Adjusted ADM形式による数値計算の安定化

大阪工業大学情報科学部　真貝寿明
早稲田大学理工学部　米田　元

- 数値相対論の定式化問題（Formulation problem）
- Adjusted ADM形式（ADM+Lagrange乗数補正）で計算は安定化する
- Teukolsky waveの3+1次元発展計算
  ⇒ Standard ADMの計算寿命より　1.5〜4倍

数値相対論

目的

- 中性子星合体，ブラックホール合体による重力波の波形を理論的に計算する
- 宇宙論，高次元時空での非線形現象の解明

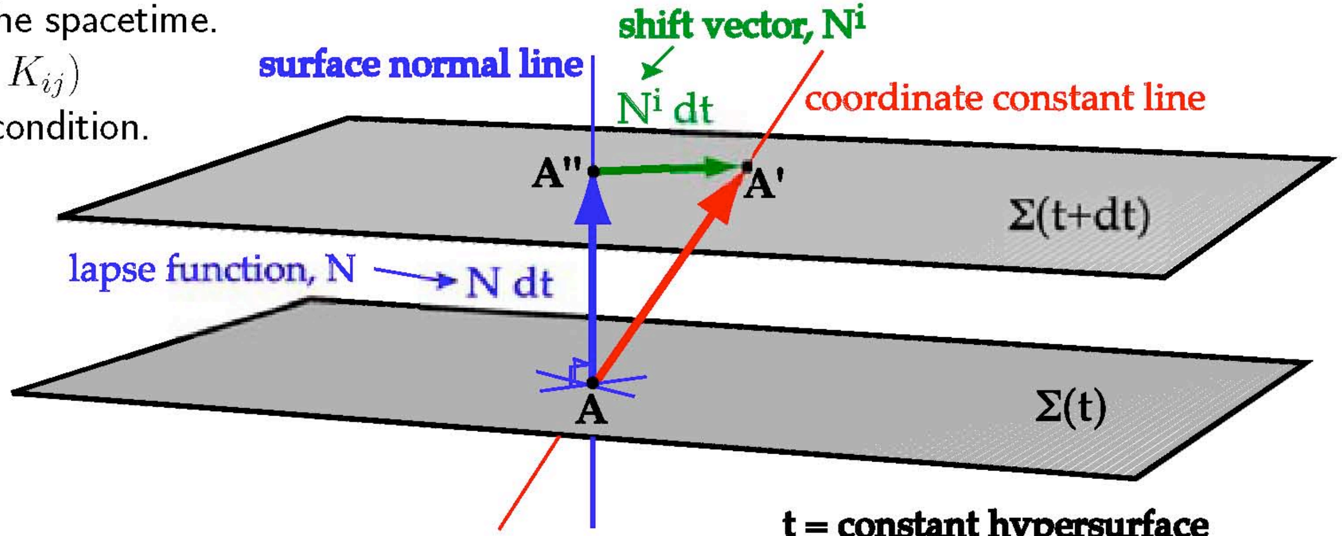
現状

- 中性子星合体 (Shibata et al)
- ブラックホール合体
  2005年-2006年に大きな進展
  (Pretorius, UTB, NASA, PSU, LSU)
  利用する方程式：BSSN formulation
  ラプス関数：　1+log slicing
  シフトベクトル：Gamma-freezing driver
  初期条件：　puncture initial data

これらの
組み合わせで
何故か
上手くいく

ADM 形式 (Arnowitt-Deser-Misner, 1962; York 1978)

3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.



	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$ div $\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} = \text{rot } \mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -\text{rot } \mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

BSSN 形式

(Nakamura et al, 1987; Shibata-Nakamura 1995, Baumgarte-Shapiro 1999)

– define new variables $(\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$, instead of the ADM's (γ_{ij}, K_{ij}) where

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},$$

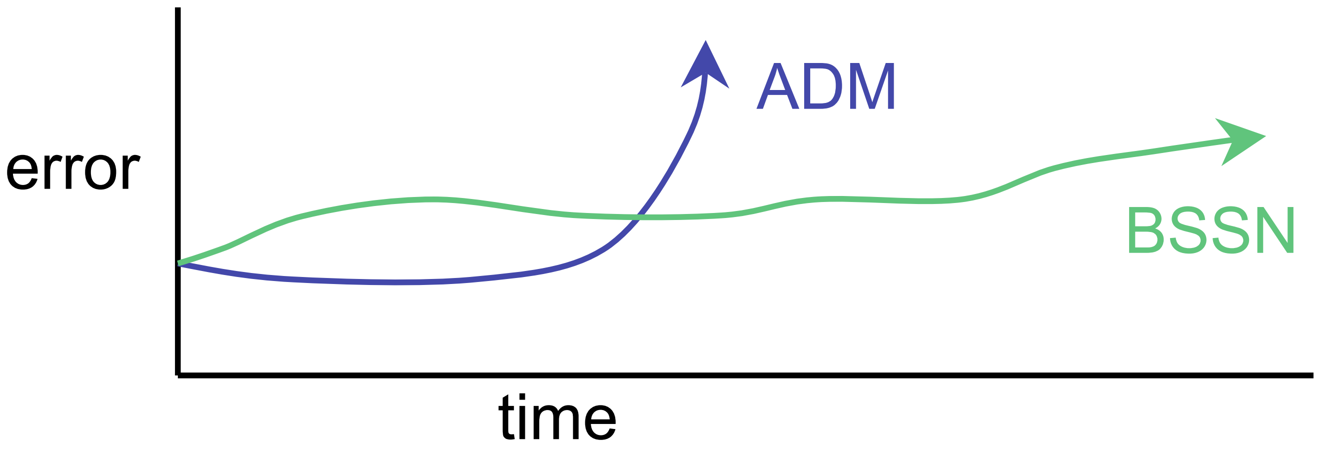
use momentum constraint in Γ^i-eq., and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

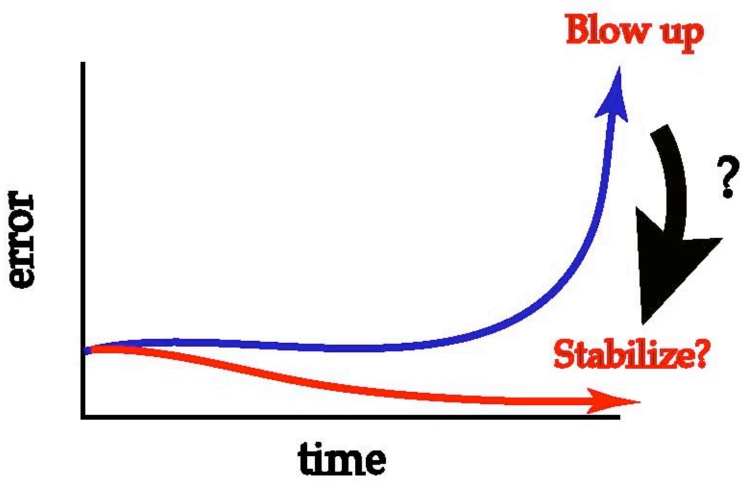
– The set of evolution equations become

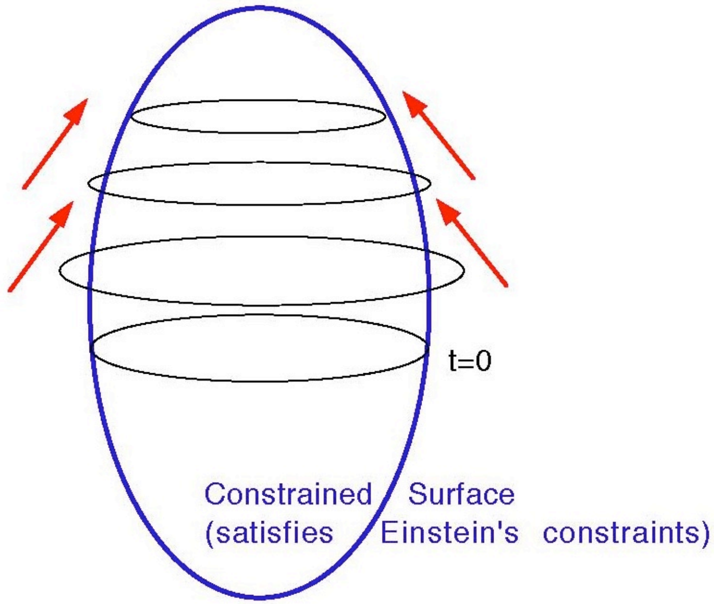
$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

$$\begin{aligned}
R_{ij} &= \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^\phi_{ij} \\
&\quad R^\phi_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi) \\
&\quad \tilde{R}_{ij} = -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}lm\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}
\end{aligned}$$

数値計算とblow-up







Adjusted Systems

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) \underbrace{+F(C^a, \partial_b C^a, \cdots)}$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$
   to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \cdots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) \underbrace{+G(C^a, \partial_b C^a, \cdots)}$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = \underbrace{A(\hat{C}^a)}\, \hat{C}^k$

発展方程式に「+0」の補正を行う ⇒ 安定性の改良

ADM vs BSSN | Adjusted ADM | Adjusted BSSN

より安定な定式化を得る指針（１）

- Constraint propagationの固有値解析

Conjecture on Constraint Amplification Factors (CAFs):

$$\partial_t \begin{pmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{pmatrix} = \begin{pmatrix} \text{Constraint} \\ \text{Propagation} \\ \text{Matrix} \end{pmatrix} \begin{pmatrix} \hat{C}_1 \\ \vdots \\ \hat{C}_N \end{pmatrix},$$

Eigenvalues = CAFs

We see more stable evolution, if CAFs have

(A) negative real-part (the constraints are forced to be diminished), or

(B) non-zero imaginary-part (the constraints are propagating away).

Adjusted ADM | ADM vs BSSN | Adjusted BSSN

・Standard ADMにはconstraint violating modeが存在
・ADMを基礎にした，より安定な定式化が存在する
・現在のBSSNよりもさらに安定な定式化が存在する

Adjusted ADM formulation (1)

We adjust the standard ADM system using constraints as:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \quad (1)$$

$$+P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \quad (2)$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, \quad (3)$$

$$+R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \quad (4)$$

with constraint equations

$$\mathcal{H} := R^{(3)} + K^2 - K_{ij} K^{ij}, \quad (5)$$

$$\mathcal{M}_i := \nabla_j K^j{}_i - \nabla_i K. \quad (6)$$

We can write the adjusted constraint propagation equations as

$$\partial_t \mathcal{H} = \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], \quad (7)$$

$$\partial_t \mathcal{M}_i = \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. \quad (8)$$

Adjusted ADM formulation (2)

Table 3. List of adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers in terms of (13) and (14). The column '1st?' and 'TRS' are the same as in table 1. The effects to amplification factors (when $\kappa > 0$) are commented for each coordinate system and for real/imaginary parts of AFs, respectively. The 'N/A' means that there is no effect due to the coordinate properties; 'not apparent' means the adjustment does not change the AFs effectively according to our conjecture; 'enl./red./min.' means enlarge/reduce/minimize, and 'Pos./Neg.' means positive/negative, respectively. These judgements are made at the $r \sim O(10M)$ region on their $t = 0$ slice.

					Schwarzschild/isotropic coordinates			iEF/PG coordinates	
No	No in table 1	Adjustment		1st?	TRS	Real	Imaginary	Real	Imaginary
0	0	–	no adjustments	yes	–	–	–	–	–
P-1	2-P	P_{ij}	$-\kappa_L \alpha^3 \gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-2	3	P_{ij}	$-\kappa_L \alpha \gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-3	–	P_{ij}	$P_{rr} = -\kappa$ or $P_{rr} = -\kappa\alpha$	no	no	slightly enl.Neg.	not apparent	slightly enl.Neg.	not apparent
P-4	–	P_{ij}	$-\kappa \gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-5	–	P_{ij}	$-\kappa \gamma_{rr}$	no	no	red. Pos./enl.Neg.	not apparent	red.Pos./enl.Neg.	not apparent
Q-1	–	$Q^k{}_{ij}$	$\kappa \alpha \beta^k \gamma_{ij}$	no	no	N/A	N/A	$\kappa \sim 1.35$ min. vals.	not apparent
Q-2	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
Q-3	–	$Q^k{}_{ij}$	$Q^r{}_{ij} = \kappa \gamma_{ij}$ or $Q^r{}_{ij} = \kappa \alpha \gamma_{ij}$	no	yes	red. abs vals.	not apparent	enl.Neg.	enl. vals.
Q-4	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa \gamma_{rr}$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
R-1	1	R_{ij}	$\kappa_F \alpha \gamma_{ij}$	yes	yes	$\kappa_F = -1/4$ min. abs vals.		$\kappa_F = -1/4$ min. vals.	
R-2	4	R_{ij}	$R_{rr} = -\kappa_\mu \alpha$ or $R_{rr} = -\kappa_\mu$	yes	no	not apparent	not apparent	red.Pos./enl.Neg.	enl. vals.
R-3	–	R_{ij}	$R_{rr} = -\kappa \gamma_{rr}$	yes	no	enl. vals.	not apparent	red.Pos./enl.Neg.	enl. vals.
S-1	2-S	$S^k{}_{ij}$	$\kappa_L \alpha^2 [3(\partial_{(i} \alpha) \delta^k_{j)} - (\partial_l \alpha) \gamma_{ij} \gamma^{kl}]$	yes	no	not apparent	not apparent	not apparent	not apparent
S-2	–	$S^k{}_{ij}$	$\kappa \alpha \gamma^{lk} (\partial_l \gamma_{ij})$	yes	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
p-1	–	$p^k{}_{ij}$	$p^r{}_{ij} = -\kappa \alpha \gamma_{ij}$	no	no	red. Pos.	red. vals.	red. Pos.	enl. vals.
p-2	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa \alpha$	no	no	red. Pos.	red. vals.	red.Pos/enl.Neg.	enl. vals.
p-3	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa \alpha \gamma_{rr}$	no	no	makes 2 Neg.	enl. vals.	red. Pos. vals.	red. vals.
q-1	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{ij} = \kappa \alpha \gamma_{ij}$	no	no	$\kappa = 1/2$ min. vals.	red. vals.	not apparent	enl. vals.
q-2	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{rr} = -\kappa \alpha \gamma_{rr}$	no	yes	red. abs vals.	not apparent	not apparent	not apparent
r-1	–	$r^k{}_{ij}$	$r^r{}_{ij} = \kappa \alpha \gamma_{ij}$	no	yes	not apparent	not apparent	not apparent	enl. vals.
r-2	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa \alpha$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
r-3	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa \alpha \gamma_{rr}$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
s-1	2-s	$s^{kl}{}_{ij}$	$\kappa_L \alpha^3 [\delta^k_{(i} \delta^l_{j)} - (1/3) \gamma_{ij} \gamma^{kl}]$	no	no	makes 4 Neg.	not apparent	makes 4 Neg.	not apparent
s-2	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{ij} = -\kappa \alpha \gamma_{ij}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.
s-3	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{rr} = -\kappa \alpha \gamma_{rr}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.

HS, Yoneda, CQG 19 (2002) 1027

より安定な定式化を得る指針（２）

・ Constraint propagationの右辺非線形項の発生を避ける

- If RHS of the constraint propagation accidentally includes C^2 terms,

$$\partial_t C = -aC + bC^2$$

the solution will blow-up as

$$C = \frac{-aC_0 \exp(-at)}{-a + bC_0 - bC_0 \exp(-at)}$$

In the ADM system, we have not to put too much confidence for the adjustments using p, q, P, Q-terms for the ADM formulation.

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i \\
&\quad + P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k)K_{kj} + (\nabla_j \beta^k)K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l),
\end{aligned}$$

Numerical Tests (method)

- Cactus-based original “GR” code
  http://www.cactuscode.org/
  [CactusBase+CactusPUGH+GR]
- 3+1dim, linear wave evolution
  (Teukolsky wave)
- harmonic slice
- periodic boundary, [-3,+3]
- iterative Crank-Nicholson method
- 12^3, 24^3, 48^3, 96^3

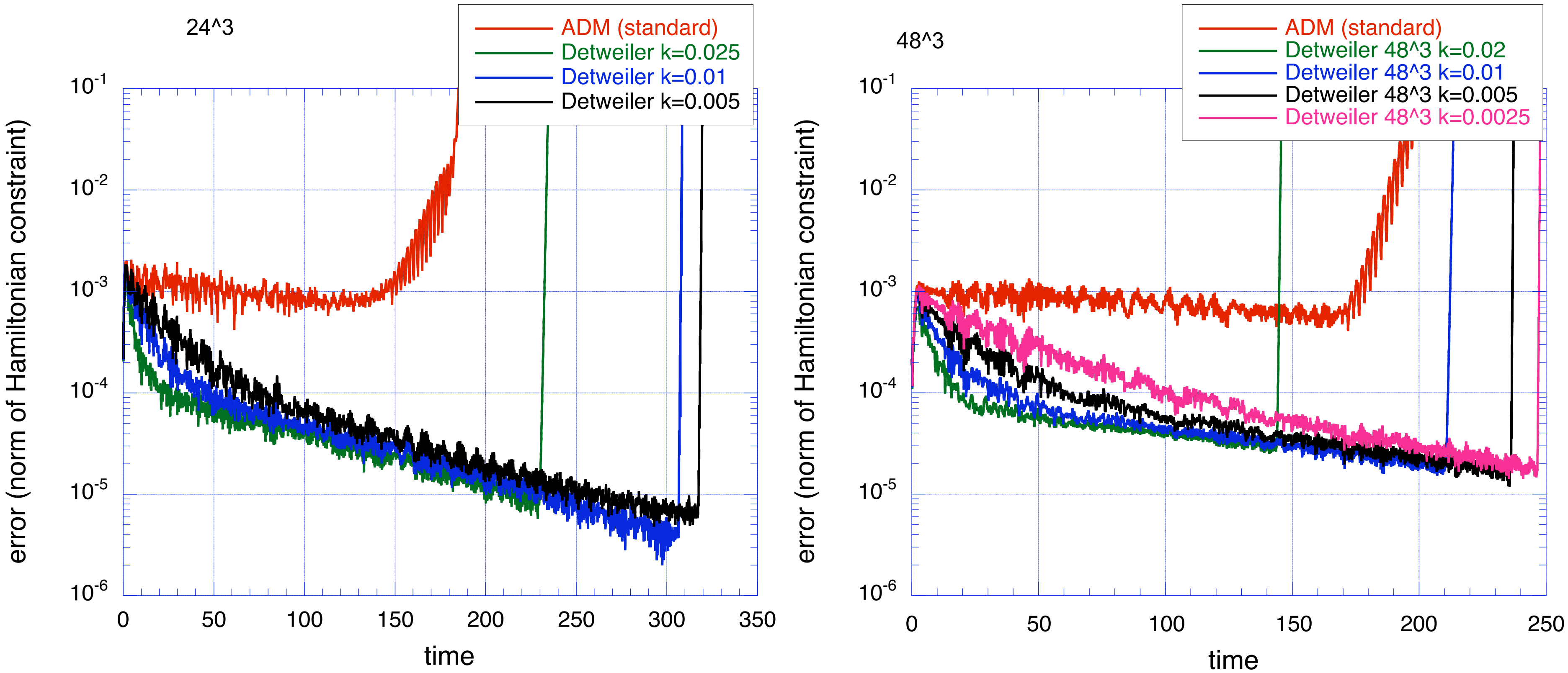
Towards standard testbeds for numerical relativity
Mexico Numerical Relativity Workshop 2002 Participants
CQG 21 (2004) 589-613

Numerical Tests (Detweiler-type)

PRD35(1987)1095

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H}$$

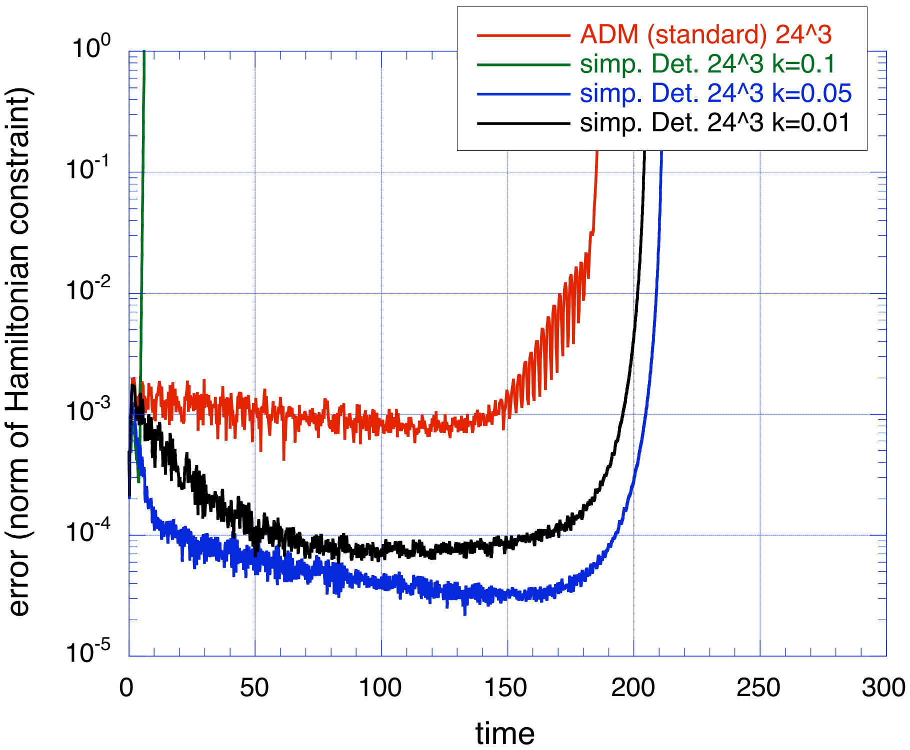
$$\begin{aligned}\partial_t K_{ij} = {} & \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\ & + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}] \mathcal{M}_k \\ & + \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}] (\nabla_k \mathcal{M}_l)\end{aligned}$$

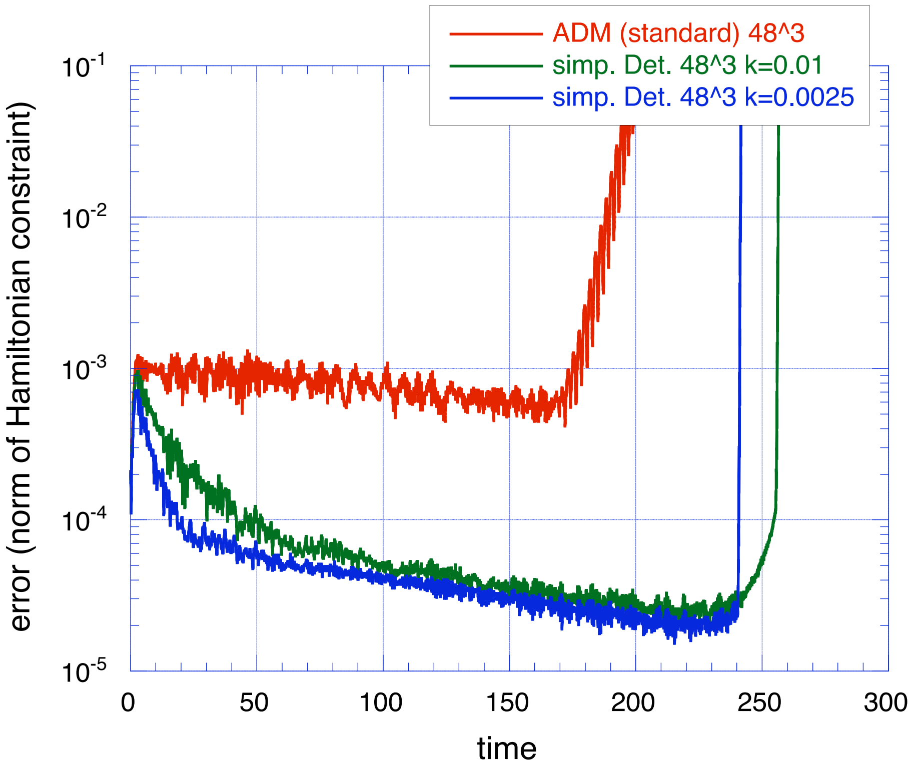


Numerical Tests (Simplified Detweiler)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha \gamma_{ij} \mathcal{H}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$

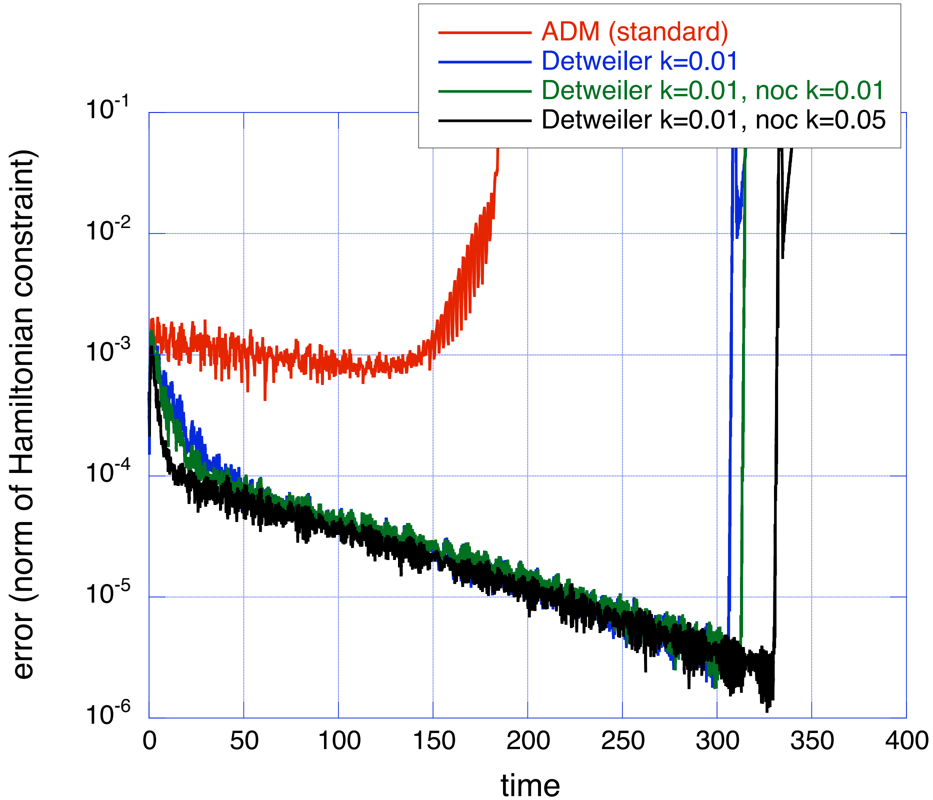




Numerical Tests (Modified Detweiler)

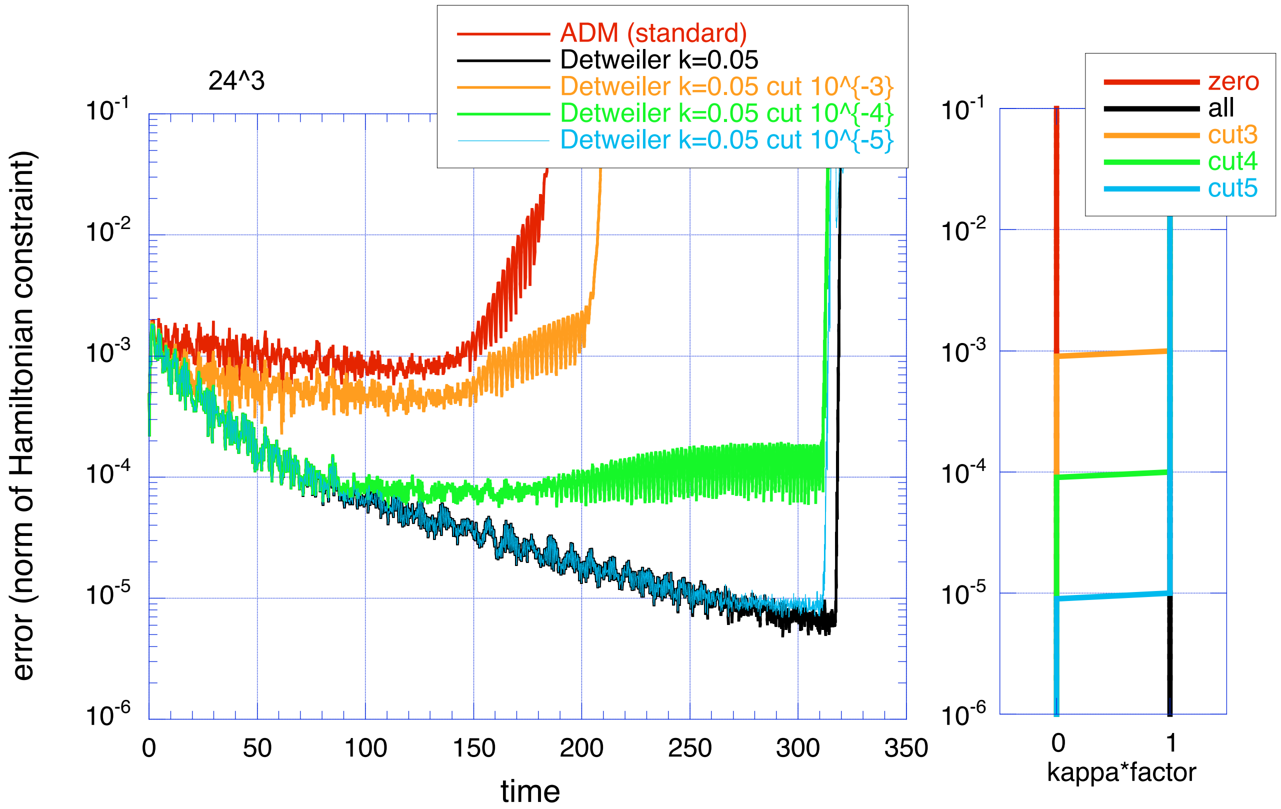
$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_1 \alpha^3 \gamma_{ij} \mathcal{H}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$
$$+\kappa_1 \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_2 \alpha \gamma_{ij} \gamma^{kl} \partial_k \mathcal{M}_l$$
$$+\kappa_1 \alpha^2 [3(\partial_{(i}\alpha) \delta^k_{j)} - (\partial_l \alpha) \gamma_{ij} \gamma^{kl}] \mathcal{M}_k + \kappa_1 \alpha^3 [\delta^k_{(i} \delta^l_{j)} - (1/3) \gamma_{ij} \gamma^{kl}] (\nabla_k \mathcal{M}_l)$$



Numerical Tests (Detweiler, k-adjust)

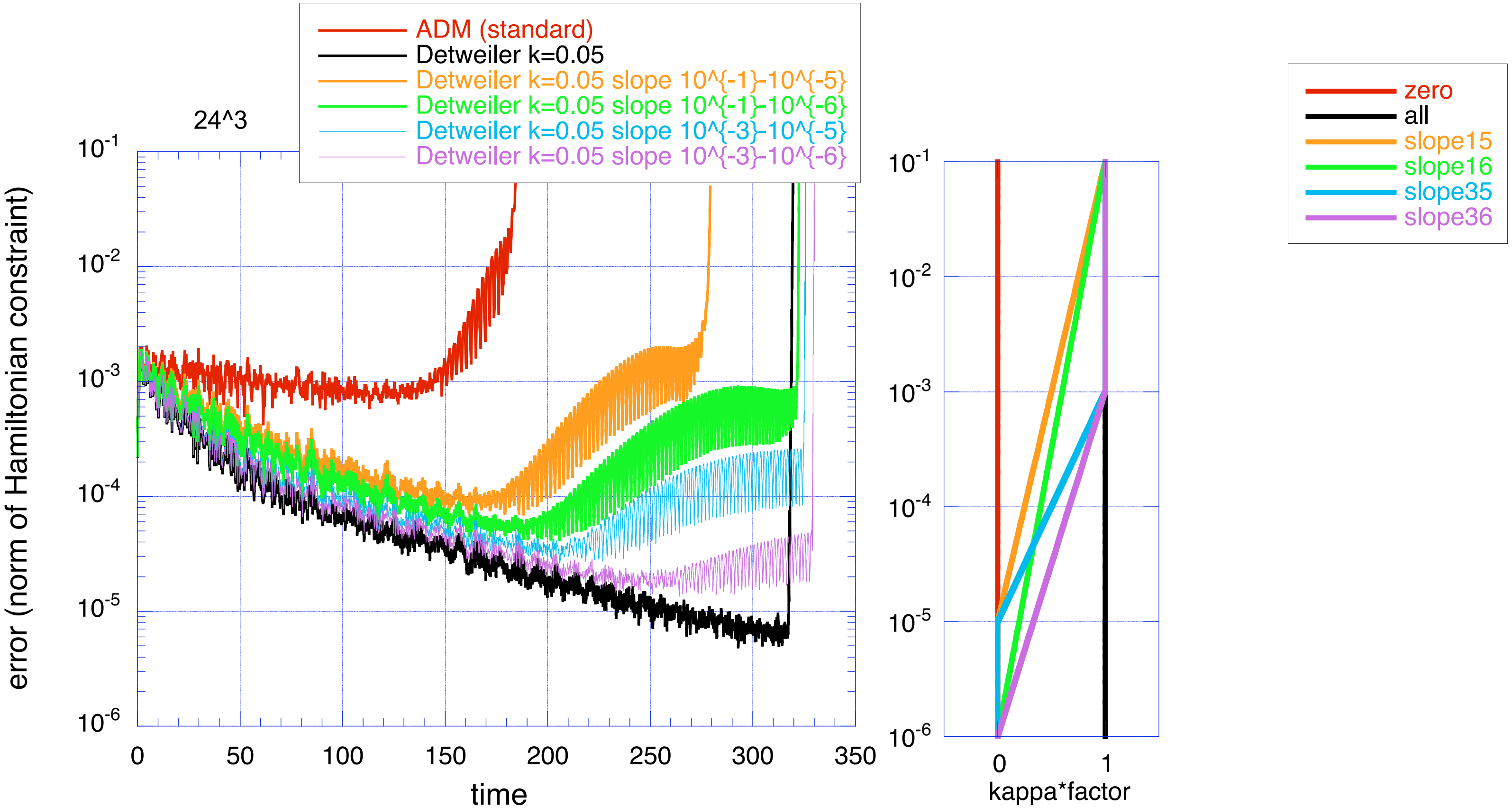
$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H} \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i} \alpha) \delta^k_{j)} - (\partial_l \alpha) \gamma_{ij} \gamma^{kl}] \mathcal{M}_k \\
&\quad + \kappa_L \alpha^3 [\delta^k_{(i} \delta^l_{j)} - (1/3) \gamma_{ij} \gamma^{kl}] (\nabla_k \mathcal{M}_l)
\end{aligned}
$$



Numerical Tests (Detweiler, k-adjust)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H}$$

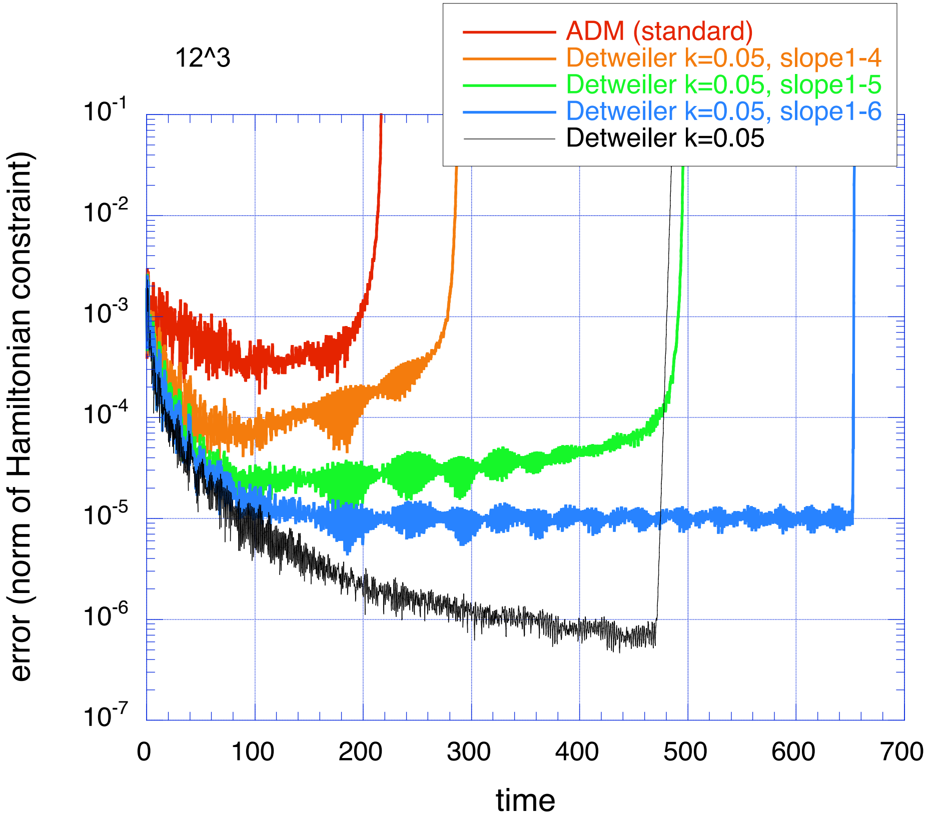
$$\begin{aligned}\partial_t K_{ij} =& \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\ & + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i} \alpha) \delta^k_{j)} - (\partial_l \alpha) \gamma_{ij} \gamma^{kl}] \mathcal{M}_k \\ & + \kappa_L \alpha^3 [\delta^k_{(i} \delta^l_{j)} - (1/3) \gamma_{ij} \gamma^{kl}] (\nabla_k \mathcal{M}_l)\end{aligned}$$

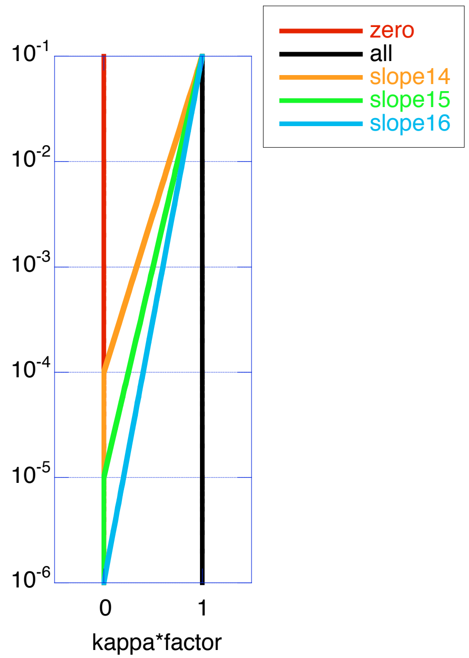


Numerical Tests (Detweiler, k-adjust)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$
$$+ \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}] \mathcal{M}_k$$
$$+ \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}] (\nabla_k \mathcal{M}_l)$$

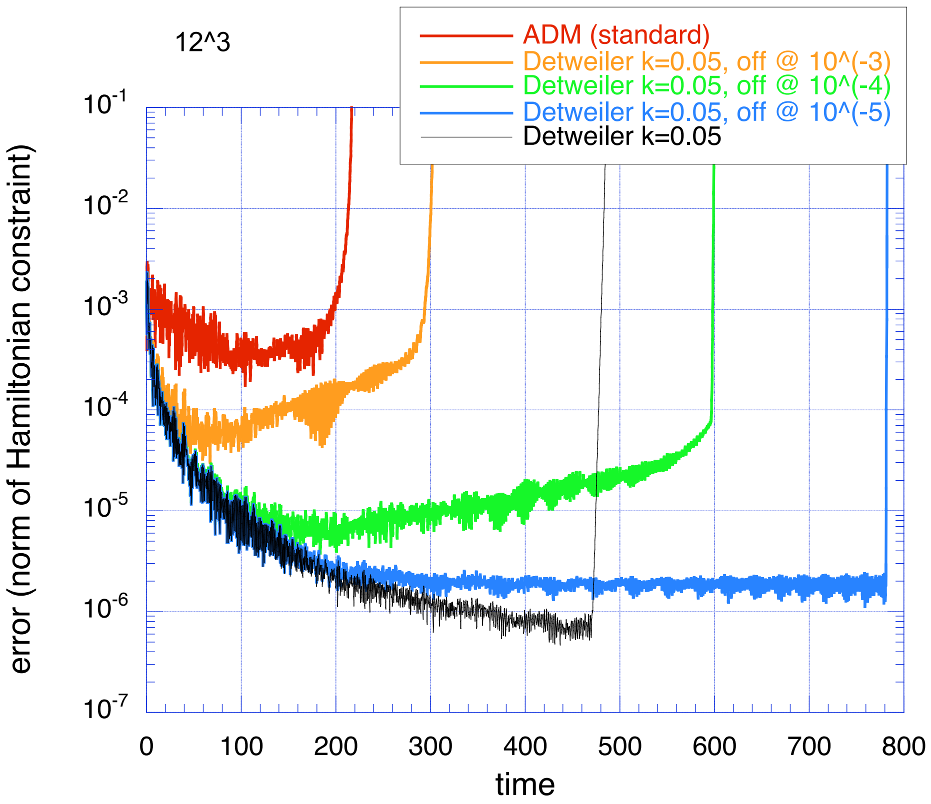


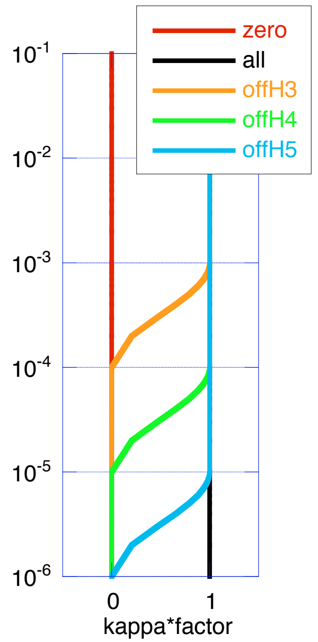


Numerical Tests (Detweiler, k-adjust)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$
$$+\kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i} \alpha) \delta^k_{j)} - (\partial_l \alpha) \gamma_{ij} \gamma^{kl}] \mathcal{M}_k$$
$$+\kappa_L \alpha^3 [\delta^k_{(i} \delta^l_{j)} - (1/3) \gamma_{ij} \gamma^{kl}] (\nabla_k \mathcal{M}_l)$$





Summary & Outlook

・数値相対論の定式化問題（Formulation problem）
・Adjusted ADM形式（ADM+Lagrange乗数補正）
　で計算は安定化する
・Teukolsky waveの3+1次元発展計算
　⇒　Standard ADMの計算寿命より　1.5〜4倍

• Constraint propagationに非線形項の出現を
　抑える工夫で若干計算寿命が延伸
• Errorをゼロにするよりも微小値で抑えるほうが
　計算が長時間継続
　⇒ Lagrange乗数補正 k の自動応答制御機能を
　　開発できれば，職人芸のいらない数値相対論へ
　⇒ 同様の結論は，adjusted BSSNシステムでも
　　成立するはず.