Wormhole dynamics
in Gauss-Bonnet gravity

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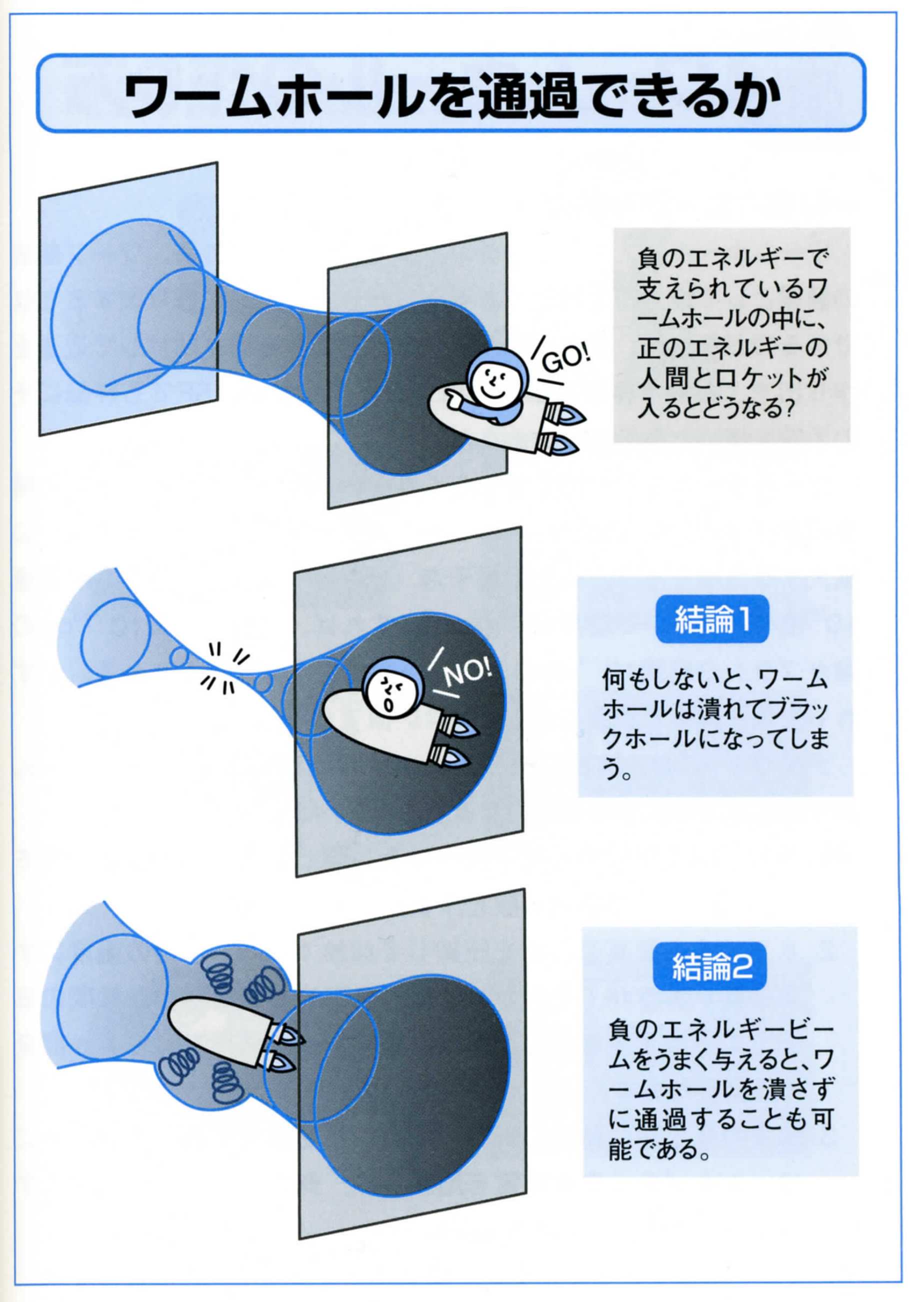


「図解雑学　タイムマシンと時空の科学」（ナツメ社）

Part I

4次元GRでのWH時間発展の復習

Part II

1. N次元時空GRでのEllis解を求めた
2. 摂動計算では不安定のようだ
3. 5次元GRでのシミュレーション結果
4. 5次元Gauss-Bonnet重力理論での シミュレーション結果

Part I 4次元 GRでのワームホールの復習

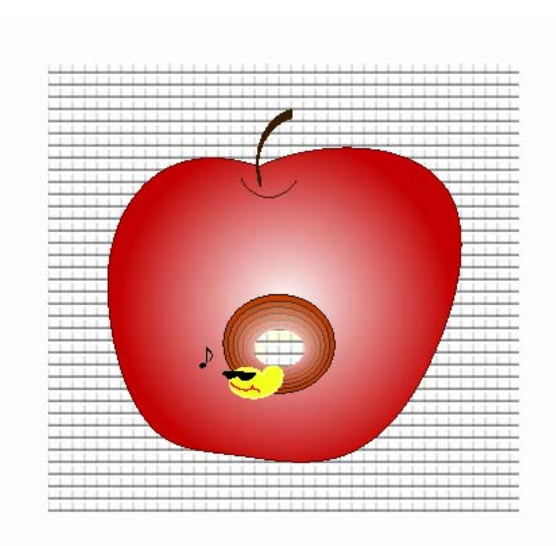
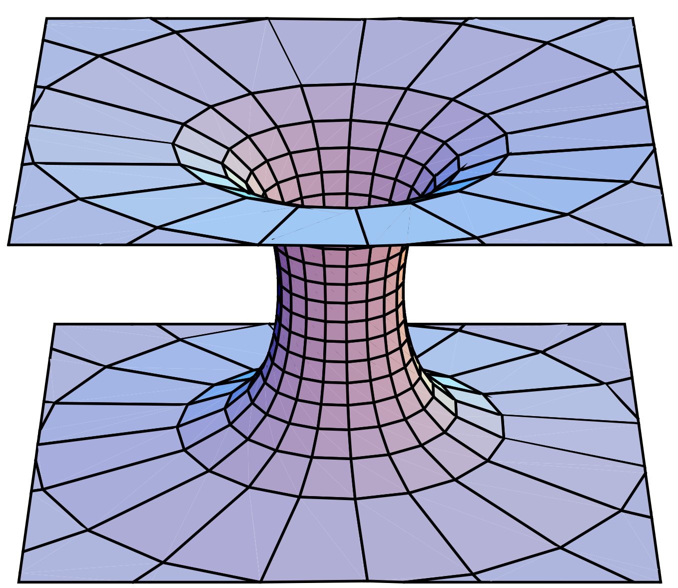
HS & Hayward, PRD66 (2002) 044005

1 Why Wormhole?

- They make great science fiction – short cuts between otherwise distant regions. Morris & Thorne 1988, Sagan "Contact" etc
- They increase our understanding of gravity when the usual energy conditions are not satisfied, due to quantum effects (Casimir effect, Hawking radiation) or alternative gravity theories, brane-world models etc.
- They are very similar to black holes –both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH).

Wormhole ≡ Hypersurface foliated by marginally trapped surfaces

- BH and WH are interconvertible?
  New duality?

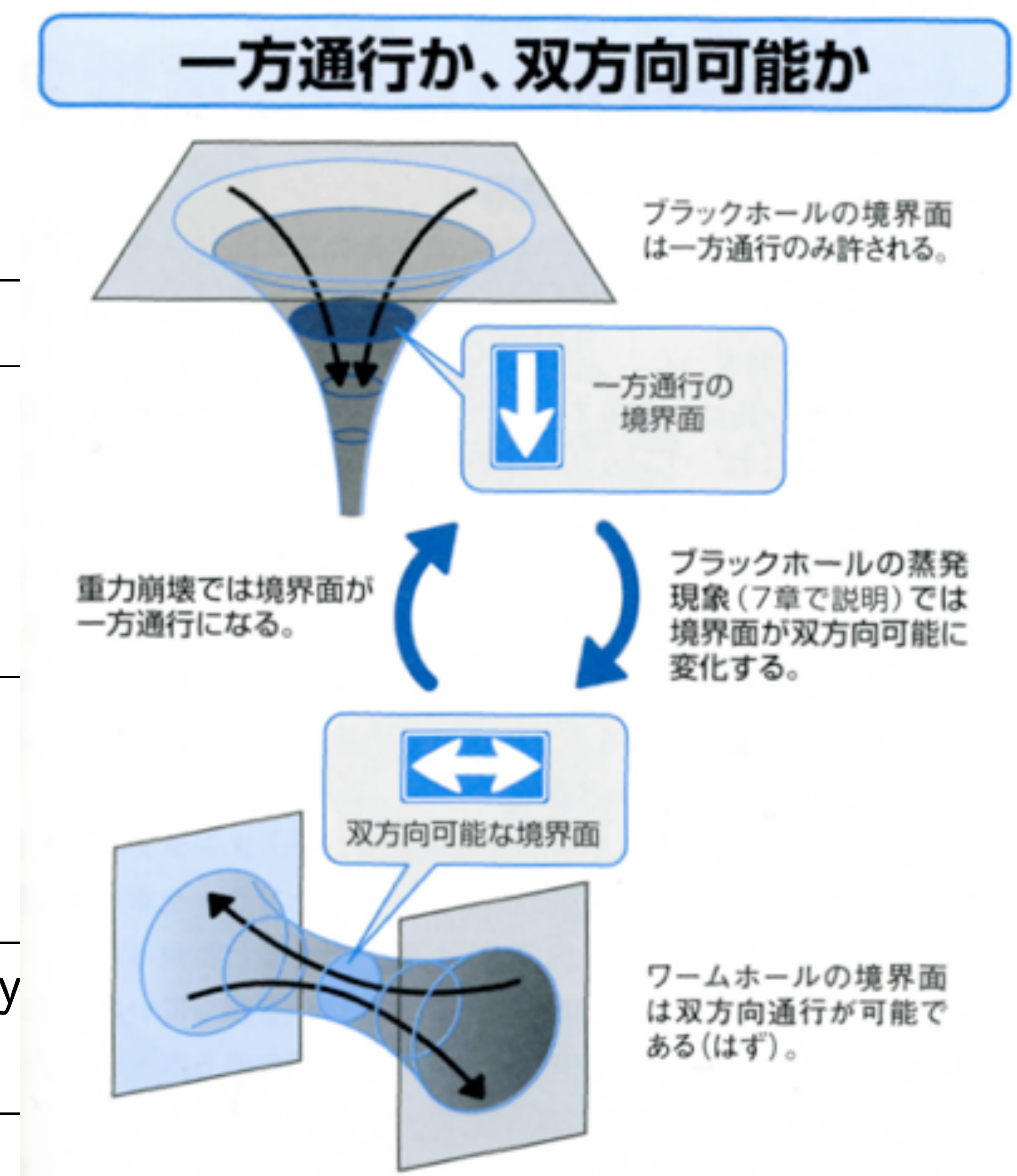




BH and WH are interconvertible ? (New Duality?)

S.A. Hayward, Int. J. Mod. Phys. D 8 (1999) 373

- They are very similar – both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH)
- Only the causal nature of the THs differs, whether THs evolve in plus / minus density.

	Black Hole	Wormhole
Locally defined by	Achronal(spatial/null) outer TH ⇒ 1-way traversable	Temporal (timelike) outer THs ⇒ 2-way traversable
Einstein eqs.	Positive energy density normal matter (or vacuum)	Negative energy density "exotic" matter
Appearance	occur naturally	Unlikely to occur naturally but constructible ???



Part I　4次元 GRでのワームホールの時間発展

PHYSICAL REVIEW D **66**, 044005 (2002)

Fate of the first traversible wormhole: Black-hole collapse or inflationary expansion

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Fate of Morris-Thorne (Ellis) wormhole?

- "Dynamical wormhole" defined by local trapping horizon
- spherically symmetric, both normal/ghost KG field
- apply dual-null formulation in order to seek horizons
- Numerical simulation

ghost/normal Klein-Gordon fields

$$T_{\mu\nu} = T_{\mu\nu}(\psi) + T_{\mu\nu}(\phi) = \underbrace{\left[\psi_{,\mu}\psi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}(\nabla\psi)^2 + V_1(\psi)\right)\right]}_{\text{normal}} + \underbrace{\left[-\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(-\frac{1}{2}(\nabla\phi)^2 + V_2(\phi)\right)\right]}_{\text{ghost}}$$

$$\Box\psi = \frac{dV_1(\psi)}{d\psi}, \qquad \Box\phi = \frac{dV_2(\phi)}{d\phi}. \qquad \text{(Hereafter, we set } V_1(\psi) = 0, V_2(\phi) = 0)$$

dual-null formulation, spherically symmetric spacetime (4D)

- The spherically symmetric line-element:

$$ds^2 = -2e^{-f}dx^+dx^- + r^2 dS^2, \quad \text{where} \quad r = r(x^+, x^-), f = f(x^+, x^-), \cdots$$

- To obtain a system accurate near $\Im^{\pm}$, we introduce the conformal factor $\boxed{\Omega = 1/r}$. We also define first-order variables, the conformally rescaled momenta

expansions	$\vartheta_{\pm} = 2\partial_{\pm}r = -2\Omega^{-2}\partial_{\pm}\Omega$	$(\theta_{\pm} = 2r^{-1}\partial_{\pm}r)$	(1)
inaffinities	$\nu_{\pm} = \partial_{\pm}f$		(2)
momenta of ϕ	$\wp_{\pm} = r\partial_{\pm}\phi = \Omega^{-1}\partial_{\pm}\phi$		(3)
momenta of ψ	$\pi_{\pm} = r\partial_{\pm}\psi = \Omega^{-1}\partial_{\pm}\psi$		(4)

The set of equations (remember the identity: $\partial_+\partial_- = \partial_-\partial_+$):

$$\partial_{\pm}\vartheta_{\pm} = -\nu_{\pm}\vartheta_{\pm} - 2\Omega\pi_{\pm}^2 + 2\Omega\wp_{\pm}^2, \tag{5}$$

$$\partial_{\pm}\vartheta_{\mp} = -\Omega(\vartheta_+\vartheta_-/2 + e^{-f}), \tag{6}$$

$$\partial_{\pm}\nu_{\mp} = -\Omega^2(\vartheta_+\vartheta_-/2 + e^{-f} - 2\pi_+\pi_- + 2\wp_+\wp_-), \tag{7}$$

$$\partial_{\pm}\wp_{\mp} = -\Omega\vartheta_{\mp}\wp_{\pm}/2, \tag{8}$$

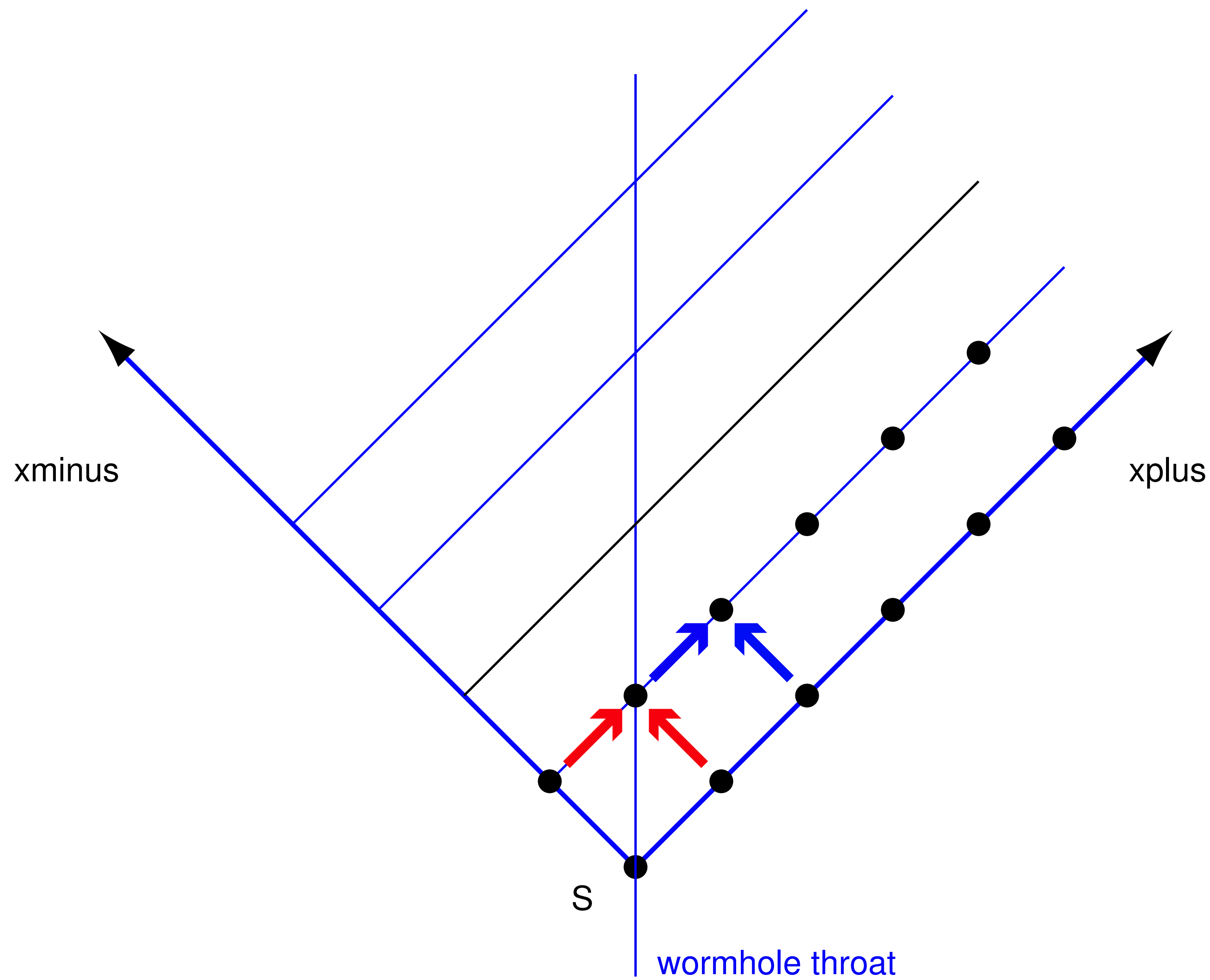
$$\partial_{\pm}\pi_{\mp} = -\Omega\vartheta_{\mp}\pi_{\pm}/2. \tag{9}$$

Initial data on $x^+ = 0$, $x^- = 0$ slices and on S

Generally, we have to set :

$$(\Omega, f, \vartheta_\pm, \phi, \psi) \quad \text{on } S\text{: } x^+ = x^- = 0$$
$$(\nu_\pm, \wp_\pm, \pi_\pm) \quad \text{on } \Sigma_\pm\text{: } x^\mp = 0,\ x^\pm \geq 0$$

Grid Structure for Numerical Evolution



Ghost pulse input -- Bifurcation of the horizons

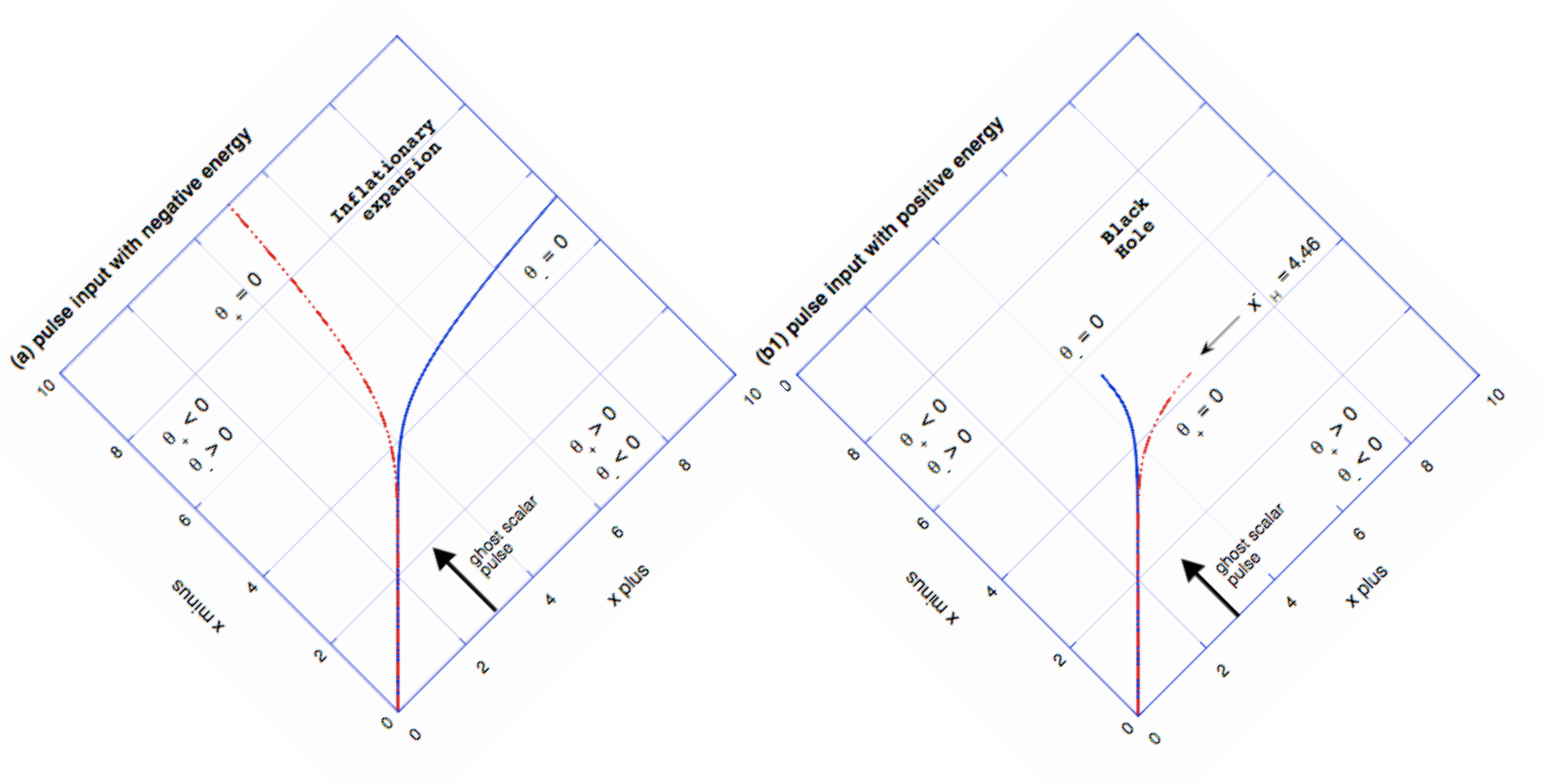
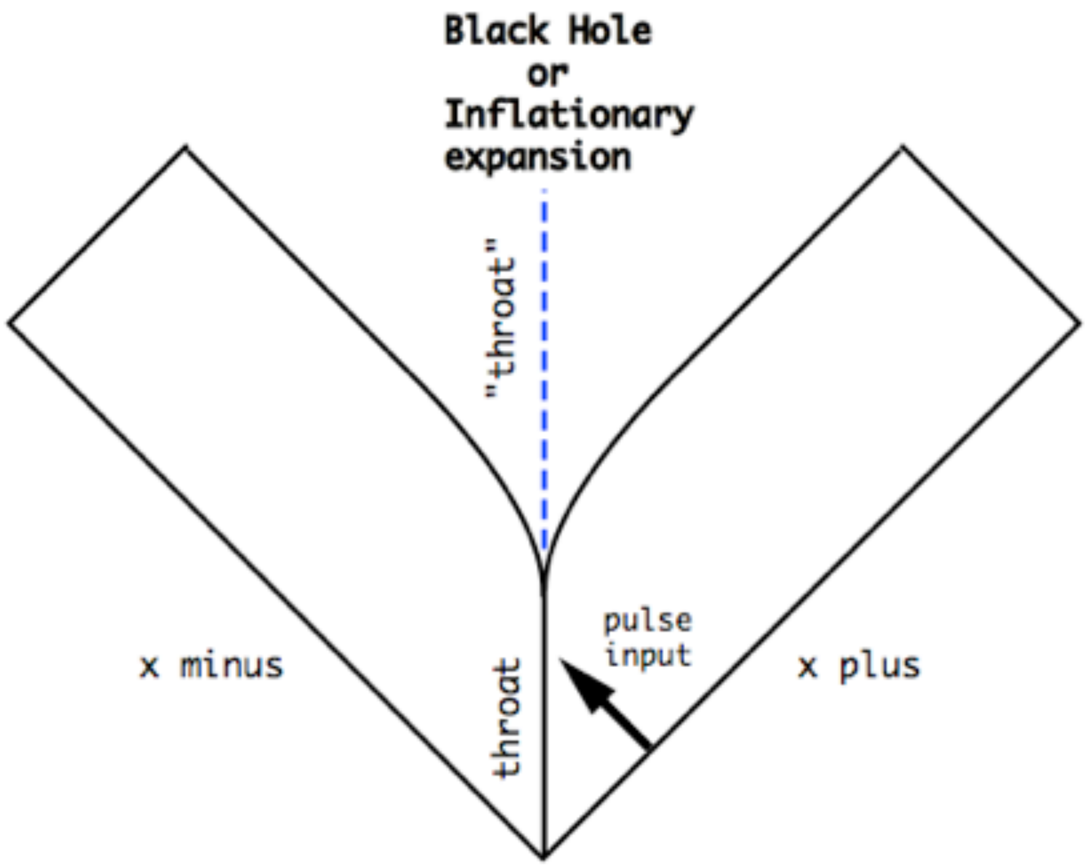


Figure 3: Horizon locations, $\vartheta_\pm = 0$, for perturbed wormhole. Fig.(a) is the case we supplement the ghost field, $c_a = 0.1$, and (b1) and (b2) are where we reduce the field, $c_a = -0.1$ and -0.01. Dashed lines and solid lines are $\vartheta_+ = 0$ and $\vartheta_- = 0$ respectively. In all cases, the pulse hits the wormhole throat at $(x^+, x^-) = (3, 3)$. A 45° counterclockwise rotation of the figure corresponds to a partial Penrose diagram.

Bifurcation of the horizons
-- go to a Black Hole or Inflationary expansion



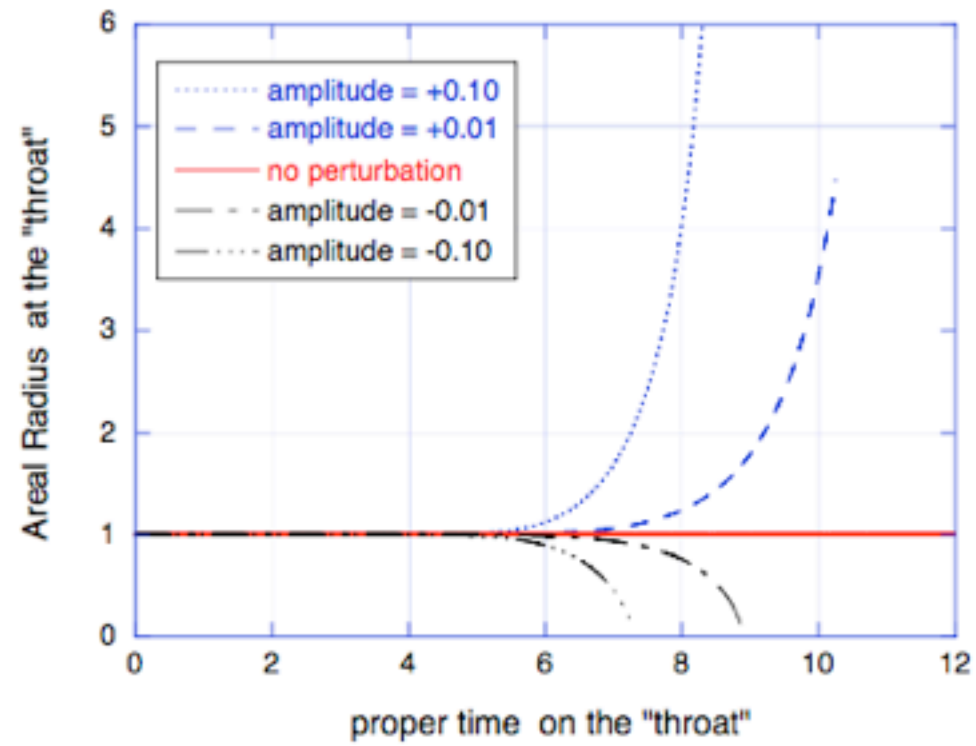


Figure 4: Partial Penrose diagram of the evolved space-time.
Figure 6: Areal radius r of the "throat" $x^+ = x^-$, plotted as a function of proper time. Additional negative energy causes inflationary expansion, while reduced negative energy causes collapse to a black hole and central singularity.

Normal pulse (a traveller) input -- Forming a Black Hole

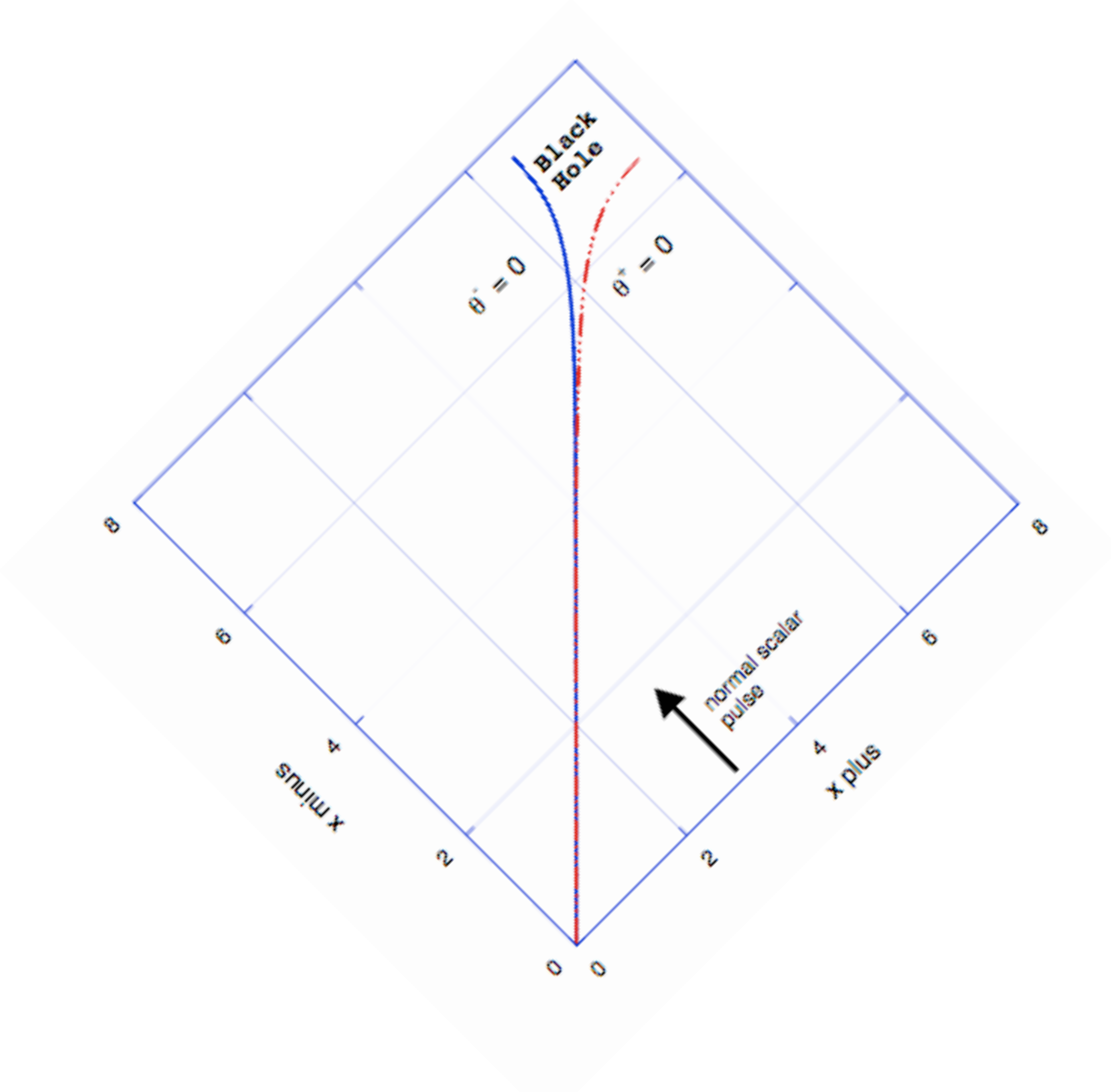
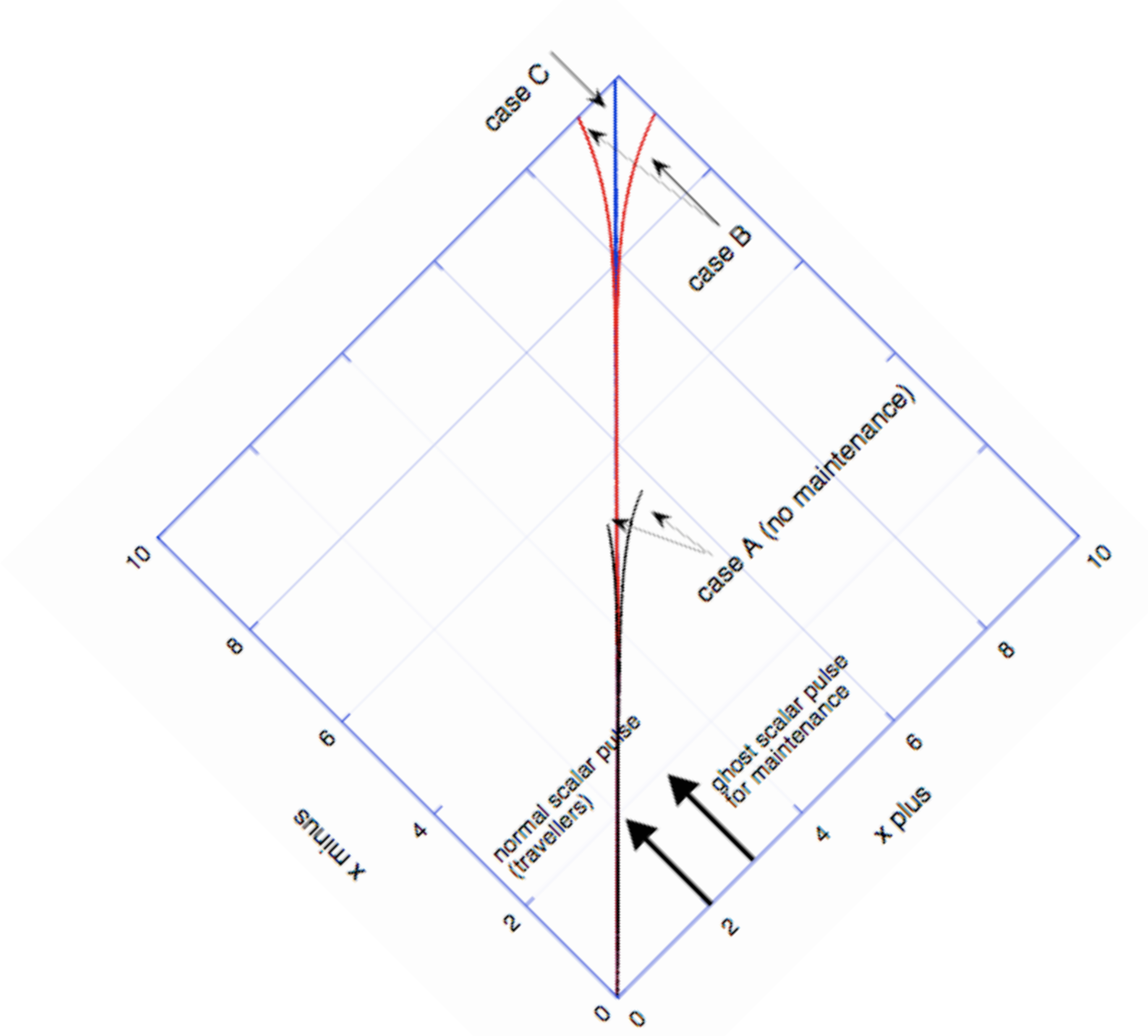


Figure 9: Evolution of a wormhole perturbed by a normal scalar field. Horizon locations: dashed lines and solid lines are $\vartheta_+ = 0$ and $\vartheta_- = 0$ respectively.

Travel through a Wormhole
-- with Maintenance Operations!



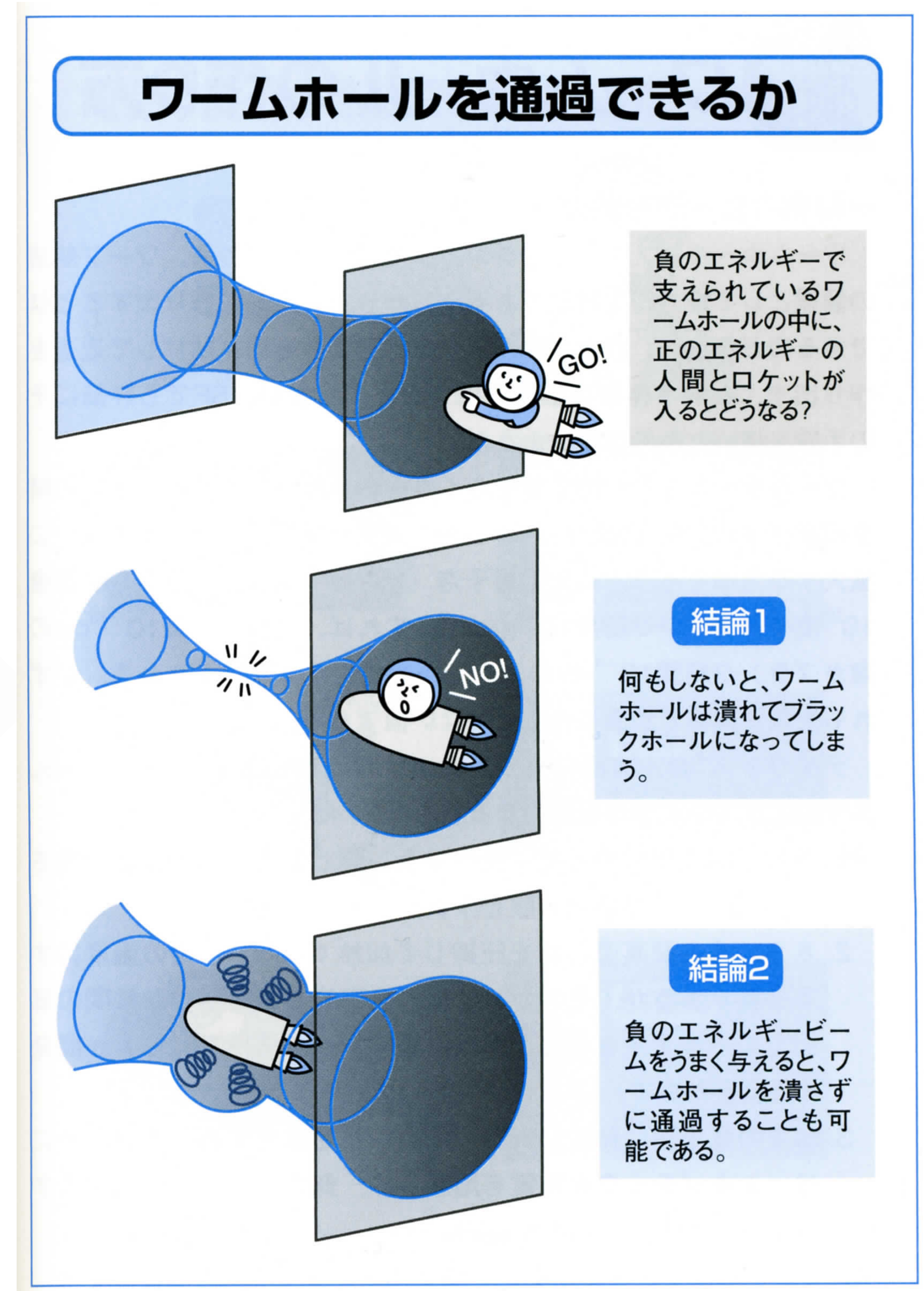


Figure 11: A trial of wormhole maintenance. After a normal scalar pulse, we signalled a ghost scalar pulse to extend the life of wormhole throat. The travellers pulse are commonly expressed with a normal scalar field pulse, $(\tilde{c}_a, \tilde{c}_b, \tilde{c}_c) = (+0.1, 6.0, 2.0)$. Horizon locations $\vartheta_+ = 0$ are plotted for three cases:
(A) no maintenance case (results in a black hole),
(B) with maintenance pulse of $(c_a, c_b, c_c) = (0.02390, 6.0, 3.0)$ (results in an inflationary expansion),
(C) with maintenance pulse of $(c_a, c_b, c_c) = (0.02385, 6.0, 3.0)$ (keep stationary structure upto the end of this range).

Summary of Part I

Dynamics of Ellis (Morris-Thorne) traversible WH

WH is Unstable

(A) with positive energy pulse ---> BH

---> confirms duality conjecture between BH and WH.

(B) with negative energy pulse ---> Inflationary expansion

---> provides a mechanism for enlarging a quantum WH to macroscopic size

(C) can be maintained by sophisticated operations

---> a round-trip is available for our hero/heroine

The basic behaviors has been confirmed by

A Doroshkevich, J Hansen, I Novikov, A Shatskiy, IJMPD 18 (2009) 1665

J A Gonzalez, F S Guzman & O Sarbach, CQG 26 (2009) 015010, 015011

J A Gonzalez, F S Guzman & O Sarbach, PRD80 (2009) 024023

O Sarbach & T Zannias, PRD 81 (2010) 047502

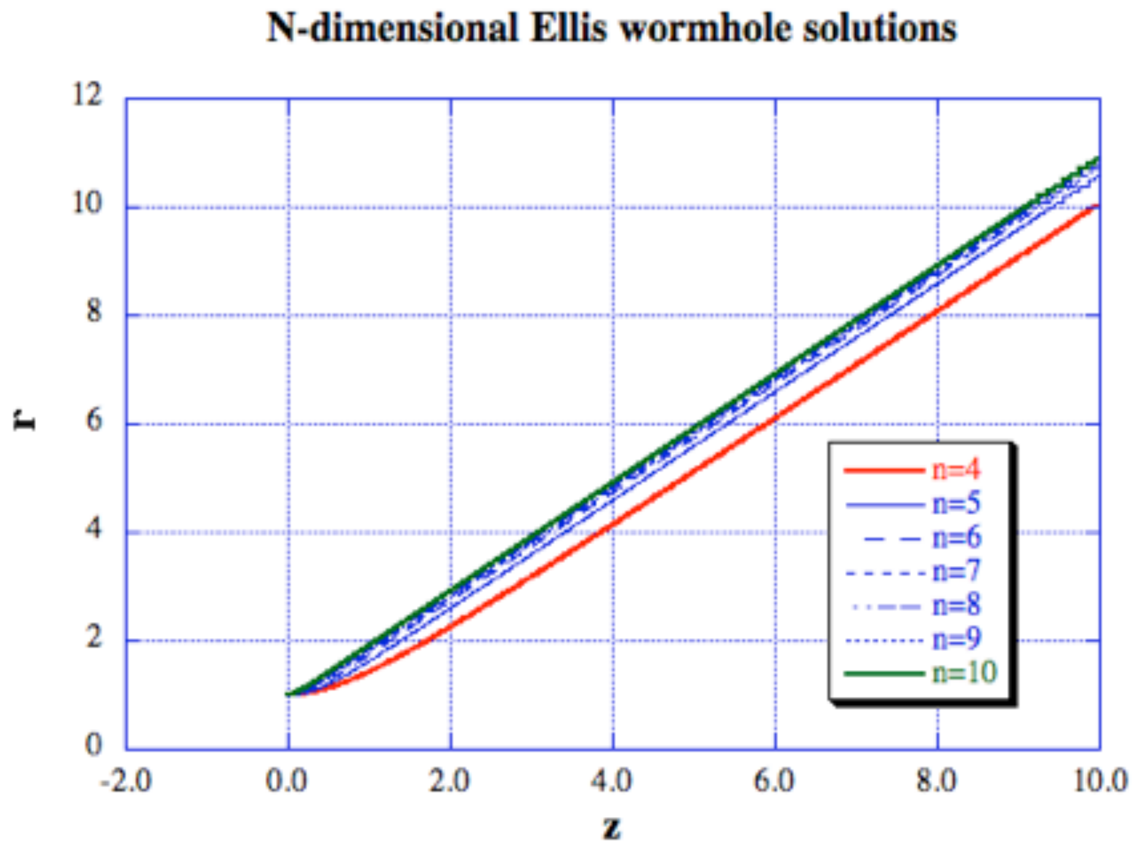
Part II 高次元ワームホール (1) 解の構築 in GR

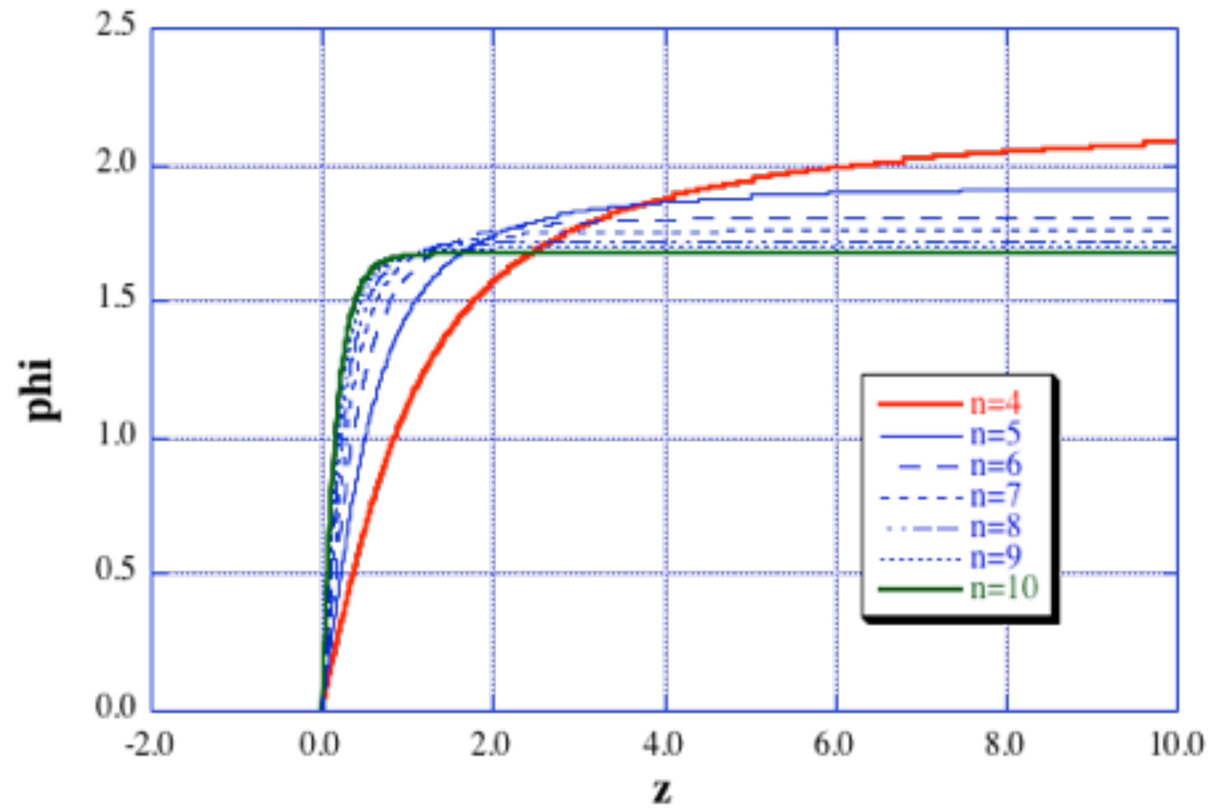
A Wormhole Solution (n-Dim, massless ghost scalar)

- massless ghost scalar field ϕ, throat radius a.
- static, spherical symmetry.

$$ds^2 = -dt^2 + dz^2 + r^2(z)d\Omega^{(n-2)}$$

$$\begin{cases} \dfrac{d^2r}{dz^2} = \dfrac{(n-3)a^{2(n-3)}}{r^{2n-5}} \\ \dfrac{d\phi}{dz} = \sqrt{(n-2)(n-3)}\dfrac{a^{n-3}}{r^{n-2}} \end{cases} \quad (1)$$





Part II　高次元ワームホール　(2) 解の摂動 in GR

Perturbation of n-dimensional Ellis solution

- n-dim. ghost scalar wormhole sols
- spherically symmetric spacetime

$$ds^2 = -f(t,r)e^{-2\delta(t,r)}dt^2 + f(t,r)^{-1}dr^2 + R(t,r)^2 h_{ij}dx^i dx^j$$

- The perturbed functions (ε is infinitesimal parameter.)

$$\begin{aligned} f(t,r) &= f_0(r) + \varepsilon f_1(r)e^{i\omega t}, \\ \delta(t,r) &= \delta_0(r) + \varepsilon \delta_1(r)e^{i\omega t}, \\ R(t,r) &= R_0(r) + \varepsilon R_1(r)e^{i\omega t}, \\ \phi(t,r) &= \phi_0(r) + \varepsilon \phi_1(r)e^{i\omega t}. \end{aligned}$$

Type I: perturbation under throat-radius fixed when $R_1(r) \equiv 0$

Type II: perturbation under throat-radius unfixed when $R_1(r) \neq 0$

Type I: perturbation under throat-radius fixed when $R_1(r) \equiv 0$

The KG equation becomes

$$-\phi_1'' - (n-2)\frac{R_0'}{R_0}\phi_1' + \frac{2(n-3)^2}{R_0^{2(n-2)}{R_0'}^2}\phi_1 = \omega^2\phi_1,$$

which can be written

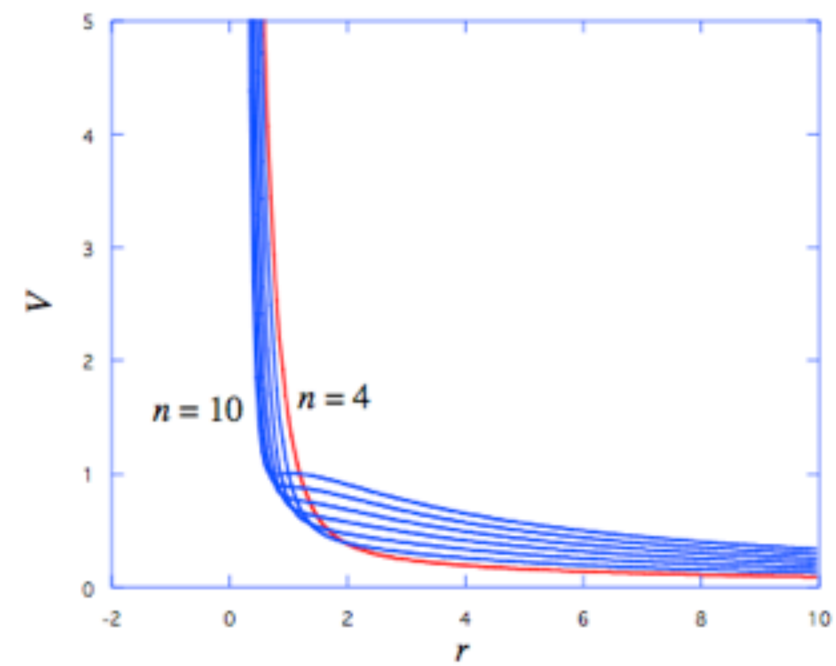
$$-\psi_1'' + V(r)\psi_1 = \omega^2\psi_1,$$

where

$$\psi_1 = \exp\left(\frac{n-2}{2}\int_0^r \frac{R_0'}{R_0}dr\right)\phi_1 = R_0^{\frac{n-2}{2}}\phi_1$$

with the potential

$$V(r) = \frac{n-2}{2}\left[\frac{n-3}{R_0^{2(n-2)}} + \frac{R_0'}{R_0}\left(1-\frac{R_0'}{R_0}\right)\right] + \frac{2(n-3)^2}{R_0^{2(n-2)}{R_0'}^2}$$



Potential $V(r)$

This means there is no negative eigenvalue ω^2, and the static solution is stable against this kind of perturbations.

Consistent with C.Armendáriz-Picon, PRD 65 (2002) 104010.

Type II: perturbation under throat-radius unfixed **when** $R_1(r) \neq 0$

- First-order equations (let $\delta(t,r) \equiv 0$)

$$R_1'' = (n-3)R_0^{-2(n-2)}R_1 + 2\sqrt{\frac{n-3}{n-2}}R_0^{-n+3}\phi_1' + \omega^2 R_1,$$

$$\phi_1'' = -\frac{A}{R_0 R_0'}\phi_1' + \frac{2(n-3)^2}{R_0^{2n-4}R_0'^2}\phi_1 + \sqrt{(n-2)(n-3)}\left[\frac{2}{R_0^{n-2}R_0'}R_1'' - \frac{A}{R_0^{n-1}R_0'^2}R_1' + (n-2)\frac{R_0'}{R_0^n}R_1\right] - \omega^2\phi_1,$$

where $A = (n-2) + (n-4)R_0^{-2(n-3)}$.

- **throat での境界条件（$r = 0$）**

odd parity（throat 半径は固定）

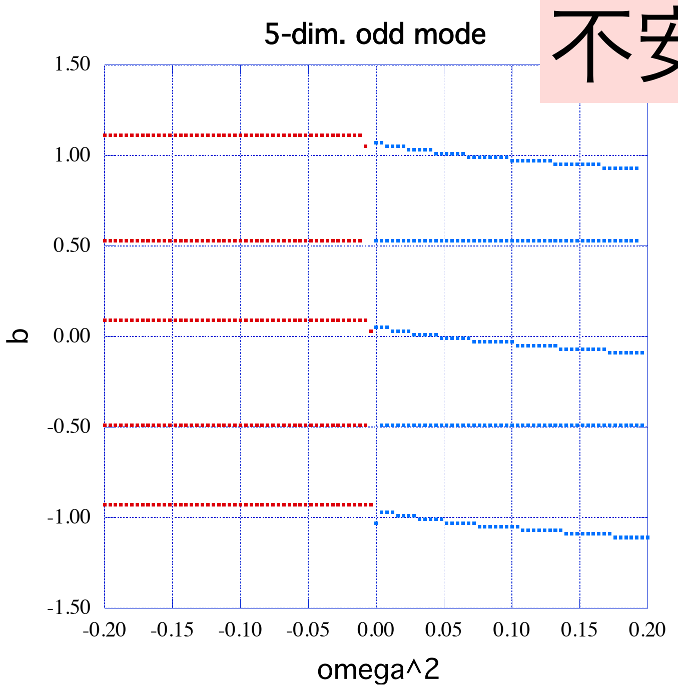
$$\phi_1(r) = a + \frac{1}{2}br^2 + O(r^4)$$

$$R_1(r) = \sqrt{\frac{n-3}{n-2}}ar + O(r^3)$$

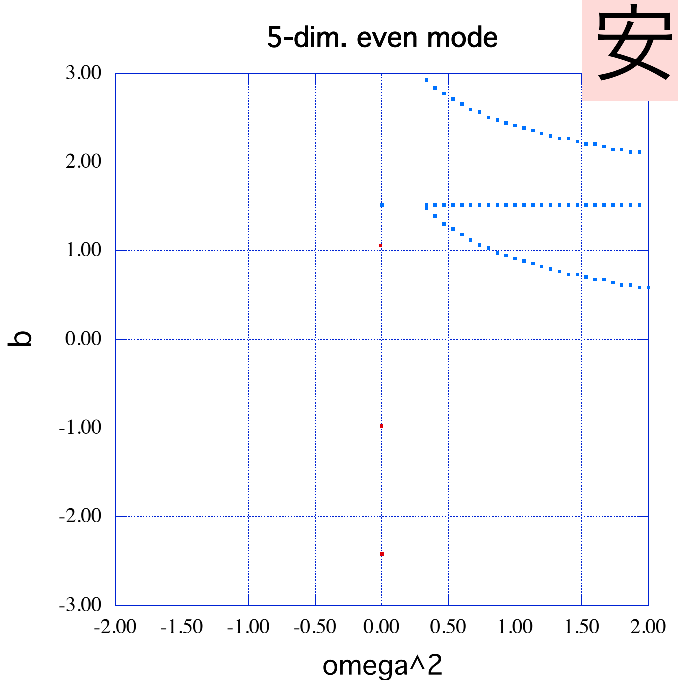
even parity（throat 半径変化）

$$\phi_1(r) = ar + O(r^3)$$

$$R_1(r) = b + O(r^2)$$



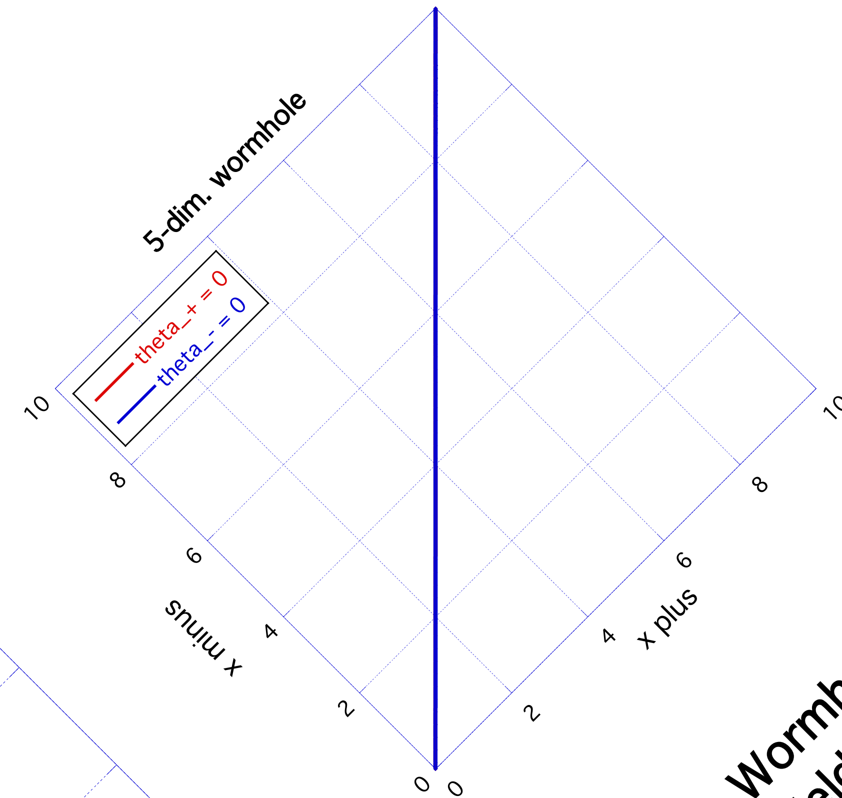
不安定モードあり



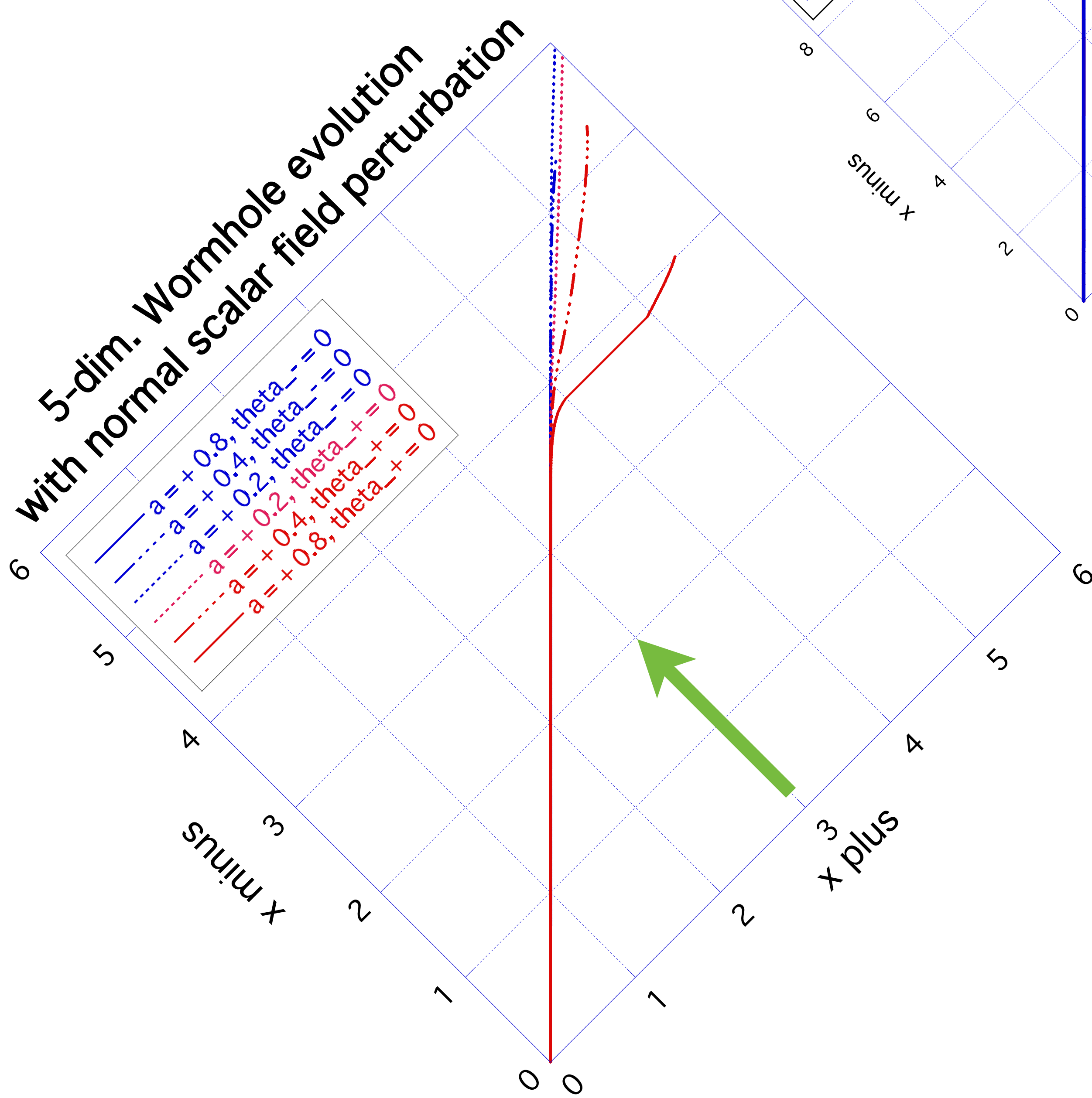
安定モードのみ

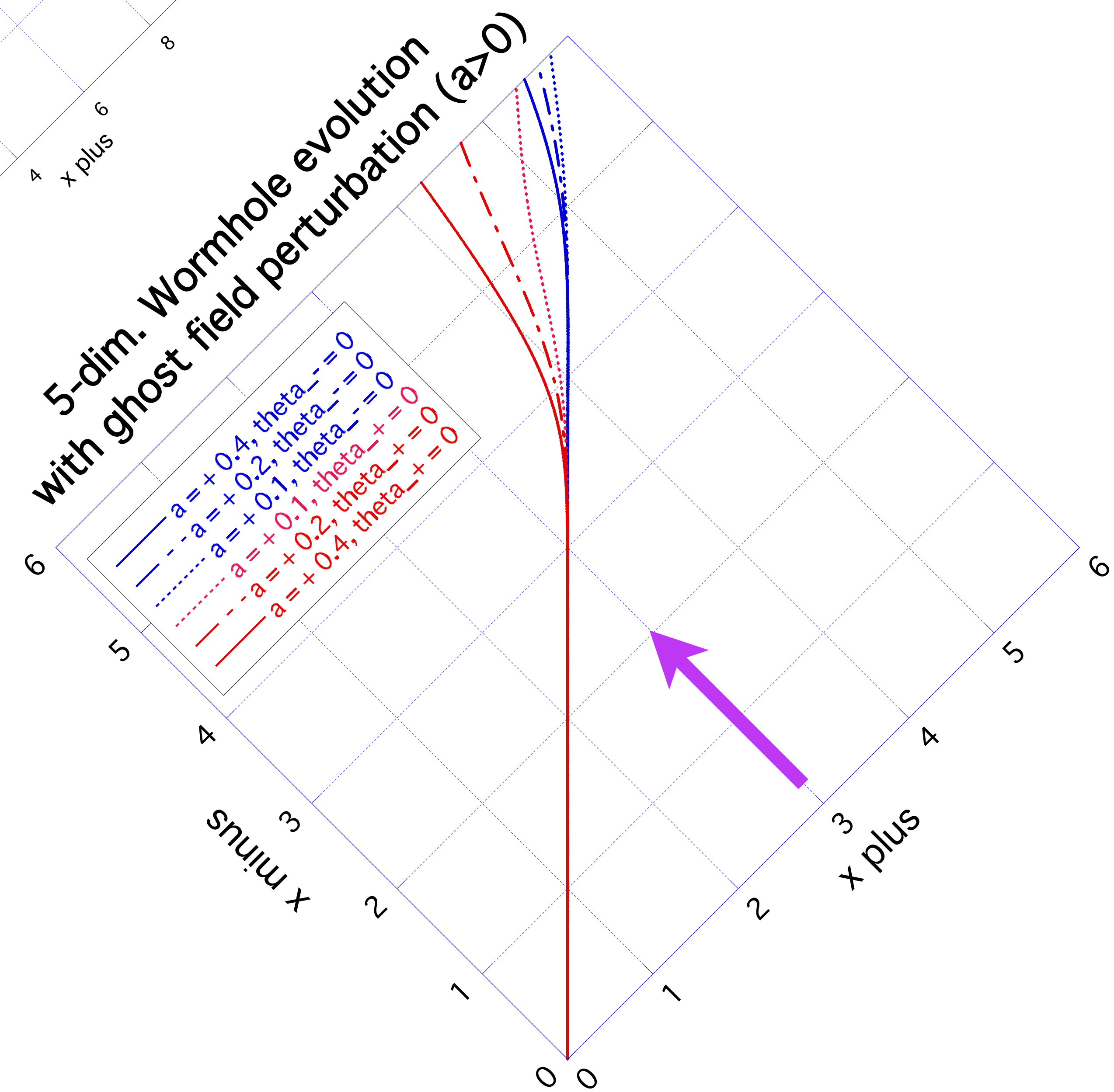
Part II　高次元ワームホール　(3) 時間発展 in GR

+normal field
→ turns to a black hole



+ghost field
→ throat expands





Part II 高次元ワームホール (4) 時間発展 in GB

Gauss-Bonnet gravity

$$S = \int_{\mathcal{M}} d^{N+1}x\sqrt{-g}\Big[\frac{1}{2\kappa^2}\{\alpha_1\mathcal{R} + \alpha_2\left(\mathcal{R}^2 - 4\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta} + \mathcal{R}_{\alpha\beta\gamma\delta}\mathcal{R}^{\alpha\beta\gamma\delta}\right)\} + \mathcal{L}_{\rm matter}\Big]$$

- has GR correction terms from String Theory.
- has two solution branches (GR/non-GR).
- is expected to have singularity avoidance feature.
  (but has never been demonstrated.)

- new topic in numerical relativity.
  (S Golod & T Piran, PRD 85 (2012) 104015;
  F Izaurieta & E Rodriguez, 1207.1496; N Deppe+ 1208.5250)

Wormholes in Einstein-Gauss-Bonnet gravity

- B Bhawal & S Kar, PRD 46 (1992) 2464
  WH sols and a-α relations.
- G Dotti, J Oliva & R Troncoso, PRD 76 (2007) 064038
  exhaustive classification of sols
- M G Richarte & C Simeone, PRD 76 (2007) 087502
  thin-shell WHs supported by ordinary matter.
- H Maeda & M Nozawa, PRD 78 (2008) 024005
  WH sols and energy conditions.
- M H Dehghani & Z Dayyani, PRD 79 (2009) 064010
  WH sols and a-α relations in Lovelock.
- S H Mazharimousavi+, CQG 28 (2011) 025004
  thin-shell WHs in Einstein-Yang-Mills-Gauss-Bonnet.
- P Kanti, B Kleihaus & J Kunz, PRL 107 (2011) 271101, PRD 85 (2012) 044007
  WH sols in Dilatonic-Gauss-Bonnet.

Field Equations

- Action

$$S = \int_{\mathcal{M}} d^{N+1}x\sqrt{-g}\Big[\frac{1}{2\kappa^2}\{\alpha_1\mathcal{R} + \alpha_2\mathcal{L}_{\rm GB}\} + \mathcal{L}_{\rm matter}\Big]$$

where $\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$

- Field equation

$$\alpha_1 G_{\mu\nu} + \alpha_2 H_{\mu\nu} + g_{\mu\nu}\Lambda = \kappa^2 T_{\mu\nu}$$

where $H_{\mu\nu} = 2[\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}_{\ \nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}^{\ \alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}] - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB}$

- matter

normal field $\psi(u,v)$ and/or ghost field $\phi(u,v)$

$$\begin{aligned} T_{\mu\nu} &= T_{\mu\nu}(\psi) + T_{\mu\nu}(\phi) \\ &= \left[\psi_{,\mu}\psi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}(\nabla\psi)^2 + V_1(\psi)\right)\right] + \left[-\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(-\frac{1}{2}(\nabla\phi)^2 + V_2(\phi)\right)\right] \end{aligned}$$

this derives Klein-Gordon equations

$$\Box\psi = \frac{dV_1}{d\psi}, \qquad \Box\phi = \frac{dV_2}{d\phi}.$$

Assumptions

n-dim., Spherical Symmetry, Dual-null coordinate

$$ds^2 = -2e^{-f(x^+,x^-)}dx^+\,dx^- + r^2(x^+,x^-)d\Omega_3^{(n-2)}$$

Space-time Variables

$$\Omega = \frac{1}{r}$$
$$\vartheta_\pm \equiv (n-2)\partial_\pm r$$
$$\nu_\pm \equiv \partial_\pm f$$

We also define η as

$$\eta = \Omega^2\left(e^{-f} + \frac{2}{(n-2)^2}\vartheta_+\vartheta_-\right)$$

Scalar field variables

$$\pi_\pm \equiv r\partial_\pm\psi = \frac{1}{\Omega}\partial_\pm\psi$$
$$p_\pm \equiv r\partial_\pm\phi = \frac{1}{\Omega}\partial_\pm\phi$$

Klein-Gordon eqs.

$$\begin{aligned}\Box\phi &= -\frac{e^f}{r}\left(2r\phi_{uv} + (n-2)r_u\phi_v + (n-2)r_v\phi_u\right)\\ &= -2e^f\phi_{uv} - e^f\Omega^2\left(\vartheta_-p_+ + \vartheta_+p_-\right)\end{aligned}$$

Energy-momentum tensor

$$T_{++} = \Omega^2(\pi_+^2 - p_+^2)$$
$$T_{--} = \Omega^2(\pi_-^2 - p_-^2)$$
$$T_{+-} = -e^{-f}\left(V_1(\psi) + V_2(\phi)\right)$$
$$T_{zz} = e^f(\pi_+\pi_- - p_+p_-) - \frac{1}{\Omega^2}\left(V_1(\psi) - V_2(\phi)\right)$$

Dual-null equations in 5-D with Gauss-Bonnet corrections

$$\alpha_1 G_{\mu\nu} + \alpha_2 H_{\mu\nu} + g_{\mu\nu}\Lambda = \kappa^2 T_{\mu\nu}$$

$$\eta = \Omega^2 \left(e^{-f} + \frac{2}{9}\vartheta_+\vartheta_- \right), \quad \tilde{A} = \alpha_1 + 4\alpha_2 \eta e^f, \quad B = \kappa^2 T_{+-} + e^{-f}\Lambda$$

x^+-direction

$$\partial_+\Omega = -\frac{1}{3}\vartheta_+\Omega^2 \tag{1}$$

$$\partial_+\vartheta_+ = -\nu_+\vartheta_+ - \frac{1}{\tilde{A}\Omega}\kappa^2 T_{++} \tag{2}$$

$$\partial_+\vartheta_- = \frac{1}{\tilde{A}\Omega}(-3\alpha_1\eta + B) \tag{3}$$

$$\partial_+ f = \nu_+ \tag{4}$$

$$\partial_+\nu_- = \frac{\alpha_1}{\tilde{A}}\left\{\eta - \frac{4(3\alpha_1\eta - B)}{3\tilde{A}}\right\} + \frac{(\kappa^2 T_{zz}\Omega^2 - \Lambda)}{\tilde{A}e^f} + \frac{8\alpha_2}{9\tilde{A}^3}\left\{e^f (3\alpha_1\eta - B)^2 - \kappa^4 T_{++}T_{--}\right\} \tag{5}$$

$$\partial_+\psi = \Omega\pi_+ \tag{6}$$

$$\partial_+\phi = \Omega p_+ \tag{7}$$

$$\partial_+\pi_- = -\frac{1}{6}\Omega\vartheta_+\pi_- - \frac{1}{2}\Omega\vartheta_-\pi_+ - \frac{1}{2e^f\Omega}\frac{dV_1}{d\psi} \tag{8}$$

$$\partial_+ p_- = -\frac{1}{6}\Omega\vartheta_+ p_- - \frac{1}{2}\Omega\vartheta_- p_+ - \frac{1}{2e^f\Omega}\frac{dV_2}{d\phi} \tag{9}$$

x^--direction

$$\partial_-\Omega = -\frac{1}{3}\vartheta_-\Omega^2 \tag{10}$$

$$\partial_-\vartheta_+ = \frac{1}{\tilde{A}\Omega}(-3\alpha_1\eta + B) \tag{11}$$

$$\partial_-\vartheta_- = -\nu_-\vartheta_- - \frac{1}{\tilde{A}\Omega}\kappa^2 T_{--} \tag{12}$$

$$\partial_- f = \nu_- \tag{13}$$

$$\partial_-\nu_+ = (5) \tag{14}$$

$$\partial_-\psi = \Omega\pi_- \tag{15}$$

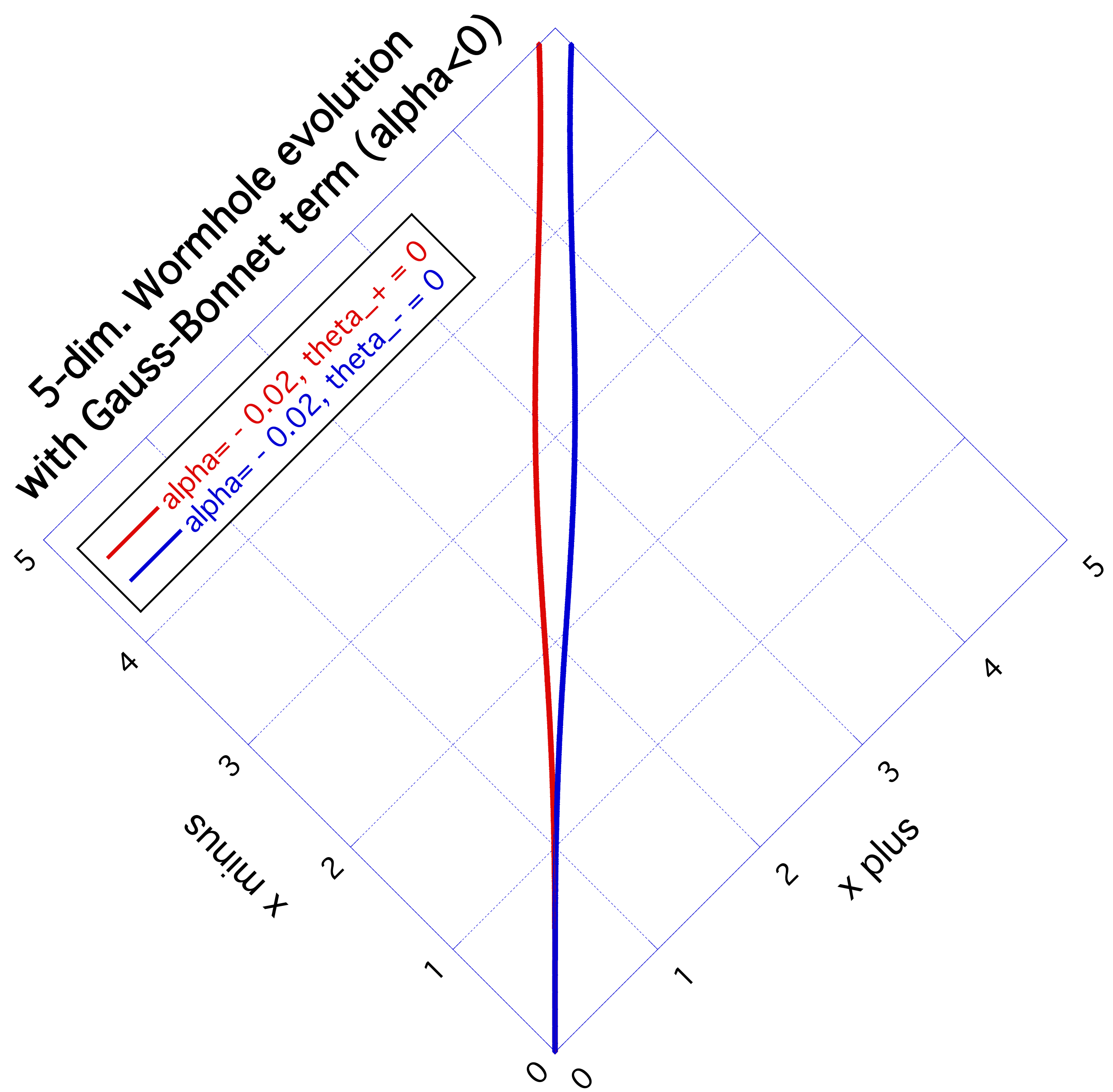
$$\partial_-\phi = \Omega p_- \tag{16}$$

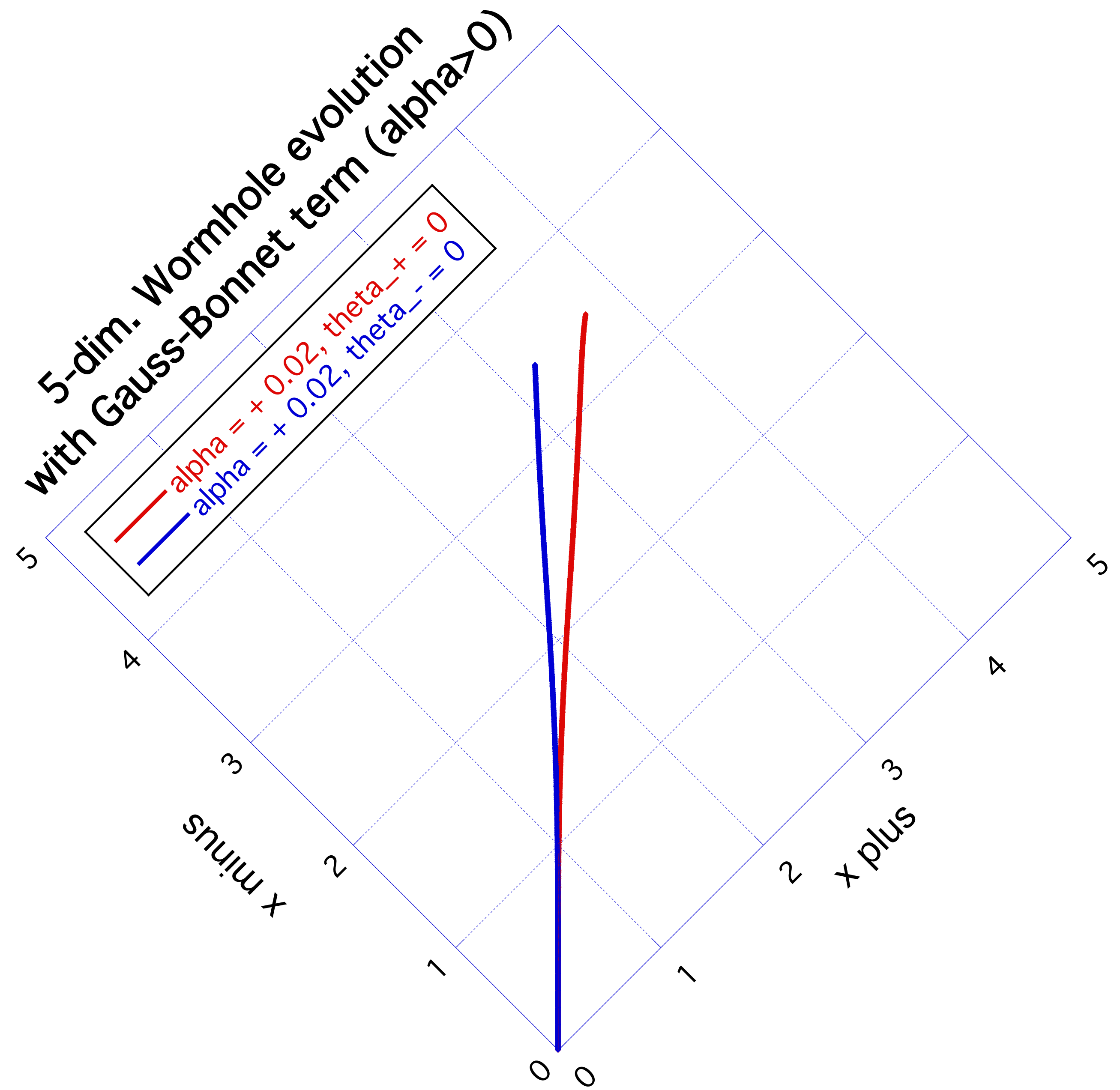
$$\partial_-\pi_+ = -\frac{1}{2}\Omega\vartheta_+\pi_- - \frac{1}{6}\Omega\vartheta_-\pi_+ - \frac{1}{2e^f\Omega}\frac{dV_1}{d\psi} \tag{17}$$

$$\partial_- p_+ = -\frac{1}{2}\Omega\vartheta_+ p_- - \frac{1}{6}\Omega\vartheta_- p_+ - \frac{1}{2e^f\Omega}\frac{dV_2}{d\phi} \tag{18}$$

WH evolution in 5D Gauss-Bonnet gravity

positive GB term accelerates BH collapse





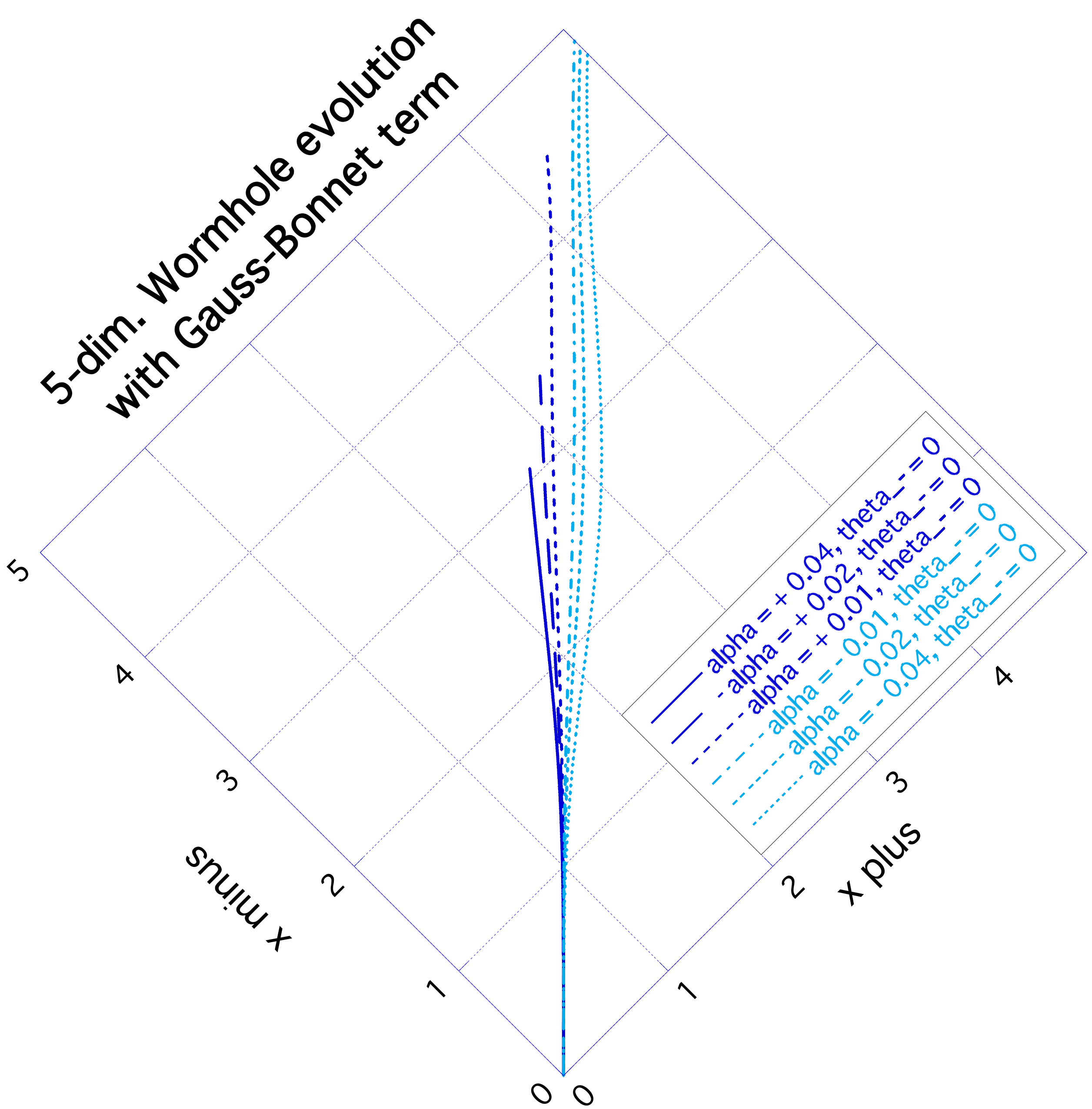
$$S = \int_{\mathcal{M}} d^{N+1}x\sqrt{-g}\left[\frac{1}{2\kappa^2}\{\alpha_1\mathcal{R} + \alpha_2\mathcal{L}_{\rm GB}\} + \mathcal{L}_{\rm matter}\right]$$

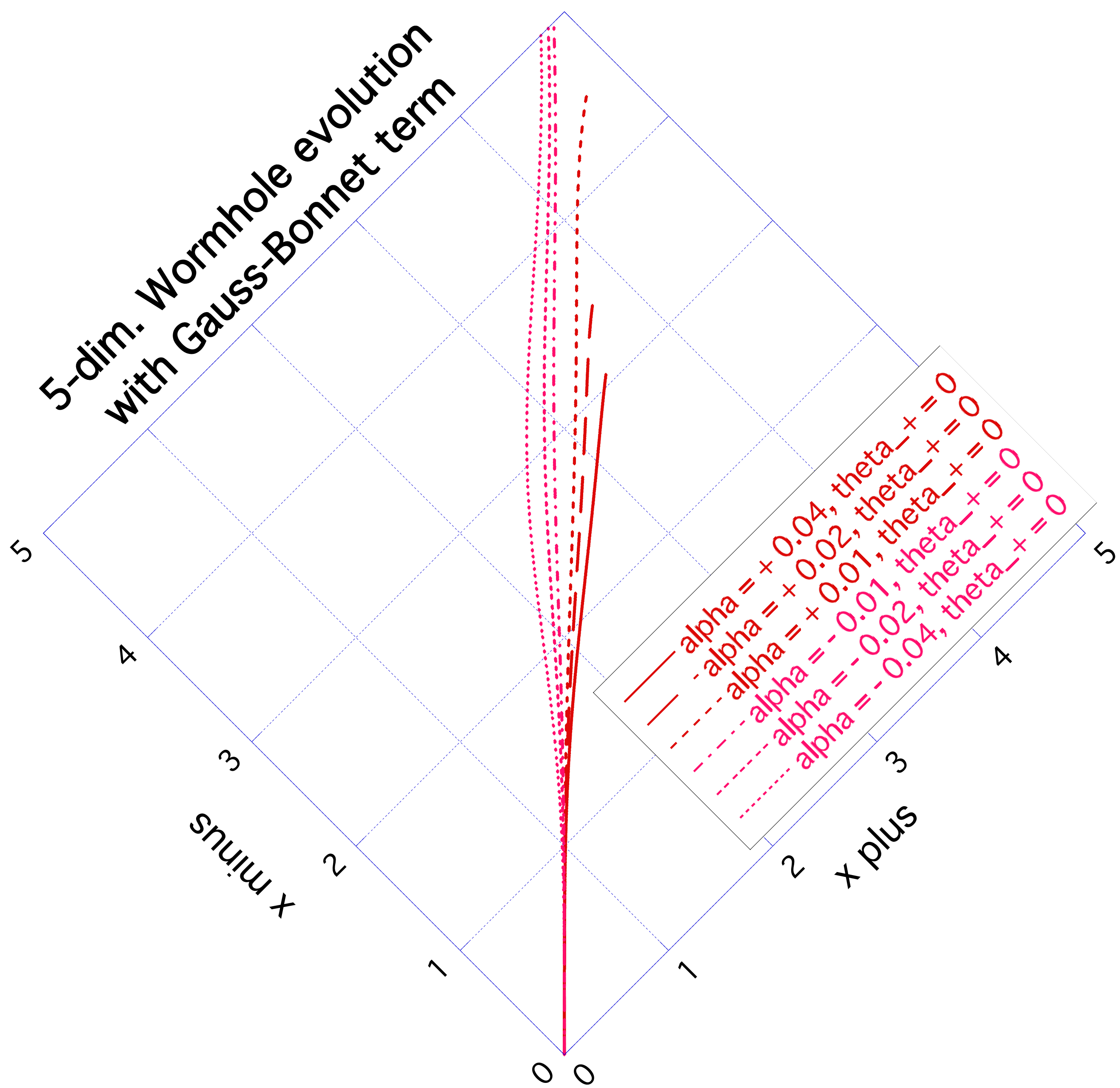
where $\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$

注：初期値は5dim. GR解

WH evolution in 5D Gauss-Bonnet gravity

positive GB term accelerates BH collapse





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where $\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$

注：初期値は5dim. GR解

Summary of Part I (4D)

Ellis (Morris-Thorne) traversible WH解　時間発展

WH は不安定である

(A) 正のエネルギーパルス ---> BH

(B) 負のエネルギーパルス ---> Inflationary expansion

(C) 頑張ればメンテナンス可能

by J.A.Wheeler

Summary of Part II (higher-dim.)

N次元GRでのWH解

得られた

摂動計算：スロートが動くことを許すと不安定モードが存在

5次元GRでのWH解　時間発展

基本的な運命は4次元と同じ

5次元 Gauss-Bonnet 項入り発展方程式での時間発展

負αの GB term --> prevents BH collapse　注：初期値は5dim. GR解

正αの GB term --> accelerates BH collapse