Constraint Propagation Revisited
– Adjusted ADM Formulations for Numerical Relativity –

Hisa-aki Shinkai
真貝寿明　@　大阪工業大学　情報科学部
Dept of Information Science, Osaka Institute of Technology, Japan
shinkai@is.oit.ac.jp

Outline

A search of a formulation for stable numerical evolution

- Adjust ADM formulation with constraints $\Longrightarrow$ Attractor System
- A New Criteria for adjusting rules
- Numerical test with 3D Teukolsky wave evolution $\Longrightarrow$ Longer Stability

Work with **Gen Yoneda** Math. Sci. Dept., Waseda Univ., Japan

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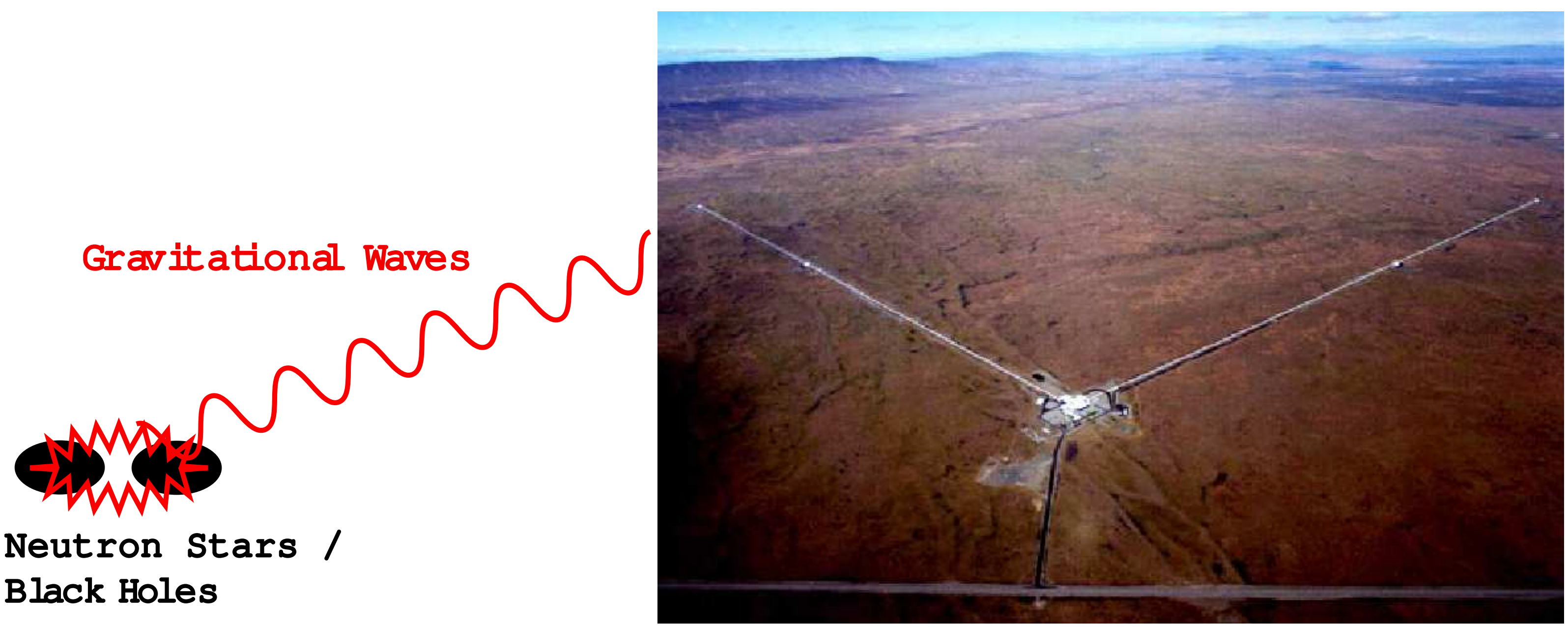
1 Numerical Relativity and "Formulation" Problem

Numerical Relativity – Necessary for unveiling the nature of strong gravity

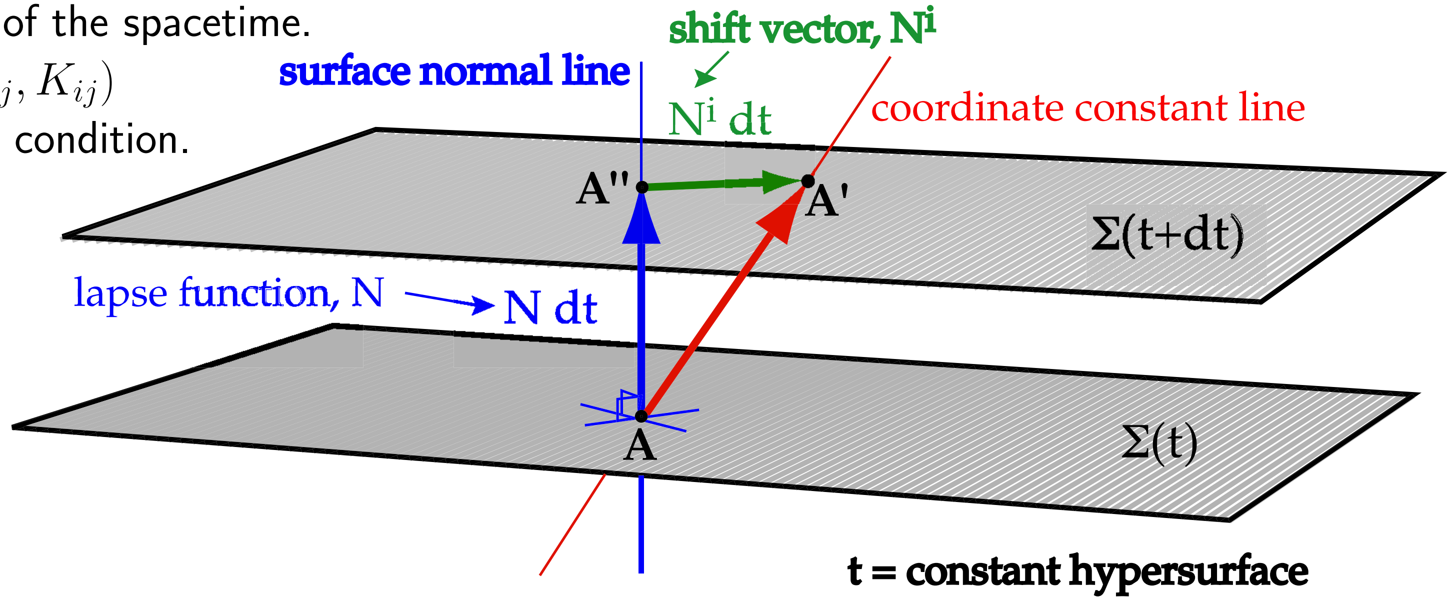
- Gravitational Wave from colliding Black Holes, Neutron Stars, Supernovae, ...
- Relativistic Phenomena like Cosmology, Active Galactic Nuclei, ...
- Mathematical feedbacks to Singularity, Exact Solutions, Chaotic behavior, ...
- Labratory of Gravitational theories, Higher dimensional models, ...



LIGO/VIRGO/GEO/TAMA, ...

The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

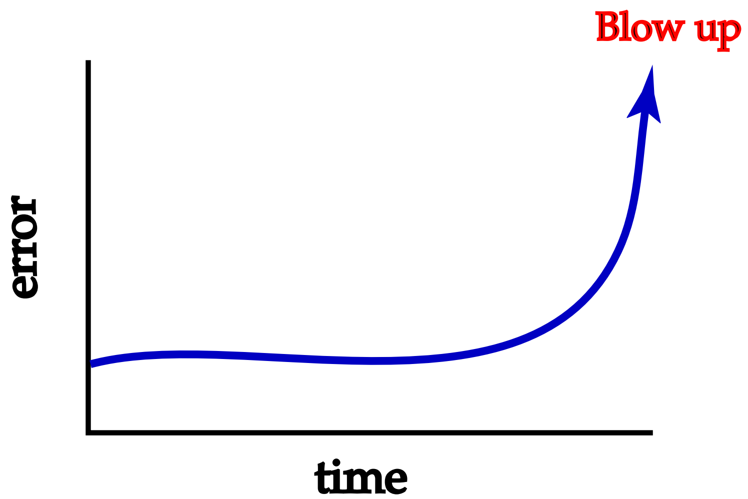
3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.

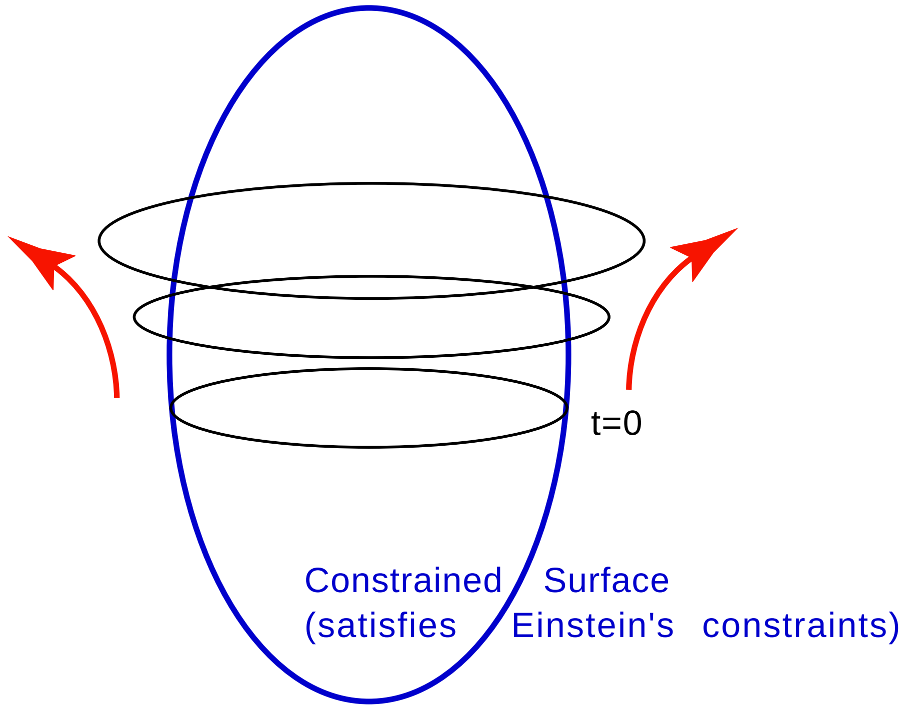


	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$ div $\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} =$ rot $\mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -$rot $\mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

Best Einstein formulation for long-term stable and accurate simulation?

Many (too many) trials and errors, not yet a systematical understanding.





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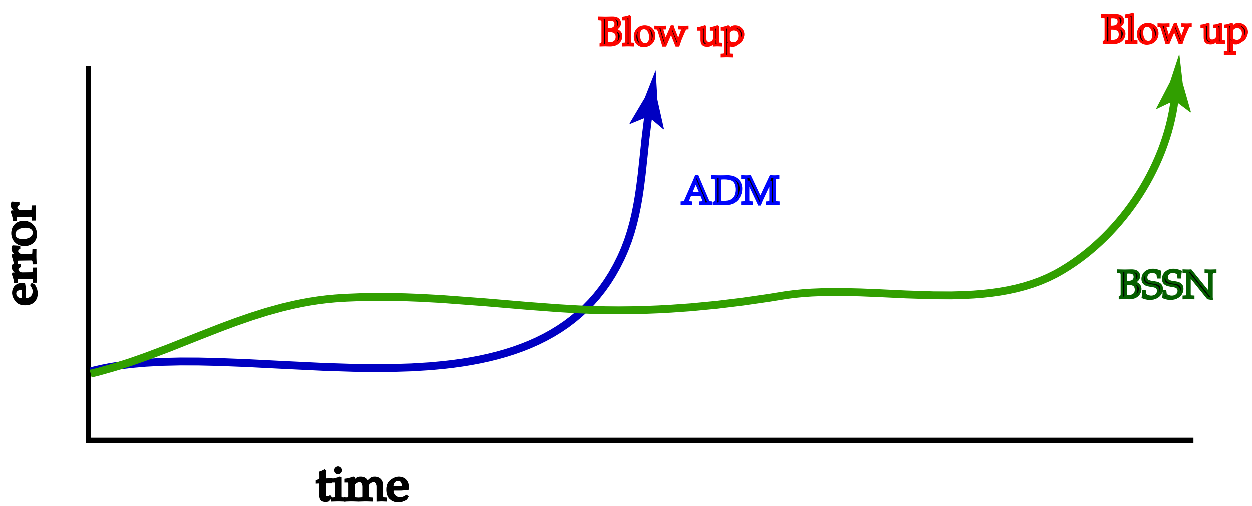


Mathematically equivalent formulation, but differ in its stability!

strategy 0: Arnowitt-Deser-Misner formulation

strategy 1: Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

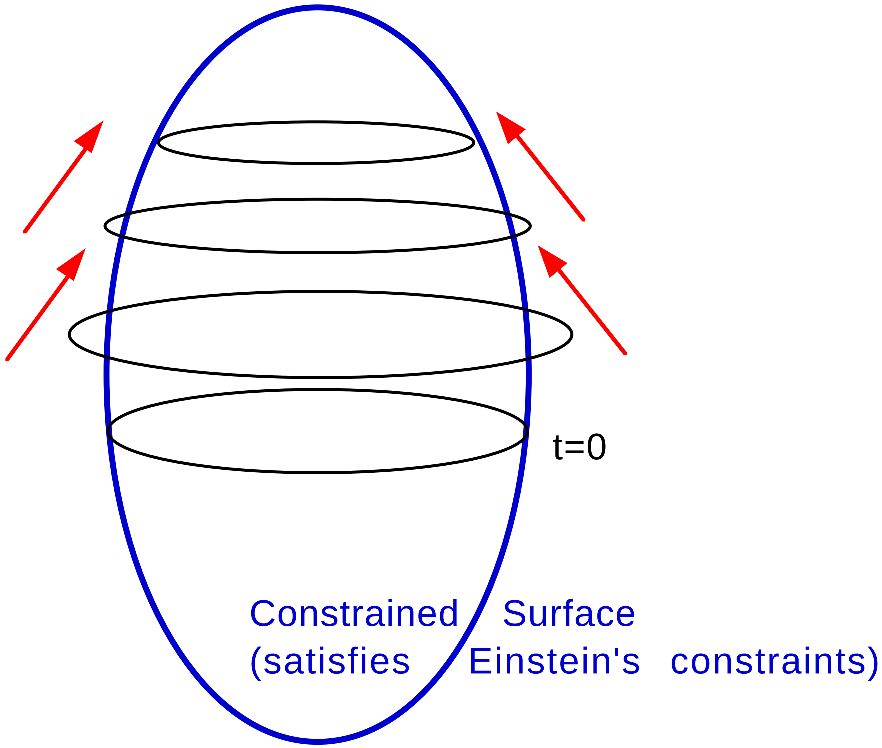
strategy 2: Apply a formulation which reveals a hyperbolicity explicitly

strategy 3: Formulate a system which is "asymptotically constrained" against a violation of constraints

Key Fact: By adding constraints in RHS, we can kill error growing modes.

2 Idea of "Adjusted system" and Our Conjecture

Formulate a system which is "asymptotically constrained" against a violation of constraints
"Asymptotically Constrained System"– Constraint Surface as an Attractor



method 1: λ-system (Brodbeck et al, 2000)

- Add aritificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may prodict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. $\Rightarrow$
  for the ADM/BSSN formulation, too!!

The Idea

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) + F(C^a, \partial_b C^a, \cdots)$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$
   to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \cdots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) + G(C^a, \partial_b C^a, \cdots)$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = A(\hat{C}^a)\, \hat{C}^k$

Conjecture: Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs. If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

3 Adjusted ADM systems

We adjust the standard ADM system using constraints as:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \tag{1}$$

$$+P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \tag{2}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, \tag{3}$$

$$+R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \tag{4}$$

with constraint equations

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij} K^{ij}, \tag{5}$$

$$\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K. \tag{6}$$

We can write the adjusted constraint propagation equations as

$$\partial_t \mathcal{H} = \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], \tag{7}$$

$$\partial_t \mathcal{M}_i = \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. \tag{8}$$

The constraint propagation equations of the original ADM equation:

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (1)

$$\begin{aligned} \partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) + \alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \\ \partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) + \alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j. \end{aligned}$$

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (2)

$$\begin{aligned} \partial_t \mathcal{H} &= \beta^l \partial_l \mathcal{H} + 2\alpha K \mathcal{H} - 2\alpha\gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l \alpha)\mathcal{M}^l \\ &= \beta^l \nabla_l \mathcal{H} + 2\alpha K \mathcal{H} - 2\alpha(\nabla_l \mathcal{M}^l) - 4(\nabla_l \alpha)\mathcal{M}^l, \\ \partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^l \nabla_l \mathcal{M}_i + \alpha K \mathcal{M}_i + (\nabla_i \beta_l)\mathcal{M}^l \\ &= -(1/2)\alpha(\nabla_i \mathcal{H}) - (\nabla_i \alpha)\mathcal{H} + \beta^l \nabla_l \mathcal{M}_i + \alpha K \mathcal{M}_i + (\nabla_i \beta_l)\mathcal{M}^l, \end{aligned}$$

- Expression using $\mathcal{H}$ and $\mathcal{M}_i$ (3): by using Lie derivatives along αn^μ,

$$\begin{aligned} \pounds_{\alpha n^\mu} \mathcal{H} &= +2\alpha K \mathcal{H} - 2\alpha\gamma^{-1/2}\partial_l(\sqrt{\gamma}\mathcal{M}^l) - 4(\partial_l \alpha)\mathcal{M}^l, \\ \pounds_{\alpha n^\mu} \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \alpha K \mathcal{M}_i. \end{aligned}$$

- Expression using γ_{ij} and K_{ij}

$$\begin{aligned} \partial_t \mathcal{H} &= H_1^{mn}(\partial_t \gamma_{mn}) + H_2^{imn}\partial_i(\partial_t \gamma_{mn}) + H_3^{ijmn}\partial_i \partial_j(\partial_t \gamma_{mn}) + H_4^{mn}(\partial_t K_{mn}), \\ \partial_t \mathcal{M}_i &= M_{1i}{}^{mn}(\partial_t \gamma_{mn}) + M_{2i}{}^{jmn}\partial_j(\partial_t \gamma_{mn}) + M_{3i}{}^{mn}(\partial_t K_{mn}) + M_{4i}{}^{jmn}\partial_j(\partial_t K_{mn}), \end{aligned}$$

where

$$H_1^{mn} := -2R^{(3)mn} - \Gamma^p_{kj}\Gamma^k_{pi}\gamma^{mi}\gamma^{nj} + \Gamma^m\Gamma^n$$

$$
\begin{aligned}
& +\gamma^{ij}\gamma^{np}(\partial_i\gamma^{mk})(\partial_j\gamma_{kp}) - \gamma^{mp}\gamma^{ni}(\partial_i\gamma^{kj})(\partial_j\gamma_{kp}) - 2KK^{mn} + 2K^n{}_jK^{mj}, \\
H_2^{imn} &:= -2\gamma^{mi}\Gamma^n - (3/2)\gamma^{ij}(\partial_j\gamma^{mn}) + \gamma^{mj}(\partial_j\gamma^{in}) + \gamma^{mn}\Gamma^i, \\
H_3^{ijmn} &:= -\gamma^{ij}\gamma^{mn} + \gamma^{in}\gamma^{mj}, \\
H_4^{mn} &:= 2(K\gamma^{mn} - K^{mn}), \\
M_{1i}{}^{mn} &:= \gamma^{nj}(\partial_i K^m{}_j) - \gamma^{mj}(\partial_j K^n{}_i) + (1/2)(\partial_j\gamma^{mn})K^j{}_i + \Gamma^n K^m{}_i, \\
M_{2i}{}^{jmn} &:= -\gamma^{mj}K^n{}_i + (1/2)\gamma^{mn}K^j{}_i + (1/2)K^{mn}\delta_i^j, \\
M_{3i}{}^{mn} &:= -\delta_i^n\Gamma^m - (1/2)(\partial_i\gamma^{mn}), \\
M_{4i}{}^{jmn} &:= \gamma^{mj}\delta_i^n - \gamma^{mn}\delta_i^j,
\end{aligned}
$$

where we expressed $\Gamma^m = \Gamma^m_{ij}\gamma^{ij}$.

4 Additional Idea (NEW)

In order to avoid blow-up in the last stage, we prohibid the adjustments which simply produce self-growing terms (C^2) in constraint propagation, $\partial_t C$.

- If RHS of the constraint propagation accidentally includes C^2 terms,

$$\partial_t C = -aC + bC^2$$

the solution will blow-up as

$$C = \frac{-aC_0 \exp(-at)}{-a + bC_0 - bC_0 \exp(-at)}$$

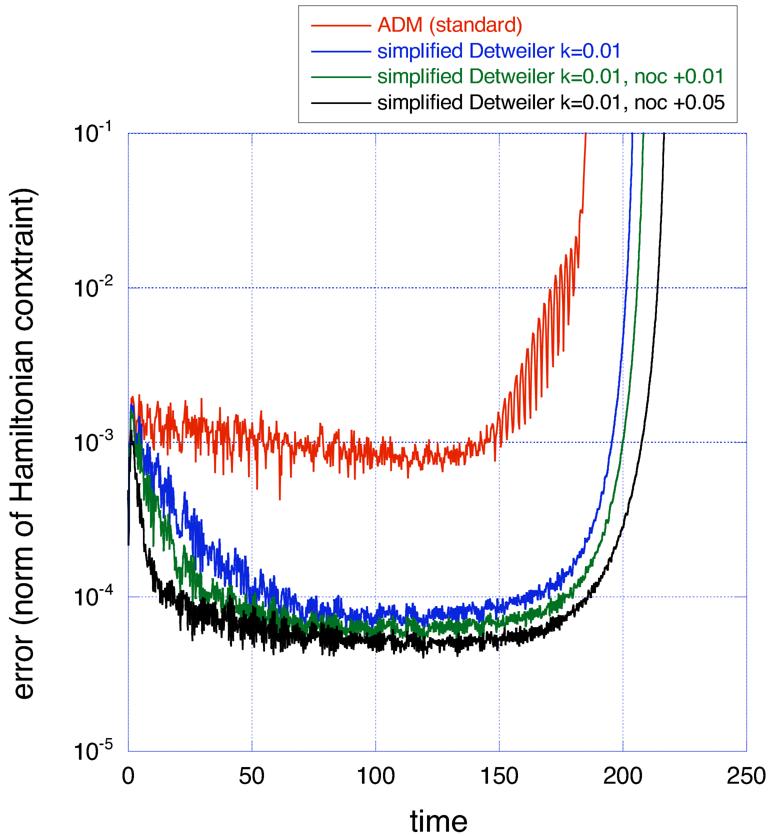
In the ADM system, we have not to put too much confidence for the adjustments using p, q, P, Q-terms for the ADM formulation.

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \\
&\quad + P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l),
\end{aligned}$$

5 Numerical Test

Comparisons of Adjusted ADM systems (linear wave)

Original GR code based on Cactus framework.



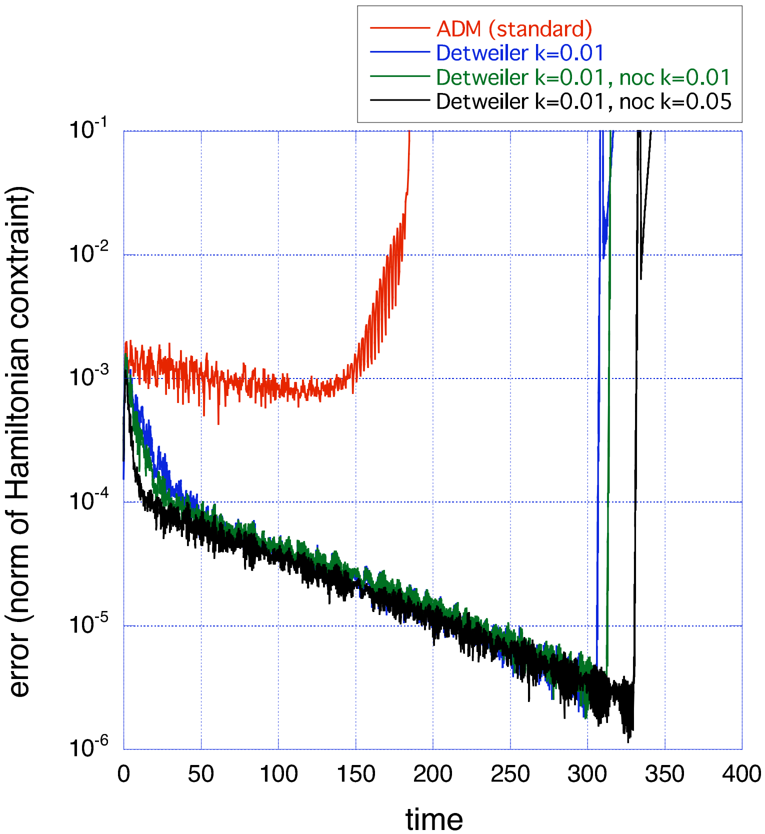
Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR code was used. Grid $= 24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method.

$\Longrightarrow$ Newly added term works effectively. 10% longer evolution is available, but not yet perfect... (to be continued)

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_1 \alpha \gamma_{ij} \mathcal{H} \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + \kappa_2 \alpha \gamma_{ij} \gamma^{kl} \partial_k \mathcal{M}_l
\end{aligned}$$

Comparisons of Adjusted ADM systems (linear wave)

Original GR code based on Cactus framework.



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$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_1 \alpha^3 \gamma_{ij} \mathcal{H} \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + \kappa_1 \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_2 \alpha \gamma_{ij} \gamma^{kl} \partial_k \mathcal{M}_l \\
&\quad + \kappa_1 \alpha^2 [3(\partial_{(i} \alpha) \delta^k_{j)} - (\partial_l \alpha) \gamma_{ij} \gamma^{kl}] \mathcal{M}_k + \kappa_1 \alpha^3 [\delta^k_{(i} \delta^l_{j)} - (1/3) \gamma_{ij} \gamma^{ki}] (\nabla_k \mathcal{M}_l)
\end{aligned}
$$