reduced negative energy causes collapse to a black hole and central singularity.

Local Energy Measure – Determination of the Black Hole Mass

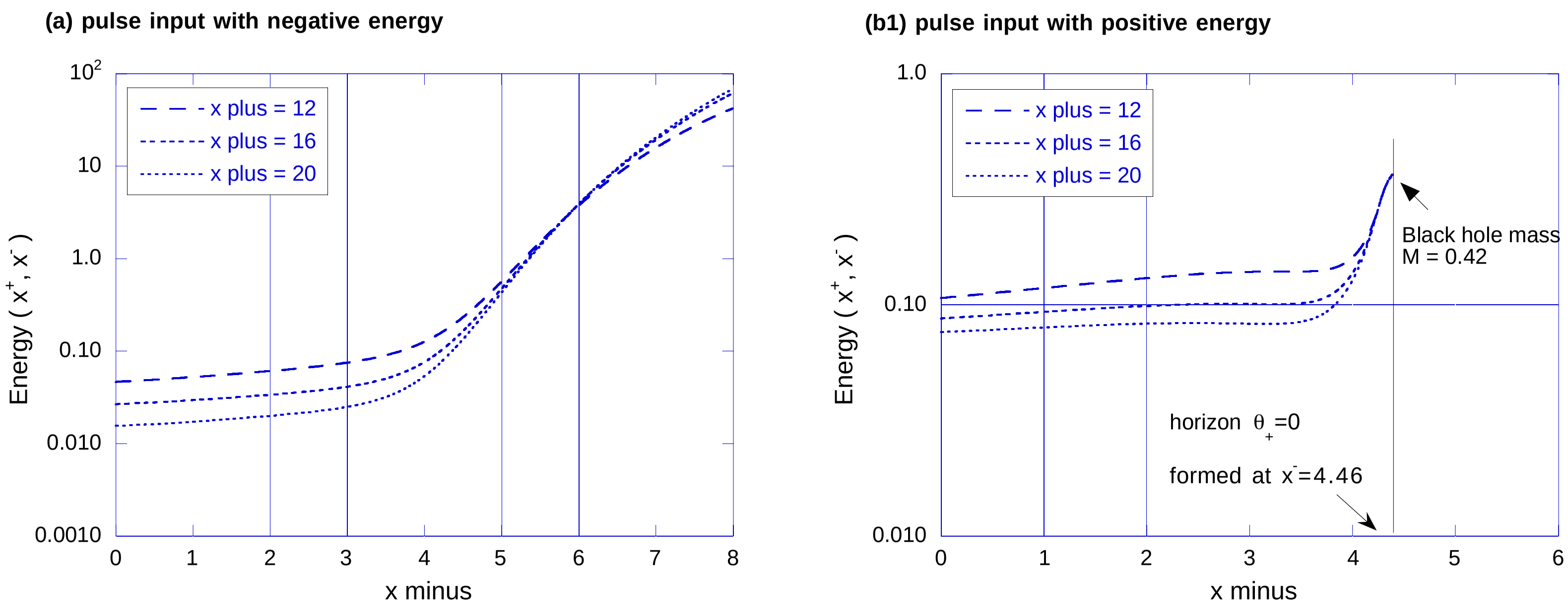
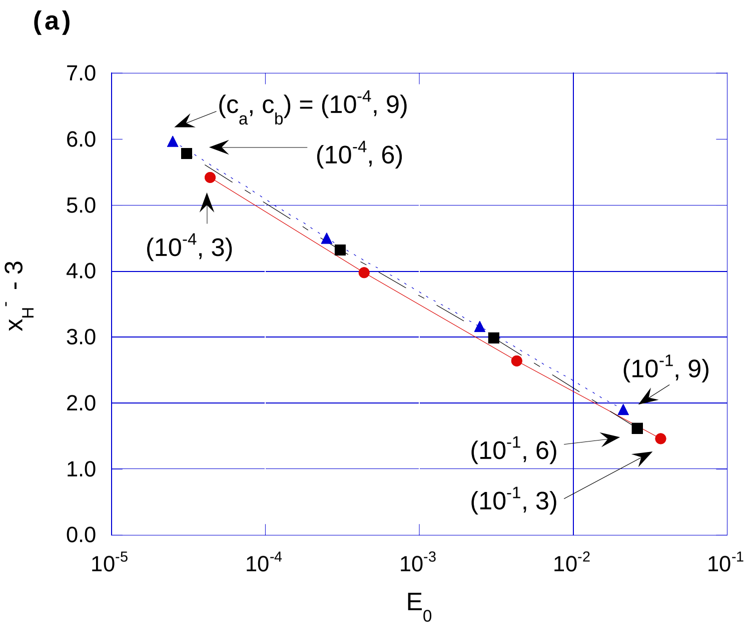


Figure 7: Energy $E(x^+, x^-)$ as a function of x^-, for $x^+ = 12, 16, 20$. Here c_a is (a) 0.05, (b1) -0.1 and (b2) -0.01. The energy for different x^+ coincides at the final horizon location x_H^-, indicating that the horizon quickly attains constant mass $M = E(\infty, x_H^-)$. This is the final mass of the black hole or cosmological horizon.

Is there a Minimum Black Hole Mass to be formed?



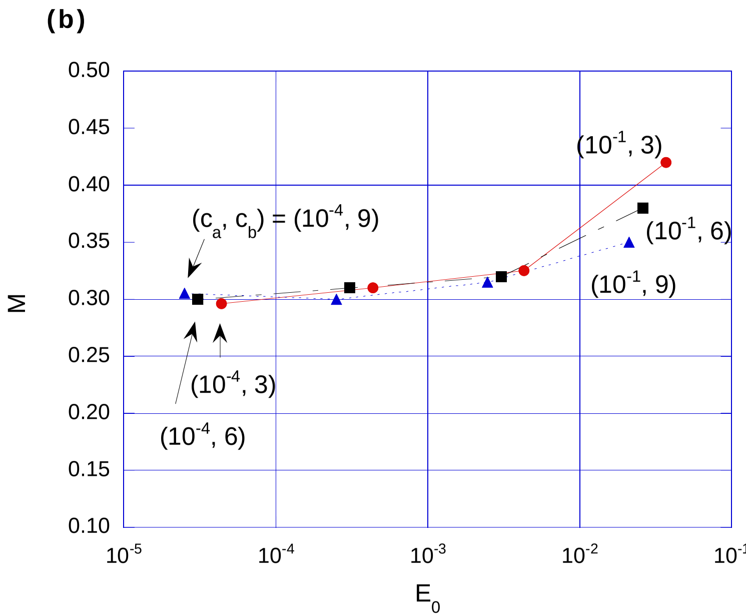


Figure 8: Relation between the initial perturbation and the final mass of the black hole. (a) The trapping horizon ($\vartheta_+ = 0$) coordinate, $x_H^- - 3$ (since we fixed $c_c = 3$), versus initial energy of the perturbation, E_0. We plotted the results of the runs of $c_a = 10^{-1}, \cdots, 10^{-4}$ with $c_b = 3, 6,$ and 9. They lie close to one line. (b) The final black hole mass M for the same examples. We see that M appears to reach a non-zero minimum for small perturbations.

Normal Pulse (a traveller) Input – Forming a Black Hole

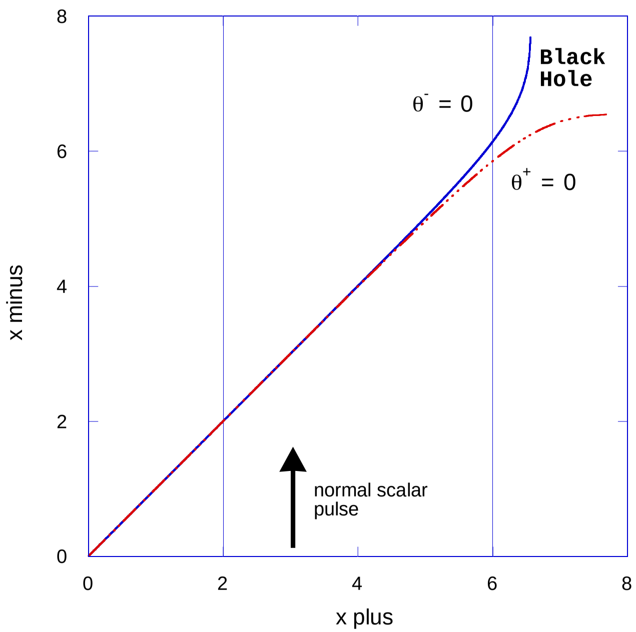


Figure 9: Evolution of a wormhole perturbed by a normal scalar field. Horizon locations: dashed lines and solid lines are $\vartheta_+ = 0$ and $\vartheta_- = 0$ respectively.

Critical Minimum Black Hole Mass again

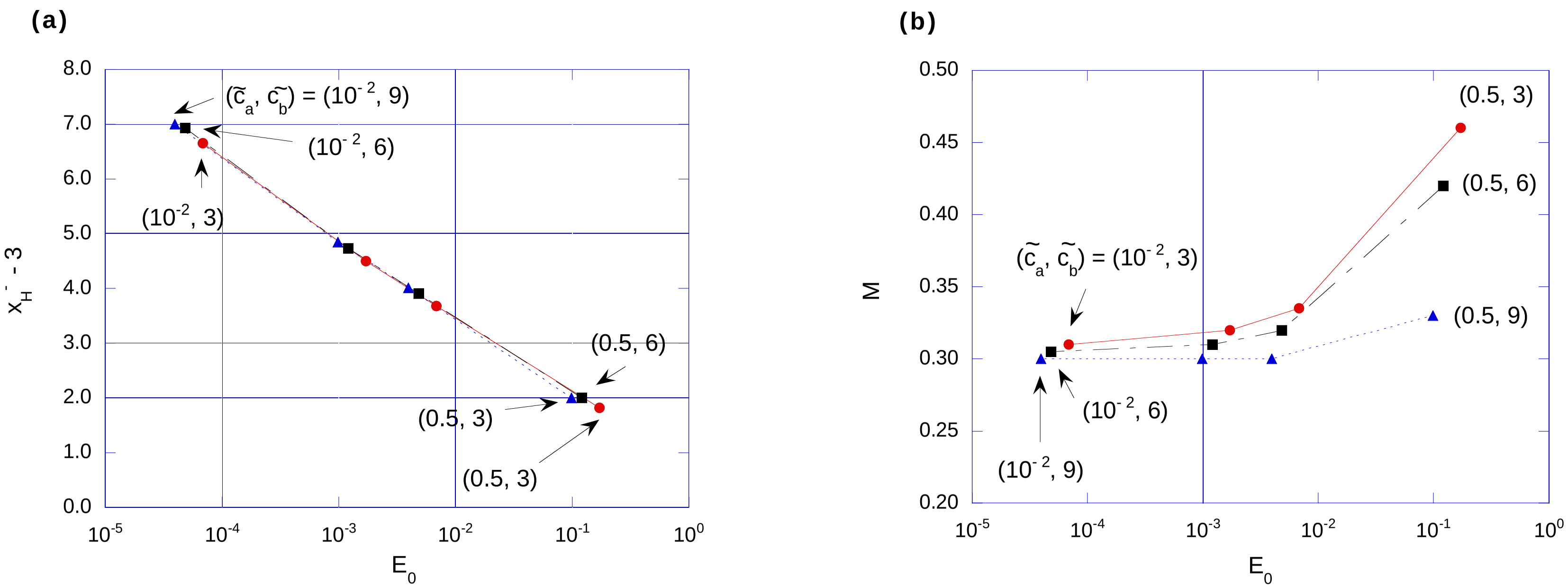


Figure 10: The same plots with Fig.**??** for the small conventional field pulses. (a) The trapping horizon ($\vartheta_+ = 0$) coordinate, $x_H^- - 3$ (since we fixed $\tilde{c}_c = 3$), versus initial energy of the perturbation, E_0. We plotted the results of the runs of $\tilde{c}_a = 0.5, \cdots, 10^{-2}$ with $\tilde{c}_b = 3, 6$, and 9. They lie close to one line. (b) The final black hole mass M for the same examples.

Travel through a Wormhole – with Maintenace Operations!

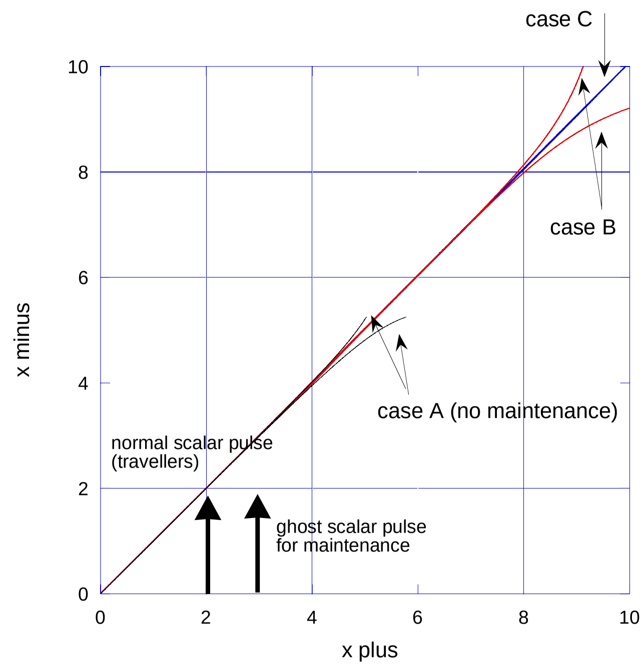


Figure 11: A trial of wormhole maintenance. After a normal scalar pulse, we signalled a ghost scalar pulse to extend the life of wormhole throat. The travellers pulse are commonly expressed with a normal scalar field pulse, $(\tilde{c}_a, \tilde{c}_b, \tilde{c}_c) = (+0.1, 6.0, 2.0)$. Horizon locations $\vartheta_+ = 0$ are plotted for three cases:
(A) no maintenance case (results in a black hole),
(B) with maintenance pulse of $(c_a, c_b, c_c) = (0.02390, 6.0, 3.0)$ (results in an inflationary expansion),
(C) with maintenance pulse of $(c_a, c_b, c_c) = (0.02385, 6.0, 3.0)$ (keep stationary structure upto the end of this range).

Discussion

Dynamics of the Ellis-Morris-Thorne traversible wormhole

$\Rightarrow$ WH is Unstable

(A) with positive energy pulse $\Rightarrow$ Black Hole

(B) with negative energy pulse $\Rightarrow$ Inflationary expansion

$\Rightarrow$ (A) confirms duality conjecture between BH and WH.

$\Rightarrow$ (B) provides a mechanism for enlarging a quantum wormhole to macroscopic size.

- We answered to the question of :
  what happens if our hero (or heroine) attempts to traverse the wormhole.
- New discoveries of the critical behaviour.

"Science can be stranger than science fiction."