

拘束条件式の発散を防ぐ数値相対論の定式化

--- Adjusted ADM formulation for Numerical Relativity ---

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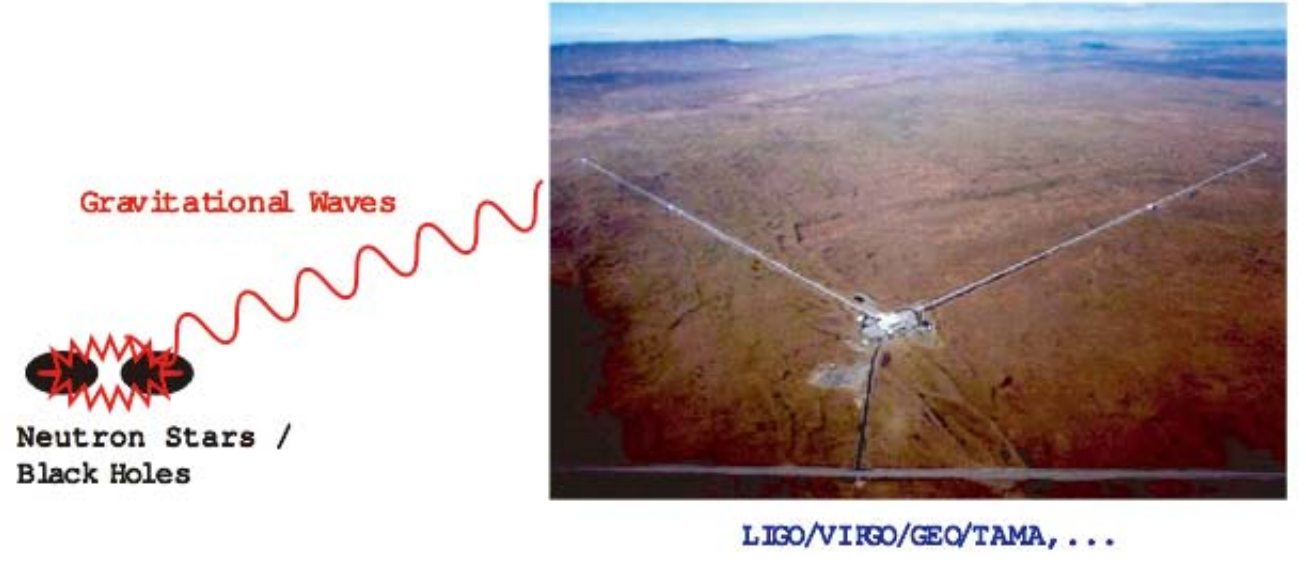
Outline

数値相対論では、Einstein 方程式の定式化によって、計算の安定性が変わることが明らかになっている。我々は、定式化の手法として、ADM 形式を出発点にして、どのように発展方程式を補正すれば、拘束多様体に計算が収束してゆくか、という観点から、拘束条件式の時間発展式を固有値解析することを提案している。今回は、これまでに提案した一部の Lagrange 補正が、拘束条件式の発展で発散項を含みうることを明らかにし、そのような補正を行わないための一般的な指針と定式化の提案を行う。新たな補正では、数値計算の寿命がこれまでより、約 2 割伸び、通常の ADM 形式よりは 2 倍以上の安定な計算時間を実現する例が得られた。

1 Numerical Relativity and "Formulation" Problem

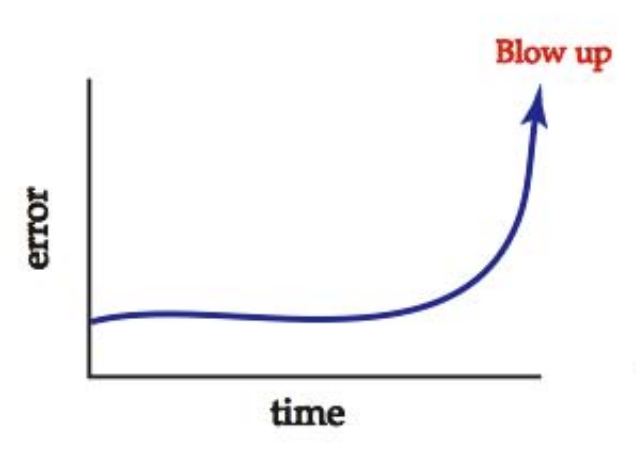
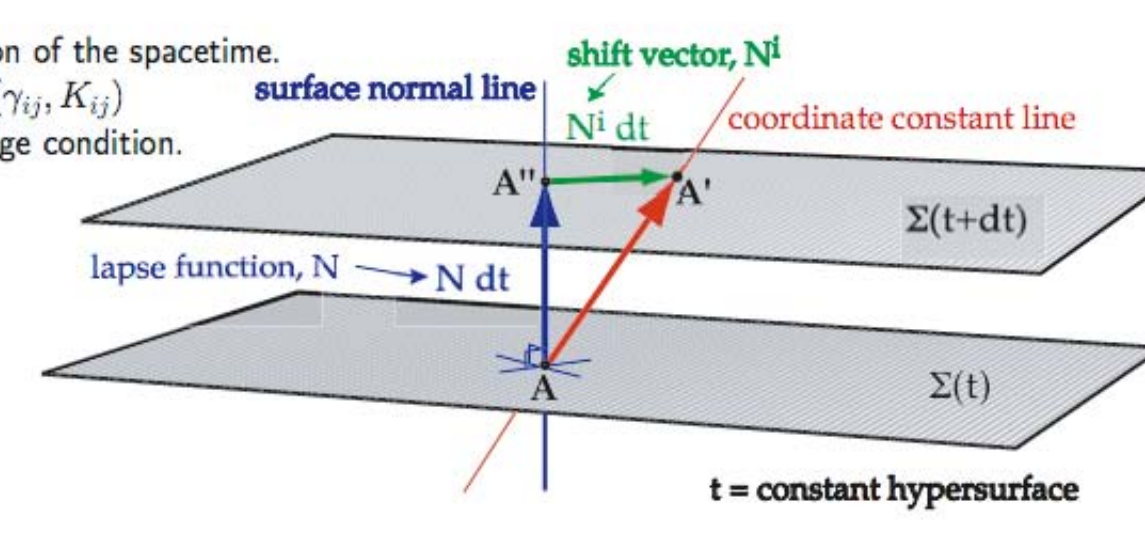
Numerical Relativity - Necessary for unveiling the nature of strong gravity

- Gravitational Wave from colliding Black Holes, Neutron Stars, Supernovae, ...
- Relativistic Phenomena like Cosmology, Active Galactic Nuclei, ...
- Mathematical feedbacks to Singularity, Exact Solutions, Chaotic behavior, ...
- Laboratory of Gravitational theories, Higher dimensional models, ...



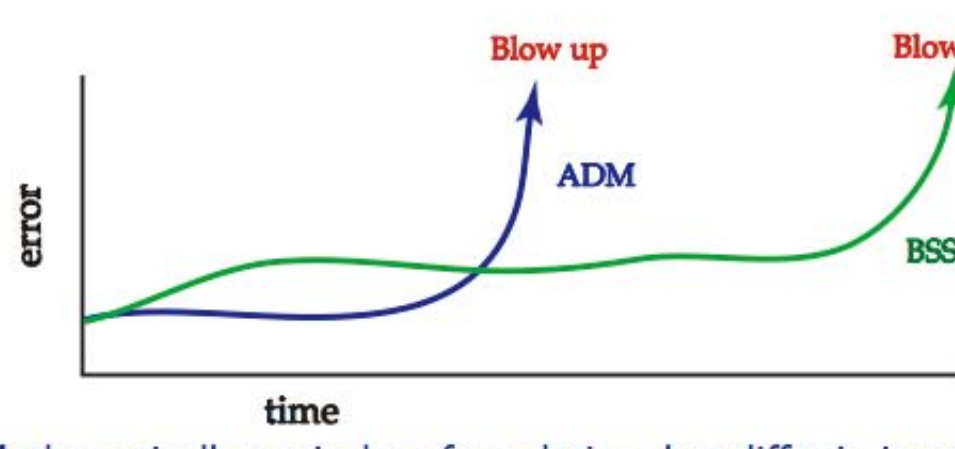
Best Einstein formulation for long-term stable and accurate simulation?

Many (too many) trials and errors, not yet a systematical understanding.

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Mathematically equivalent formulation, but differ in its stability!

strategy 0: Arnowitt-Deser-Misner formulation
 strategy 1: Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM
 strategy 2: Apply a formulation which reveals a hyperbolicity explicitly
 strategy 3: Formulate a system which is "asymptotically constrained" against a violation of constraints

Key Fact: By adding constraints in RHS, we can kill error growing modes.

3 Adjusted ADM systems

We adjust the standard ADM system using constraints as:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i$$

$$+ P_{ij} \mathcal{H} + Q_{ij}^k M_k + p_{ij}^k (\nabla_k \mathcal{H}) + q_{ij}^k (\nabla_k M_i)$$

$$\partial_t K_{ij} = \alpha R_{ij}^{(3)} + \alpha K K_{ij} - 2\alpha K_{ik} K_j^k - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$

$$+ R_{ij} \mathcal{H} + S_{ij}^k M_k + r_{ij}^k (\nabla_k \mathcal{H}) + s_{ij}^k (\nabla_k M_i)$$

with constraint equations

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij} K^{ij}$$

$$M_i := \nabla_j K_j^i - \nabla_i K$$

We can write the adjusted constraint propagation equations as

$$\partial_t \mathcal{H} = (\text{original terms}) + H_1^{(3)} \mathcal{H} + H_2^{(3)} \mathcal{H} + H_3^{(3)} \mathcal{H} + H_4^{(3)} \mathcal{H}$$

$$\partial_t M_i = (\text{original terms}) + M_1^{(3)} M_i + M_2^{(3)} M_i + M_3^{(3)} M_i + M_4^{(3)} M_i$$

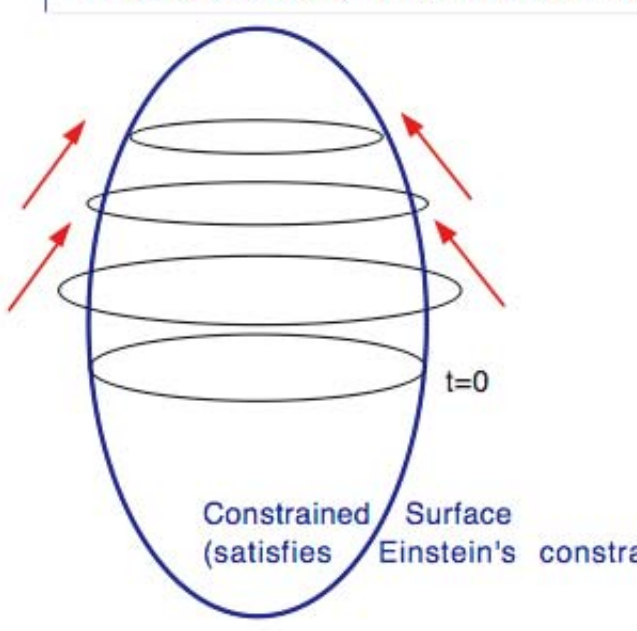
The constraint propagation equations of the original ADM equation:

- Expression using \mathcal{H} and M_i (1)
- Expression using \mathcal{H} and M_i (2)
- Expression using \mathcal{H} and M_i (3): by using Lie derivatives along α^{ab}
- Expression using γ_{ij} and K_{ij}

2 Idea of "Adjusted system" and Our Conjecture

Formulate a system which is "asymptotically constrained" against a violation of constraints

"Asymptotically Constrained System" - Constraint Surface as an Attractor



method 1: λ -system (Brodbeck et al, 2000)

- Add artificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may predict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. => for the ADM/BSSN formulation, too!

The Idea

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \dots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \dots) + F(C^a, \partial_b C^a, \dots)$
3. choose appropriate $F(C^a, \partial_b C^a, \dots)$ to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \dots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \dots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \dots) + G(C^a, \partial_b C^a, \dots)$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^a = A(\hat{C}^a) \hat{C}^a$

Conjecture: Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs. If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

4 Additional Idea (NEW)

In order to avoid blow-up in the last stage, we prohibited the adjustments which simply produce self-growing terms (C^2) in constraint propagation, $\partial_t C$.

- If RHS of the constraint propagation accidentally includes C^2 terms, $\partial_t C = -aC + bC^2$

the solution will blow-up as

$$C = \frac{-aC_0 \exp(-at)}{-a + bC_0 - bC_0 \exp(-at)}$$

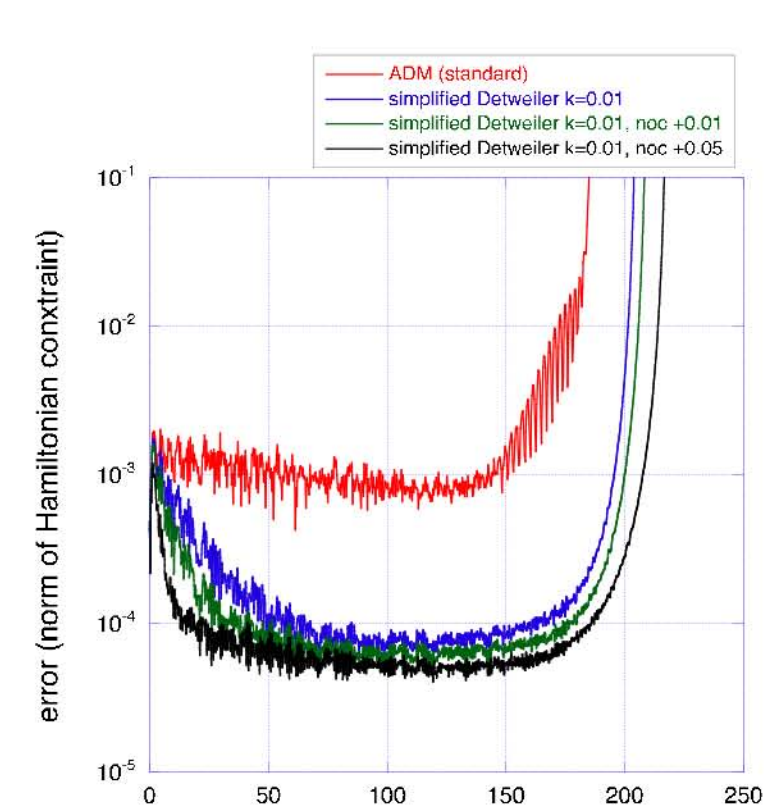
In the ADM system, we have not to put too much confidence for the adjustments using p, q, P, Q -terms for the ADM formulation.

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i + P_{ij} \mathcal{H} + Q_{ij}^k M_k + p_{ij}^k (\nabla_k \mathcal{H}) + q_{ij}^k (\nabla_k M_i)$$

$$\partial_t K_{ij} = \alpha R_{ij}^{(3)} + \alpha K K_{ij} - 2\alpha K_{ik} K_j^k - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + R_{ij} \mathcal{H} + S_{ij}^k M_k + r_{ij}^k (\nabla_k \mathcal{H}) + s_{ij}^k (\nabla_k M_i)$$

5 Numerical Test

Comparisons of Adjusted ADM systems (linear wave)

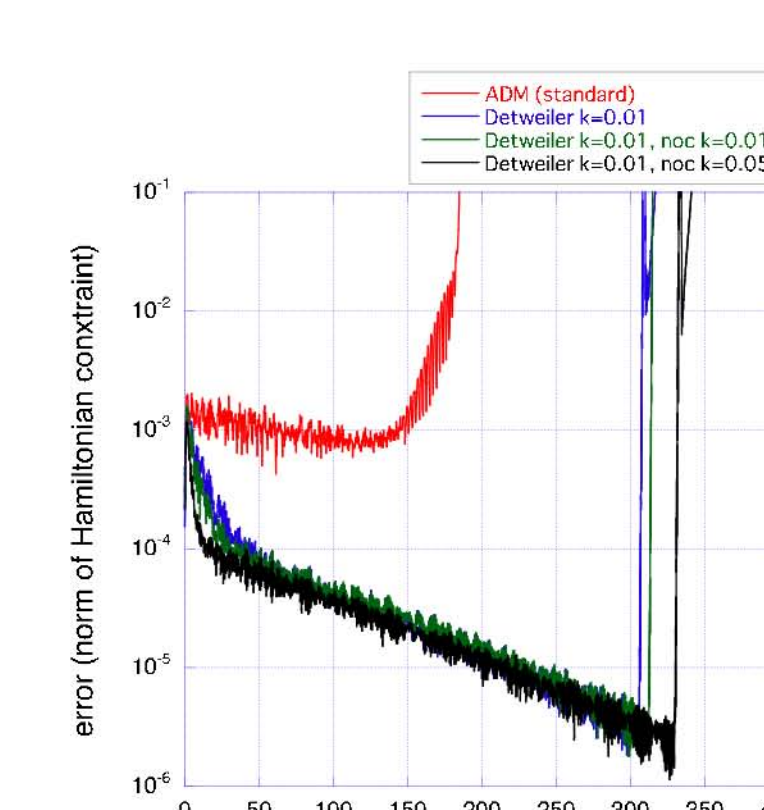


Original GR code based on Cactus framework.

Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR code was used. Grid = 24^3 , $\Delta x = 0.25$, iterative Crank-Nicholson method.

⇒ Newly added term works effectively. 10% longer evolution is available, but not yet perfect... (to be continued)

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$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_1 \alpha^3 \gamma_{ij} \mathcal{H}$$

$$\partial_t K_{ij} = \alpha R_{ij}^{(3)} + \alpha K K_{ij} - 2\alpha K_{ik} K_j^k - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + \kappa_2 \alpha^3 \gamma_{ij}^k \partial_k M_i$$

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