Constraint Propagation Revisited

--- Adjusted ADM formulation for Numerical Relativity ---

Hisaaki Shinkai (Osaka Inst. Technology)　真貝寿明（大阪工業大学）
Gen Yoneda (Waseda Univ.)　米田　元（早稲田大学）

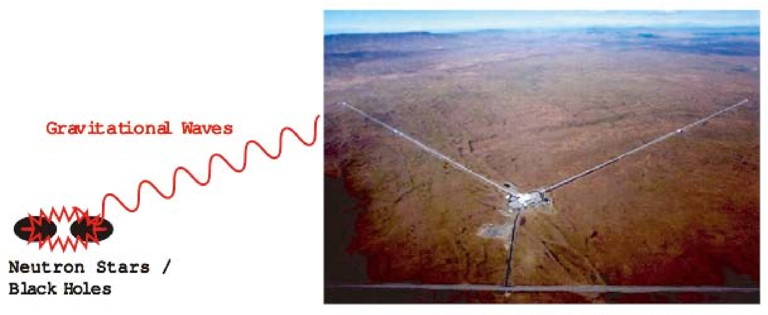
Outline

-- Proposal of a formulation for stable numerical evolution in General Relativity
-- Adjust ADM formulation with constraints ==>> Attractor System
-- A new criteria for adjusting rules
-- Numerical test with 3D Teukolsky wave evolution ==>> Better & Longer Stability

1 Numerical Relativity and "Formulation" Problem

Numerical Relativity – Necessary for unveiling the nature of strong gravity

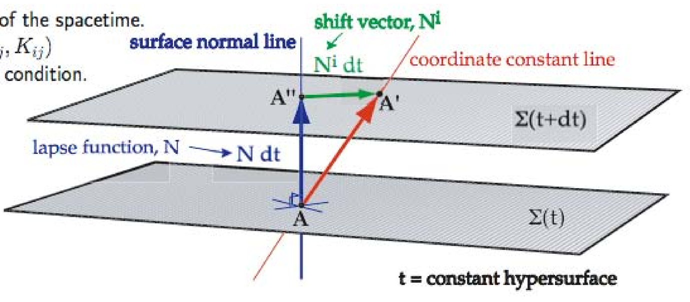
- Gravitational Wave from colliding Black Holes, Neutron Stars, Supernovae, ...
- Relativistic Phenomena like Cosmology, Active Galactic Nuclei, ...
- Mathematical feedbacks to Singularity, Exact Solutions, Chaotic behavior, ...
- Labratory of Gravitational theories, Higher dimensional models, ...



LIGO/VIRGO/GEO/TAMA, ...

The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

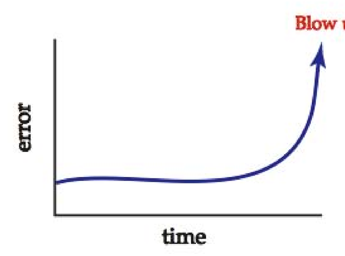
3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.

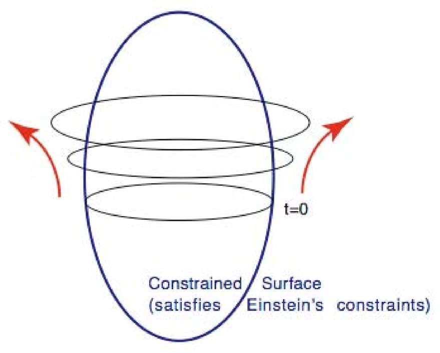


	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$ div $\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} = \mathrm{rot}\,\mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -\mathrm{rot}\,\mathbf{E}$	$\partial_t\gamma_{ij} = -2NK_{ij} + D_jN_i + D_iN_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}KK_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN + (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} - N\gamma_{ij}\Lambda - \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

Best Einstein formulation for long-term stable and accurate simulation?

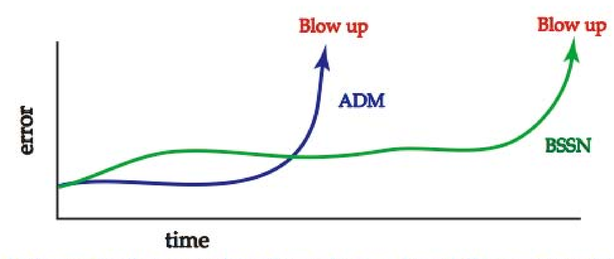
Many (too many) trials and errors, not yet a systematical understanding.





Best Einstein formulation for long-term stable and accurate simulation?

Many (too many) trials and errors, not yet a systematical understanding.



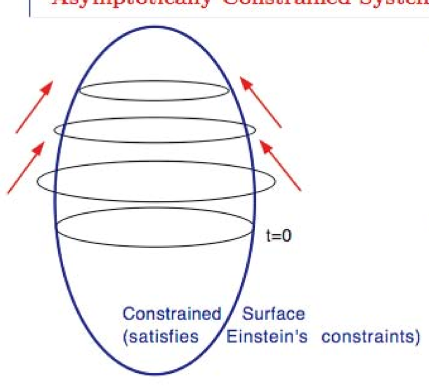
Mathematically equivalent formulation, but differ in its stability!

strategy 0: Arnowitt-Deser-Misner formulation
strategy 1: Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM
strategy 2: Apply a formulation which reveals a hyperbolicity explicitly
strategy 3: Formulate a system which is "asymptotically constrained" against a violation of constraints

Key Fact: By adding constraints in RHS, we can kill error growing modes.

2 Idea of "Adjusted system" and Our Conjecture

Formulate a system which is "asymptotically constrained" against a violation of constraints
"Asymptotically Constrained System"– Constraint Surface as an Attractor



method 1: λ-system (Brodbeck et al, 2000)

- Add aritificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may prodict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. ⇒ for the ADM/BSSN formulation, too!!

The Idea

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) + F(C^a, \partial_b C^a, \cdots)$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$ to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \cdots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) + G(C^a, \partial_b C^a, \cdots)$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = A(\hat{C}^a)\,\hat{C}^k$

Conjecture: Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs. If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

3 Adjusted ADM systems

We adjust the standard ADM system using constraints as:

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \nabla_i\beta_j + \nabla_j\beta_i, \quad (1)$$
$$+P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k\mathcal{H}) + q^{kl}{}_{ij}(\nabla_k\mathcal{M}_l), \quad (2)$$
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha KK_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i\nabla_j\alpha + (\nabla_i\beta^k)K_{kj} + (\nabla_j\beta^k)K_{ki} + \beta^k\nabla_kK_{ij} \quad (3)$$
$$+R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k\mathcal{H}) + s^{kl}{}_{ij}(\nabla_k\mathcal{M}_l), \quad (4)$$

with constraint equations

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij}K^{ij}, \quad (5)$$
$$\mathcal{M}_i := \nabla_jK^j{}_i - \nabla_iK. \quad (6)$$

We can write the adjusted constraint propagation equations as

$$\partial_t\mathcal{H} = \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], \quad (7)$$
$$\partial_t\mathcal{M}_i = \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. \quad (8)$$

The constraint propagation equations of the original ADM equation:

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (1)

$$\partial_t\mathcal{H} = \beta^j(\partial_j\mathcal{H}) + 2\alpha K\mathcal{H} - 2\alpha\gamma^{ij}(\partial_i\mathcal{M}_j) + \alpha(\partial_l\gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j\alpha)\mathcal{M}_i,$$
$$\partial_t\mathcal{M}_i = -(1/2)\alpha(\partial_i\mathcal{H}) - (\partial_i\alpha)\mathcal{H} + \beta^j(\partial_j\mathcal{M}_i) + \alpha K\mathcal{M}_i - \beta^k\gamma^{jl}(\partial_i\gamma_{lk})\mathcal{M}_j + (\partial_i\beta_k)\gamma^{kj}\mathcal{M}_j.$$

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (2)

$$\partial_t\mathcal{H} = \beta^l\partial_l\mathcal{H} + 2\alpha K\mathcal{H} - 2\alpha\gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l\alpha)\mathcal{M}^l$$
$$= \beta^l\nabla_l\mathcal{H} + 2\alpha K\mathcal{H} - 2\alpha(\nabla_l\mathcal{M}^l) - 4(\nabla_l\alpha)\mathcal{M}^l,$$
$$\partial_t\mathcal{M}_i = -(1/2)\alpha(\partial_i\mathcal{H}) - (\partial_i\alpha)\mathcal{H} + \beta^l\nabla_l\mathcal{M}_i + \alpha K\mathcal{M}_i + (\nabla_i\beta_l)\mathcal{M}^l$$
$$= -(1/2)\alpha(\nabla_i\mathcal{H}) - (\nabla_i\alpha)\mathcal{H} + \beta^l\nabla_l\mathcal{M}_i + \alpha K\mathcal{M}_i + (\nabla_i\beta_l)\mathcal{M}^l,$$

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (3): by using Lie derivatives along αn^μ,

$$\pounds_{\alpha n^\mu}\mathcal{H} = +2\alpha K\mathcal{H} - 2\alpha\gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l\alpha)\mathcal{M}^l,$$
$$\pounds_{\alpha n^\mu}\mathcal{M}_i = -(1/2)\alpha(\partial_i\mathcal{H}) - (\partial_i\alpha)\mathcal{H} + \alpha K\mathcal{M}_i.$$

- Expression using γ_{ij} and K_{ij}

$$\partial_t\mathcal{H} = H_1^{mn}(\partial_t\gamma_{mn}) + H_2^{imn}\partial_i(\partial_t\gamma_{mn}) + H_3^{ijmn}\partial_i\partial_j(\partial_t\gamma_{mn}) + H_4^{mn}(\partial_tK_{mn}),$$
$$\partial_t\mathcal{M}_i = M_{1i}{}^{mn}(\partial_t\gamma_{mn}) + M_{2i}{}^{jmn}\partial_j(\partial_t\gamma_{mn}) + M_{3i}{}^{mn}(\partial_tK_{mn}) + M_{4i}{}^{jmn}\partial_j(\partial_tK_{mn}),$$

where

$$H_1^{mn} := -2R^{(3)mn} - \Gamma^p_{kj}\Gamma^k_{pi}\gamma^{mi}\gamma^{nj} + \Gamma^m\Gamma^n$$
$$+\gamma^{ij}\gamma^{np}(\partial_i\gamma^{mk})(\partial_j\gamma_{kp}) - \gamma^{mp}\gamma^{ni}(\partial_i\gamma^{kj})(\partial_j\gamma_{kp}) - 2KK^{mn} + 2K^n{}_jK^{mj},$$
$$H_2^{imn} := -2\gamma^{mi}\Gamma^n - (3/2)\gamma^{ij}(\partial_j\gamma^{mn}) + \gamma^{mj}(\partial_j\gamma^{in}) + \gamma^{mn}\Gamma^i,$$
$$H_3^{ijmn} := -\gamma^{ij}\gamma^{mn} + \gamma^{in}\gamma^{mj},$$
$$H_4^{mn} := 2(K\gamma^{mn} - K^{mn}),$$
$$M_{1i}{}^{mn} := \gamma^{nj}(\partial_iK^m{}_j) - \gamma^{mj}(\partial_jK^n{}_i) + (1/2)(\partial_j\gamma^{mn})K^j{}_i + \Gamma^nK^m{}_i,$$
$$M_{2i}{}^{jmn} := -\gamma^{mj}K^n{}_i + (1/2)\gamma^{mn}K^j{}_i + (1/2)K^{mn}\delta^j_i,$$
$$M_{3i}{}^{mn} := -\delta^n_i\Gamma^m - (1/2)(\partial_i\gamma^{mn}),$$
$$M_{4i}{}^{jmn} := \gamma^{mj}\delta^n_i - \gamma^{mn}\delta^j_i,$$

where we expressed $\Gamma^m = \Gamma^m_{ij}\gamma^{ij}$.

4 Additional Idea (NEW)

In order to avoid blow-up in the last stage, we prohibid the adjustments which simply produce self-growing terms (C^2) in constraint propagation, $\partial_t C$.

- If RHS of the constraint propagation accidentally includes C^2 terms,

$$\partial_t C = -aC + bC^2$$

the solution will blow-up as

$$C = \frac{-aC_0\exp(-at)}{-a + bC_0 - bC_0\exp(-at)}$$

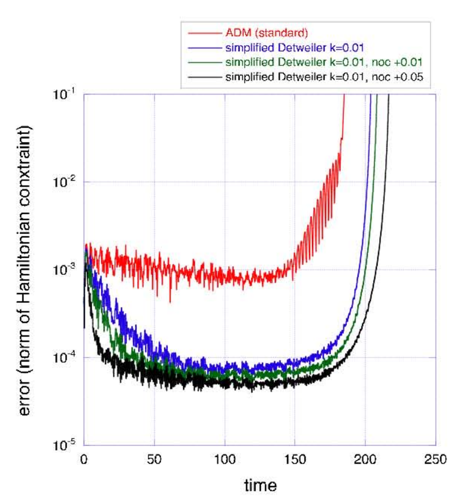
In the ADM system, we have not to put too much confidence for the adjustments using p, q, P, Q-terms for the ADM formulation.

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \nabla_i\beta_j + \nabla_j\beta_i$$
$$+P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k\mathcal{H}) + q^{kl}{}_{ij}(\nabla_k\mathcal{M}_l),$$
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha KK_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i\nabla_j\alpha + (\nabla_i\beta^k)K_{kj} + (\nabla_j\beta^k)K_{ki} + \beta^k\nabla_kK_{ij}$$
$$+R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k\mathcal{H}) + s^{kl}{}_{ij}(\nabla_k\mathcal{M}_l),$$

5 Numerical Test

Comparisons of Adjusted ADM systems (linear wave)

Original GR code based on Cactus framework.



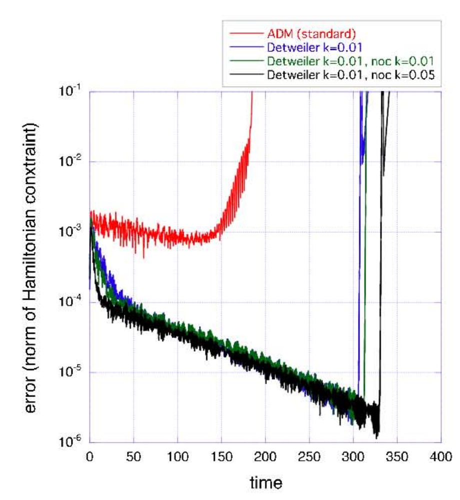
Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR code was used. Grid $= 24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method.

$\Longrightarrow$ Newly added term works effectively. 10% longer evolution is available, but not yet perfect... (to be continued)

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \nabla_i\beta_j + \nabla_j\beta_i - \kappa_1\alpha\gamma_{ij}\mathcal{H}$$
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha KK_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i\nabla_j\alpha + (\nabla_i\beta^k)K_{kj} + (\nabla_j\beta^k)K_{ki} + \beta^k\nabla_kK_{ij}$$
$$+\kappa_2\alpha\gamma_{ij}\gamma^{kl}\partial_k\mathcal{M}_l$$

Comparisons of Adjusted ADM systems (linear wave)

Original GR code based on Cactus framework.



Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR code was used. Grid $= 24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method.

$\Longrightarrow$ Newly added term works effectively. 10% longer evolution is available, but not yet perfect... (to be continued)

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \nabla_i\beta_j + \nabla_j\beta_i - \kappa_1\alpha^3\gamma_{ij}\mathcal{H}$$
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha KK_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i\nabla_j\alpha + (\nabla_i\beta^k)K_{kj} + (\nabla_j\beta^k)K_{ki} + \beta^k\nabla_kK_{ij}$$
$$+\kappa_1\alpha^3(K_{ij} - (1/3)K\gamma_{ij})\mathcal{H} + \kappa_2\alpha\gamma_{ij}\gamma^{kl}\partial_k\mathcal{M}_l$$
$$+\kappa_1\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}]\mathcal{M}_k + \kappa_1\alpha^3[\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{ki}](\nabla_k\mathcal{M}_l)$$

shinkai@is.oit.ac.jp
http://www.is.oit.ac.jp/~shinkai/

@ 16th JGRG Workshop, Niigata, 2006/11/27-12/1