Towards the dynamics in Einstein-Gauss-Bonnet gravity:
Initial Value Problem

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★Towards the investigation of the full dynamics in higher-dimensional and/or stringy gravitational model, we present the basic equations of the Einstein-Gauss-Bonnet gravity theory.

★We show (N+1)-dimensional version of the ADM decomposition including Gauss-Bonnet terms, which shall be the standard approach to treat the space-time as a Cauchy problem.

★Due to the quasi-linear property of the Gauss-Bonnet gravity, we find that the evolution equations can be in a treatable form in numerics.

★We also show the conformally-transformed constraint equations for constructing an initial data.

★We discuss how the constraints can be simplified by tuning the powers of conformal factors.

★Our equations can be used both for timelike and spacelike foliations.

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$N+1$ formalism in Einstein-Gauss-Bonnet gravity

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Einstein-Gauss-Bonnet action

- $(N+1)$-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$

$$S = \int_{\mathcal{M}} d^{N+1}X\sqrt{-g}\left[\frac{1}{2\kappa^2}(\mathcal{R} - 2\Lambda + \alpha_{GB}\mathcal{L}_{GB}) + \mathcal{L}_{\rm matter}\right] \tag{1}$$

$$\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

- The action gives the gravitational equation

$$\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu} = \kappa^2\mathcal{T}_{\mu\nu} \tag{2}$$

where

$$\begin{aligned}
\mathcal{G}_{\mu\nu} &= \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu},\\
\mathcal{H}_{\mu\nu} &= 2\left[\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}{}^{\alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}\right] - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB},\\
\mathcal{T}_{\mu\nu} &\equiv -2\frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\rm matter}.
\end{aligned}$$

$N+1$ Einstein-Gauss-Bonnet equations

Substituting (11)-(13) into (3) or (4)-(6), we find:

(a) dynamical equations for γ_{ij}:

$$\begin{aligned}
&M_{ij} - \frac{1}{2}M\gamma_{ij} - \varepsilon(-K_{ia}K^a{}_j + \gamma_{ij}K_{ab}K^{ab} - \pounds_n K_{ij} + \gamma_{ij}\gamma^{ab}\pounds_n K_{ab})\\
&+2\alpha_{GB}[H_{ij} + \varepsilon(M\pounds_n K_{ij} - 2M_i{}^a\pounds_n K_{aj} - 2M_j{}^a\pounds_n K_{ai} - W_{ij}{}^{ab}\pounds_n K_{ab})] = \kappa^2\mathcal{T}_{\mu\nu}\gamma^\mu{}_i\gamma^\nu{}_j
\end{aligned}$$

(b) Hamiltonian constraint equation:

$$M + \alpha_{GB}(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}) = -2\varepsilon\kappa^2\mathcal{T}_{\mu\nu}n^\mu n^\nu$$

(c) momentum constraint equation:

$$N_i + 2\alpha_{GB}\left(MN_i - 2M_i{}^aN_a + 2M^{ab}N_{iab} - M_i{}^{cab}N_{abc}\right) = -\kappa^2\mathcal{T}_{\mu\nu}n^\mu\gamma^\nu{}_i$$

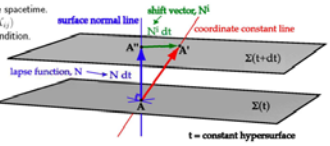
$$\begin{aligned}
M_{ijkl} &= R_{ijkl} - \varepsilon(K_{ik}K_{jl} - K_{il}K_{jk})\\
M_{ij} &= \gamma^{ab}M_{aibj} = R_{ij} - \varepsilon(KK_{ij} - K_{ia}K^a{}_j)\\
M &= \gamma^{ab}M_{ab} = R - \varepsilon(K^2 - K_{ab}K^{ab})\\
N_{ijk} &= D_iK_{jk} - D_jK_{ik}\\
N_i &= \gamma^{ab}N_{aib} = D_aK^a{}_i - D_iK\\
W_{ij}{}^{kl} &= M\gamma_{ij}\gamma^{kl} - 2M_{ij}\gamma^{kl} - 2\gamma_{ij}M^{kl} + 2M_{iajb}\gamma^{ak}\gamma^{bl}
\end{aligned}$$

$$\begin{aligned}
H_{ij} = {}& MM_{ij} - 2(M_{ia}M^a{}_j + M^{ab}M_{iajb}) + M_{iabc}M_j{}^{abc}\\
&-2\varepsilon\Big[-K_{ab}K^{ab}M_{ij} - \frac{1}{2}MK_{ia}K^a{}_j + K_{ia}K^a{}_bM^b{}_j + K_{ja}K^a{}_bM^b{}_i + K^{ac}K_c{}^bM_{iajb}\\
&\qquad +N_iN_j - N^a(N_{aij} + N_{aji}) - \frac{1}{2}N_{abi}N^{ab}{}_j - N_{aib}N_j{}^{ab}\Big]\\
&-\frac{1}{4}\gamma_{ij}[M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}]\\
&-\varepsilon\gamma_{ij}[K_{ab}K^{ab}M - 2M_{ab}K^{ac}K_c{}^b - 2N_aN^a + N_{abc}N^{abc}]
\end{aligned}$$

The Standard ADM Formulation

(Arnowitt-Deser-Misner, 1962; York 1978)

3+1 decomposition of the spacetime.
Evolve 12 variables (γ_{ij}, K_{ij})
with a choice of gauge condition.



	Maxwell eqs.	ADM Einstein eq.
constraints	$\mathrm{div}\,\mathbf{E} = 4\pi\rho$ $\mathrm{div}\,\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_jK^j{}_i - D_i\mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t\mathbf{E} = \mathrm{rot}\,\mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t\mathbf{B} = -\mathrm{rot}\,\mathbf{E}$	$\partial_t\gamma_{ij} = -2NK_{ij} + D_jN_i + D_iN_j,$ $\partial_tK_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}KK_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN$ $+ (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

Projections to Hypersurface Σ_N (spacelike or timelike) (1)

- the projection operator,

$$\perp_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu, \quad n_\mu n^\mu = \varepsilon$$

where n_μ is the unit-normal vector to Σ with n_μ is timelike (if $\varepsilon = -1$) or spaceli
Σ is spacelike (timelike) if n_μ is timelike (spacelike).

- The projections of the gravitational equation:

$$\begin{aligned}
(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\,n^\mu n^\nu &= \kappa^2\mathcal{T}_{\mu\nu}\,n^\mu n^\nu =: \kappa^2\rho_H,\\
(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\,n^\mu\perp^\nu_\rho &= \kappa^2\mathcal{T}_{\mu\nu}\,n^\mu\perp^\nu_\rho =: -\kappa^2J_\rho,\\
(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\perp^\mu_\rho\perp^\nu_\sigma &= \kappa^2\mathcal{T}_{\mu\nu}\perp^\mu_\rho\perp^\nu_\sigma =: \kappa^2S_{\rho\sigma},
\end{aligned}$$

where we defined

$$\mathcal{T}_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}, \quad \mathcal{T} = -\rho_H + S^\ell{}_\ell$$

- Introduce the extrinsic curvature K_{ij}

$$K_{ij} := -\frac{1}{2}\pounds_n h_{ij} = -\perp^\alpha{}_i\perp^\beta{}_j\nabla_\alpha n_\beta,$$

where $\pounds_n$ denotes the Lie derivative in the n-direction and ∇ and D_i is the co
entiation with respect to $g_{\mu\nu}$ and γ_{ij}, respectively.

Projections to Hypersurface Σ_N (spacelike or timelike) (2)

- Projection of the $(N+1)$-dimensional Riemann tensor onto Σ_N

Gauss eq.	$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha{}_i\perp^\beta{}_j\perp^\gamma{}_k\perp^\delta{}_l = R_{ijkl} - \varepsilon K_{ik}K_{jl} + \varepsilon K_{il}K_{jk},$	(8)
Codacci eq.	$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha{}_i\perp^\beta{}_j\perp^\gamma{}_k n^\delta = -2D_{[i}K_{j]k},$	(9)
	$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha{}_i\perp^\gamma{}_k n^\beta n^\delta = \pounds_n K_{ik} + K_{i\ell}K^\ell{}_k\,,$	(10)

- Curvature relations

$$\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma} = {}& R_{\mu\nu\rho\sigma} - \varepsilon(K_{\mu\rho}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\rho} - n_\mu D_\rho K_{\nu\sigma} + n_\mu D_\sigma K_{\rho\nu} + n_\nu D_\rho K_{\sigma\mu} - n_\nu D_\sigma K_{\rho\mu}\\
&-n_\rho D_\mu K_{\nu\sigma} + n_\rho D_\nu K_{\mu\sigma} + n_\sigma D_\mu K_{\nu\rho} - n_\sigma D_\nu K_{\mu\rho})\\
&+n_\mu n_\rho K_{\nu\alpha}K^\alpha{}_\sigma - n_\mu n_\sigma K_{\nu\alpha}K^\alpha{}_\rho - n_\nu n_\rho K_{\mu\alpha}K^\alpha{}_\sigma + n_\nu n_\sigma K_{\mu\alpha}K^\alpha{}_\rho\\
&+n_\mu n_\rho\pounds_n K_{\nu\sigma} - n_\mu n_\sigma\pounds_n K_{\nu\rho} - n_\nu n_\rho\pounds_n K_{\mu\sigma} + n_\nu n_\sigma\pounds_n K_{\mu\rho},
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathcal{R}_{\mu\nu} = {}& R_{\mu\nu} - \varepsilon[KK_{\mu\nu} - 2K_{\mu\alpha}K^\alpha{}_\nu + n_\mu(D_\alpha K^\alpha{}_\nu - D_\nu K) + n_\nu(D_\alpha K^\alpha{}_\mu - D_\mu K)]\\
&+n_\mu n_\nu K_{\alpha\beta}K^{\alpha\beta} + \varepsilon\pounds_n K_{\mu\nu} + n_\mu n_\nu\gamma^{\alpha\beta}\pounds_n K_{\alpha\beta},
\end{aligned} \tag{12}$$

$$\mathcal{R} = R - \varepsilon(K^2 - 3K_{\alpha\beta}K^{\alpha\beta} - 2\gamma^{\alpha\beta}\pounds_n K_{\alpha\beta}). \tag{13}$$

$N+1$ Einstein-Gauss-Bonnet evolution equations

$$\begin{aligned}
&(1 + 2\alpha_{GB}M)\pounds_n K_{ij} - (\gamma_{ij}\gamma^{ab} + 2\alpha_{GB}W_{ij}{}^{ab})\pounds_n K_{ab} - 8\alpha_{GB}M_{(i}{}^a\pounds_n K_{|a|j)}\\
&\quad = -\varepsilon\left(M_{ij} - \frac{1}{2}M\gamma_{ij}\right) - K_{ia}K^a{}_j + \gamma_{ij}K_{ab}K^{ab} + \varepsilon\kappa^2S_{ij} - \varepsilon\gamma_{ij}\Lambda - 2\varepsilon\alpha_{GB}H_{ij}\,,
\end{aligned} \tag{20}$$

- $\pounds_n K_{\mu\nu}$ terms appear only in the linear form, due to the quasi-linear property of the Gauss-Bonnet gravity.
- Iterative scheme is necessary, but treatable in numerics.

$$\begin{pmatrix} \pounds_n\gamma_{11}\\ \pounds_n\gamma_{12}\\ \pounds_n\gamma_{13}\\ \vdots\\ \pounds_n K_{11}\\ \pounds_n K_{12}\\ \pounds_n K_{13}\\ \vdots \end{pmatrix} = \begin{pmatrix} O & O\\ O & \text{Mixing} \end{pmatrix}\begin{pmatrix} \pounds_n\gamma_{11}\\ \pounds_n\gamma_{12}\\ \pounds_n\gamma_{13}\\ \vdots\\ \pounds_n K_{11}\\ \pounds_n K_{12}\\ \pounds_n K_{13}\\ \vdots \end{pmatrix} + \begin{pmatrix} K_{11}\\ K_{12}\\ K_{13}\\ \vdots\\ \\ \text{Source}\\ \\ \end{pmatrix}$$

- Coding is in progress.

Conformal Approach to solve constraints : Eqs. for Initial Data construction

- We generalized the Conformal approach by York and ÓMurchadha (1974) to N-dim & for Gauss-Bonnet gravity.
- Conformal transformation

solution $\boxed{\gamma_{ij} = \psi^{2m}\hat\gamma_{ij} \quad \gamma^{ij} = \psi^{-2m}\hat\gamma^{ij}}$ trial metric

this gives

$$\begin{aligned}
R &= \psi^{-2m}\left\{\hat R - 2(N-1)m\psi^{-1}(\hat D^a\hat D_a\psi) + (N-1)[2-(N-2)m]m\psi^{-2}(\hat D\psi)^2\right\},\\
R_{ij} &= \hat R_{ij} - m\hat\gamma_{ij}\psi^{-1}\hat D_a\hat D^a\psi - (N-2)m\psi^{-1}\hat D_i\hat D_j\psi + (N-2)m(m+1)\psi^{-2}\hat D_i\psi\hat D_j\psi - m[(N-2)m-1]\psi^{-2}(\hat D\psi)^2\hat\gamma_{ij},\\
R_{ijkl} &= \psi^{2m}\Big[\hat R_{ijkl} + m\psi^{-1}\hat\gamma_{il}\{\hat D_j\hat D_k\psi - (m+1)\psi^{-1}\hat D_j\psi\hat D_k\psi\} - m\psi^{-1}\hat\gamma_{ik}\{\hat D_j\hat D_l\psi - (m+1)\psi^{-1}\hat D_j\psi\hat D_l\psi\}\\
&\quad + m\psi^{-1}\hat\gamma_{jk}\{\hat D_i\hat D_l\psi - (m+1)\psi^{-1}\hat D_i\psi\hat D_l\psi\} - m\psi^{-1}\hat\gamma_{jl}\{\hat D_i\hat D_k\psi - (m+1)\psi^{-1}\hat D_i\psi\hat D_k\psi\} + m^2\psi^{-2}(\hat D\psi)^2(\hat\gamma_{il}\hat\gamma_{jk} - \hat\gamma_{ik}\hat\gamma_{jl})\Big].
\end{aligned}$$

- Decompose the extrinsic curvature K_{ij} as $K_{ij} \equiv A_{ij} + \frac{1}{N}\gamma_{ij}K$, and assume

$$\begin{aligned}
A_{ij} &= \psi^\ell\hat A_{ij}, \quad A^{ij} = \psi^{\ell-4m}\hat A^{ij},\\
K &= \psi^\tau\hat K
\end{aligned}$$

- When matter exists, define also the conformal transformation

$$\rho = \psi^{-p}\hat\rho, \qquad J^i = \psi^{-q}\hat J^i$$

Hamiltonian constraint

$$\begin{aligned}
&2(N-1)m\hat D_a\hat D^a\psi - (N-1)[2-(N-2)m]m(\hat D\psi)^2\psi^{-1}\\
&= \hat R\psi - \frac{N-1}{N}\varepsilon\psi^{2m+2\tau+1}\hat K^2 + \varepsilon\psi^{-2m+2\ell+1}\hat A_{ab}\hat A^{ab} + 2\varepsilon\kappa^2\hat\rho\psi^{-p} - 2\hat\Lambda\\
&\quad + \alpha_{GB}\left(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}\right)\psi^{2m+1}.
\end{aligned} \tag{14}$$

$$\begin{aligned}
\hat\Theta &= \left(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}\right)\\
&= (N-3)m\psi^{-4m}\Big\{4(N-2)m\psi^{-2}\left[(\hat D_a\hat D^a\psi)^2 - \hat D_a\hat D_b\psi\hat D^a\hat D^b\psi\right] - 4\psi^{-1}\left[\hat M - (N-2)((N-3)m-2)m\psi^{-2}\hat D_a\psi\hat D^a\psi\right]\hat D_a\hat D^a\psi\\
&\quad + 8\psi^{-1}\left[\hat M^{ab} + (N-2)m(m+1)\psi^{-2}\hat D^a\psi\hat D^b\psi\right]\hat D_a\hat D_b\psi + (N-1)m^2[(N-4)m-4]\psi^{-4}(\hat D_a\psi\hat D^a\psi)^2 - 2\psi^{-2}[(N-4)m-2]\hat M\hat D_a\psi\hat D^a\psi\\
&\quad - 8(m+1)\psi^{-2}\hat M^{ab}\hat D_a\psi\hat D_b\psi\Big\} + \psi^{-4m}(\hat{\mathbb{T}}^2 - 4\hat{\mathbb{T}}_{ab}\hat{\mathbb{T}}^{ab} + \hat{\mathbb{T}}_{abcd}\hat{\mathbb{T}}^{abcd}),
\end{aligned}$$

where $\hat{\mathbb{T}} = \hat R - \varepsilon\left[\frac{N-1}{N}\psi^{2m+2\tau}\hat K^2 - \psi^{2\ell-2m}\hat A_{ab}\hat A^{ab}\right]$, $\hat{\mathbb{T}}_{ij} = \hat R_{ij} - \varepsilon\left[\frac{N-1}{N^2}\psi^{2m+2\tau}\hat\gamma_{ij}\hat K^2 + \frac{N-2}{N}\psi^{\ell+\tau}\hat K\hat A_{ij} - \psi^{2\ell-2m}\hat A_{ia}\hat A^a{}_j\right]$,
$\hat{\mathbb{T}}_{ijkl} = \hat R_{ijkl} - \varepsilon\left[\frac{1}{N^2}\psi^{2m+2\tau}(\hat\gamma_{ik}\hat\gamma_{jl} - \hat\gamma_{il}\hat\gamma_{jk})\hat K^2 + \frac{1}{N}\psi^{\ell+\tau}(\hat A_{ik}\hat\gamma_{jl} - \hat A_{il}\hat\gamma_{jk} + \hat A_{jl}\hat\gamma_{ik} - \hat A_{jk}\hat\gamma_{il}) + \psi^{2\ell-2m}(\hat A_{ik}\hat A_{jl} - \hat A_{il}\hat A_{jk})\right]$.

(A) If we specify $\tau = \ell - 2m$ and $m = 2/(N-2)$, then (14) becomes

$$\frac{4(N-1)}{N-2}\hat D_a\hat D^a\psi = \hat R\psi - \varepsilon\psi^{2\ell+1-4/(N-2)}(\hat K^2 - \hat K_{ab}\hat K^{ab}) + 2\varepsilon\kappa^2\hat\rho\psi^{-p} - 2\hat\Lambda + \alpha_{GB}\hat\Theta\psi^{1+4/(N-2)}, \tag{15}$$

(B) If we specify $\tau = 0$ and $m = 2/(N-2)$, then (14) becomes

$$\frac{4(N-1)}{N-2}\hat D_a\hat D^a\psi = \hat R\psi - \varepsilon\frac{N-1}{N}\psi^{1+4/(N-2)}\hat K^2 + \varepsilon\psi^{2\ell+1-4/(N-2)}\hat A_{ab}\hat A^{ab} + 2\varepsilon\kappa^2\hat\rho\psi^{-p} - 2\hat\Lambda + \alpha_{GB}\hat\Theta\psi^{1+4/(N-2)}. \tag{16}$$

Momentum constraint

- Introduce the TT part and the longitudinal part of $\hat A^{ij}$, and its vector potential as

$$\hat D_j\hat A^{ij}_{TT} = 0, \quad \hat A^{ij}_L = \hat A^{ij} - \hat A^{ij}_{TT}, \quad \hat A^{ij}_L = \hat D^iW^j + \hat D^jW^i - \frac{2}{N}\hat\gamma^{ij}\hat D_kW^k.$$

- Conformal transformations: $D_jA_i{}^j = \psi^{\ell-2m}\{\hat D_j\hat A_i{}^j + \psi^{-1}[\ell + m(N-2)]\hat A_i{}^j\hat D_j\psi\}$

$$\begin{aligned}
&\hat D_a\hat D^aW_i + \frac{N-2}{N}\hat D_i\hat D_kW^k + \hat R_{ik}W^k\\
&+\psi^{-1}[\ell + (N-2)m](\hat D^aW^b + \hat D^bW^a - \frac{2}{N}\hat\gamma^{ab}\hat D_kW^k)\hat\gamma_{bi}\hat D_a\psi\\
&-\psi^{2m-\ell}\frac{N-1}{N}\hat D_i(\psi^\tau\hat K) + \psi^{2m-\ell}2\alpha_{GB}\hat\Xi_i = \kappa^2\psi^{4m-\ell-q}\hat J_i \quad \text{(See next page for } \hat\Xi_i\text{.)}
\end{aligned} \tag{17}$$

(A) If we specify $\tau = \ell - 2m$ and $m = 2/(N-2)$, then (17) becomes

$$\begin{aligned}
&\hat D_a\hat D^aW_i + \frac{N-2}{N}\hat D_i\hat D_kW^k + \hat R_{ik}W^k + \psi^{-1}(\ell+2)\left(\hat D^aW^b + \hat D^bW^a - \frac{2}{N}\hat\gamma^{ab}\hat D_kW^k\right)\hat\gamma_{bi}\hat D_a\psi\\
&-\frac{N-1}{N}\left[\left(\ell - \frac{4}{N-2}\right)(\hat D_i\psi)\hat K + \hat D_i\hat K\right] + \psi^{-\ell+4/(N-2)}2\alpha_{GB}\hat\Xi_i = \kappa^2\psi^{8/(N-2)-\ell-q}\hat J_i
\end{aligned} \tag{18}$$

(B) If we specify $\tau = 0$ and $m = 2/(N-2)$, then (17) becomes

$$\begin{aligned}
&\hat D_a\hat D^aW_i + \frac{N-2}{N}\hat D_i\hat D_kW^k + \hat R_{ik}W^k + \psi^{-1}(\ell+2)(\hat D^aW^b + \hat D^bW^a - \frac{2}{N}\hat\gamma^{ab}\hat D_kW^k)\hat\gamma_{bi}\hat D_a\psi\\
&-\psi^{4/(N-2)-\ell}\frac{N-1}{N}\hat D_i\hat K + \psi^{4/(N-2)-\ell}2\alpha_{GB}\hat\Xi_i = \kappa^2\psi^{8/(N-2)-\ell-q}\hat J_i
\end{aligned} \tag{19}$$

$$\begin{aligned}
\hat\Xi_i = {}& \psi^{\ell-4m}\Big\{\hat R - 2(N-3)m\psi^{-1}\hat D_b\hat D^b\psi - (N-3)m[(N-4)m+2]\psi^{-2}\hat D_b\psi\hat D^b\psi - \frac{N^2-3N+4}{N^2}\varepsilon\psi^{+2m+2\tau}\hat K^2 - \varepsilon\psi^{2\ell-2m}\hat A_{bc}\hat A^{bc}\Big\}\hat D_a\hat A^a{}_i\\
&+\psi^{\ell-4m}\Big\{-2\hat R^b{}_i + 2(N-3)m\psi^{-1}\hat D^b\hat D_i\psi - 2(N-3)m(m+1)\psi^{-2}\hat D_i\psi\hat D^b\psi + \frac{2(N-3)}{N}\varepsilon\psi^{\ell+\tau}\hat K\hat A_i{}^b - 2\varepsilon\psi^{2\ell-2m}\hat A_i{}^c\hat A_c{}^b\Big\}\hat D_a\hat A^a{}_b\\
&+\psi^{\ell-4m}\Big\{2\hat R^{ab} - 2(N-3)m\psi^{-1}\hat D^b\hat D^a\psi - 2(N-1)m(m+1)\psi^{-2}\hat D^a\psi\hat D^b\psi - \frac{2(N-3)}{N}\varepsilon\psi^{\ell+\tau}\hat K\hat A^{ab} + 2\varepsilon\psi^{2\ell-2m}\hat A^a{}_c\hat A^{cb}\Big\}(\hat D_i\hat A_{ab} - \hat D_a\hat A_{ib})\\
&+2\varepsilon\psi^{3\ell-6m}\hat A_c{}^a\hat A^{bc}(\hat D_a\hat A_{bi} - \hat D_b\hat A_{ai}) + \mathbb{R}_i + \mathbb{D}_i + \mathcal{A}^{(1)}\hat D_i\psi + \mathcal{A}^{(2)}\hat D_i\hat K + \mathcal{A}^{(3)}\hat D_a\psi\hat A^a{}_i\\
&-\frac{2(N-2)\varepsilon}{N^2}\varepsilon\psi^{\ell-2m+2\tau}\hat K(\hat D_a\hat K + \tau\psi^{-1}\hat K\hat D_a\psi)\hat A^a{}_i + \frac{2(N-3)}{N}[(N-4)m + 2\ell + \tau]\varepsilon\psi^{2\ell-4m+\tau-1}\hat K\hat D_b\psi\hat A^b{}_a\hat A^a{}_i\\
&+\frac{2(N-3)}{N}\varepsilon\psi^{2\ell-4m+\tau}\hat D_b\hat K\hat A^b{}_a\hat A^a{}_i - 2[(N-6)m + 3\ell]\varepsilon\psi^{3\ell-6m-1}\hat D_c\psi\hat A^c{}_b\hat A^b{}_a\hat A^a{}_i,
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{R}_i = {}& \Big\{[(N-3)m+\ell]\psi^{\ell-4m-1}\hat A_i{}^a\hat D_a\psi - \frac{N-3}{N}\psi^{-2m+\tau}(\hat D_i\hat K + \tau\psi^{-1}\hat K\hat D_i\psi)\Big\}\hat R\\
&+\Big\{\frac{2(N-3)}{N}\tau\psi^{-2m+\tau-1}\hat K\hat D_a\psi + \frac{2(N-3)}{N}\psi^{-2m+\tau}\hat D_a\hat K - 2[(N-3)m+\ell]\psi^{\ell-4m-1}\hat A_a{}^b\hat D_b\psi\Big\}\hat R_i{}^a\\
&-2(m-\ell)\psi^{\ell-4m-1}(\hat A_{ab}\hat D_i\psi - \hat A_{ib}\hat D_a\psi)\hat R^{ab} + 2(m-\ell)\psi^{\ell-4m-1}\hat D_a\psi\hat A_{bc}\hat R_i{}^{cab},\\
\mathbb{D}_i = {}& \Big\{\frac{N^2-8N+11}{N}m\psi^{-2m+\tau-1}(\hat D_i\hat K + \tau\psi^{-1}\hat K\hat D_i\psi) - 2m[(N^2-6N+7)m + (N-3)\ell]\psi^{\ell-4m-2}\hat D_a\psi\hat A^a{}_i\Big\}\hat D_b\hat D^b\psi\\
&-\Big\{\frac{2(N-2)\varepsilon}{N}m\psi^{-2m+\tau-1}(\hat D_a\hat K + \tau\psi^{-1}\hat K\hat D_a\psi) - 2m[(N^2-4N+5)m + (N-2)(\ell-2)]\psi^{\ell-4m-2}\hat D_b\psi\hat A^b{}_a\Big\}\hat D^a\hat D_i\psi\\
&+2(N-3)m(m-\ell)\psi^{\ell-4m-2}(\hat A_{ab}\hat D_i\psi - \hat A_{ia}\hat D_b\psi)\hat D^b\hat D^a\psi,\\
\mathcal{A}^{(1)} = {}& 2\Big\{-\frac{N-2}{N}m(m+1)\psi^{-2m+2\tau-3}(\hat D_a\hat K + \tau\psi^{-1}\hat K\hat D_a\psi)(\hat D^a\hat K + \tau\psi^{-1}\hat K\hat D^a\psi)\\
&\quad + \frac{(N-2)^2}{N}m(m+1)\psi^{-2m+\tau-2}\hat D_a\hat K\hat D^a\psi + \frac{N-2}{2N}m[(N^2-4N+5)m+2]\tau\psi^{-2m+\tau-3}\hat K\hat D_a\psi\hat D^a\psi\\
&\quad - (N-2)\varepsilon m^2(m+1)\psi^{\ell-4m-3}\hat D^a\psi\hat D^b\psi\hat A_{ab} + \frac{N-3}{N}(m-\ell-\tau)\varepsilon\psi^{2\ell-4m+\tau-1}\hat K\hat A_{ab}\hat A^{ab} - (m-\ell)\varepsilon\psi^{3\ell-6m-1}\hat A_a{}^b\hat A_b{}^c\hat A_c{}^a\Big\},\\
\mathcal{A}^{(2)} = {}& \frac{1}{N}\Big\{(N-2)\varepsilon m[(N-3)m-2]\psi^{-2m+\tau-2}\hat D_a\psi\hat D^a\psi - \frac{(N-1)\varepsilon(N+1)}{N^2}\varepsilon\psi^{3\tau}\hat K^2 - (N-3)\varepsilon\psi^{2\ell-4m+\tau}\hat A_{ab}\hat A^{ab}\Big\},\\
\mathcal{A}^{(3)} = {}& -m[(N-2)^2(N-5)m^2 + (N-2)\varepsilon(\ell-2)m + (N-1)(3\ell-2)]\psi^{\ell-4m-3}\hat D_a\psi\hat D^a\psi\\
&-\frac{1}{N^2}[(N-1)(N^2-8)m + (N^2-N+2)\ell]\varepsilon\psi^{\ell-2m+2\tau-1}\hat K^2 + [(N-6)m + 3\ell]\varepsilon\psi^{3\ell-6m-1}\hat A_{ab}\hat A^{ab}.
\end{aligned}$$

(A) Hamiltonian constraint

$$\frac{4(N-1)}{N-2}\hat D_a\hat D^a\psi = \hat R\psi - \varepsilon\psi^{2\ell+1-4/(N-2)}(\hat K^2 - \hat K_{ab}\hat K^{ab}) + 2\varepsilon\kappa^2\hat\rho\psi^{-p} - 2\hat\Lambda + \alpha_{GB}\hat\Theta\psi^{1+4/(N-2)}.$$

(A) momentum constraint

$$\begin{aligned}
&\hat D_a\hat D^aW_i + \frac{N-2}{N}\hat D_i\hat D_kW^k + \hat R_{ik}W^k + \psi^{-1}(\ell+2)(\hat D^aW^b + \hat D^bW^a - \frac{2}{N}\hat\gamma^{ab}\hat D_kW^k)\hat\gamma_{bi}\hat D_a\psi\\
&-\frac{N-1}{N}\left[\left(\ell - \frac{4}{N-2}\right)(\hat D_i\psi)\hat K + \hat D_i\hat K\right] + \psi^{-\ell+4/(N-2)}2\alpha_{GB}\hat\Xi_i = \kappa^2\psi^{8/(N-2)-\ell-q}\hat J_i
\end{aligned}$$

Procedures to construct the initial hypersurface data $(\gamma_{ij}, K_{ij}, \rho, J^i)$

1. Give the initial assumption (trial values) for $\hat\gamma_{ij}$, tr$K$, $\hat A^{TT}_{ij}$ and $\hat\rho$, $\hat J$.
2. Solve above 2 equations for ψ and W^i.
3. inverse conformal transformations,

$$\begin{aligned}
\gamma_{ij} &= \psi^4\hat\gamma_{ij}, & K_{ij} &= \psi^{-2}[\hat A^{TT}_{ij} + (\hat{\mathbb{L}}W)_{ij}] + \frac{1}{N}\psi^4\hat\gamma_{ij}\mathrm{tr}K,\\
\rho &= \psi^{-n}\hat\rho, & J^i &= \psi^{-10}\hat J^i
\end{aligned}$$

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