Wormhole dynamics
in higher-dimensional space-time

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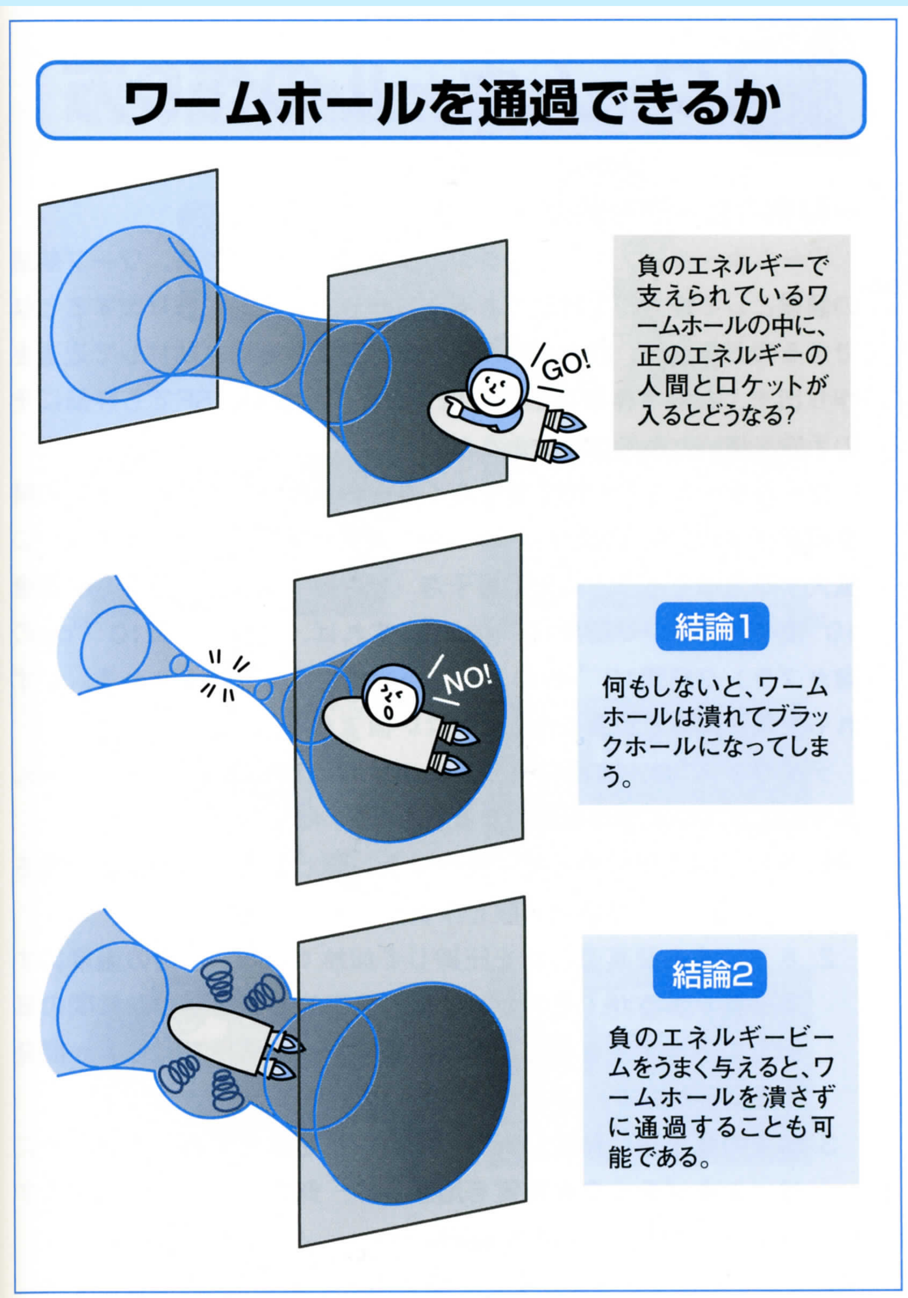


「Time-Machine & Space-time Physics」 (HS)

Part I
- Wormholes in 4-dim.

Part II
1. n-dim. exact solution
2. linear stability
3. dynamical stability

Part III
- Wormhole in Gauss-Bonnet gravity

Part I Wormhole dynamics in 4-dim GR

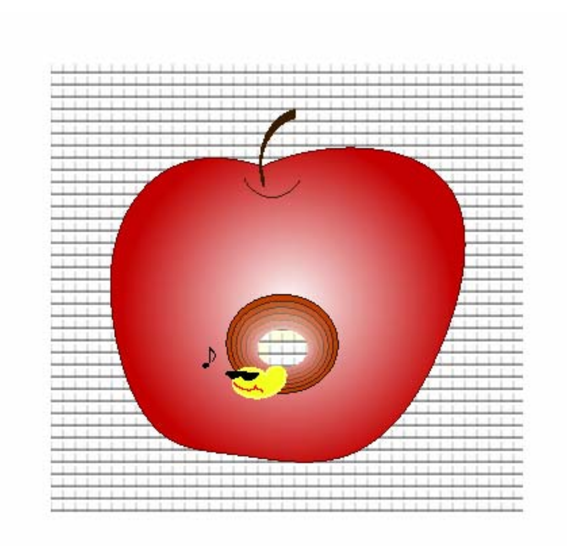
HS & Hayward, PRD66 (2002) 044005

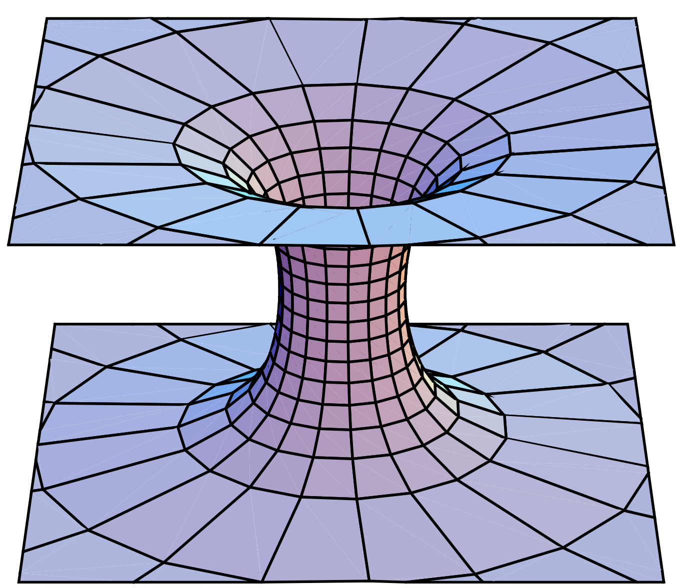
1 Why Wormhole?

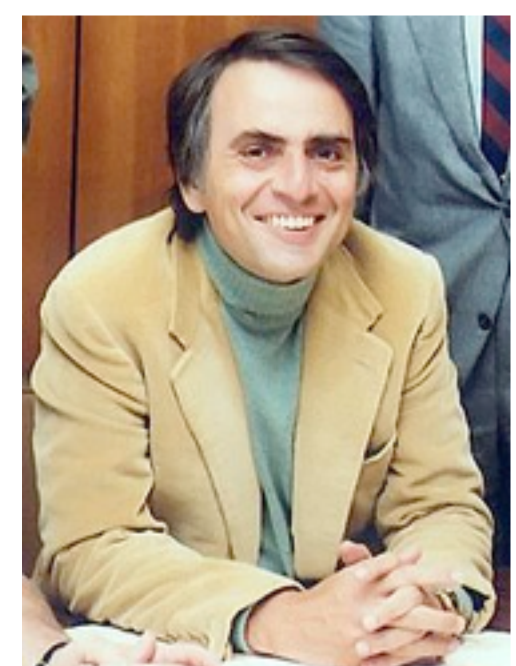
- They make great science fiction – short cuts between otherwise distant regions. Morris & Thorne 1988, Sagan "Contact" etc
- They increase our understanding of gravity when the usual energy conditions are not satisfied, due to quantum effects (Casimir effect, Hawking radiation) or alternative gravity theories, brane-world models etc.
- They are very similar to black holes –both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH).

Wormhole ≡ Hypersurface foliated by marginally trapped surfaces

- BH and WH are interconvertible?
  New duality?





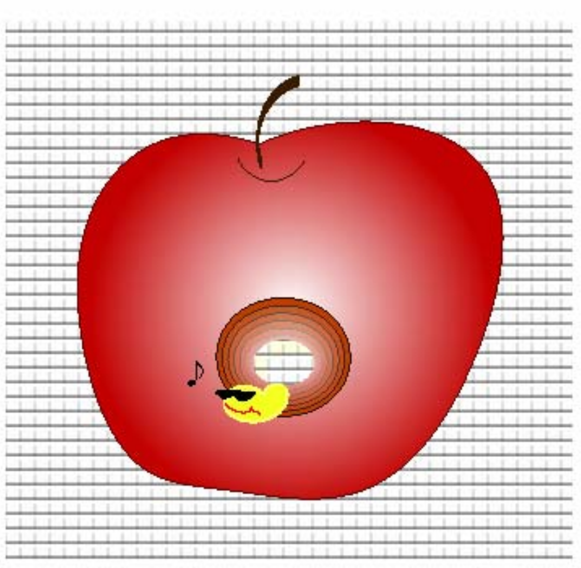


Morris-Thorne's "Traversable" wormhole

M.S. Morris and K.S. Thorne, Am. J. Phys. 56 (1988) 395
M.S. Morris, K.S. Thorne, and U. Yurtsever, PRL 61 (1988) 3182
H.G. Ellis, J. Math. Phys. 14 (1973) 104
(G. Clément, Am. J. Phys. 57 (1989) 967)

Desired properties of traversable WHs

1. Spherically symmetric and Static ⇒ M. Visser, PRD 39(89) 3182 & NPB 328 (89) 203
2. Einstein gravity
3. Asymptotically flat
4. No horizon for travel through
5. Tidal gravitational forces should be small for traveler
6. Traveler should cross it in a finite and reasonably small proper time
7. Must have a physically reasonable stress-energy tensor
   ⇒ Weak Energy Condition is violated at the WH throat.
   ⇒ (Null EC is also violated in general cases.)
8. Should be perturbatively stable
9. Should be possible to assemble



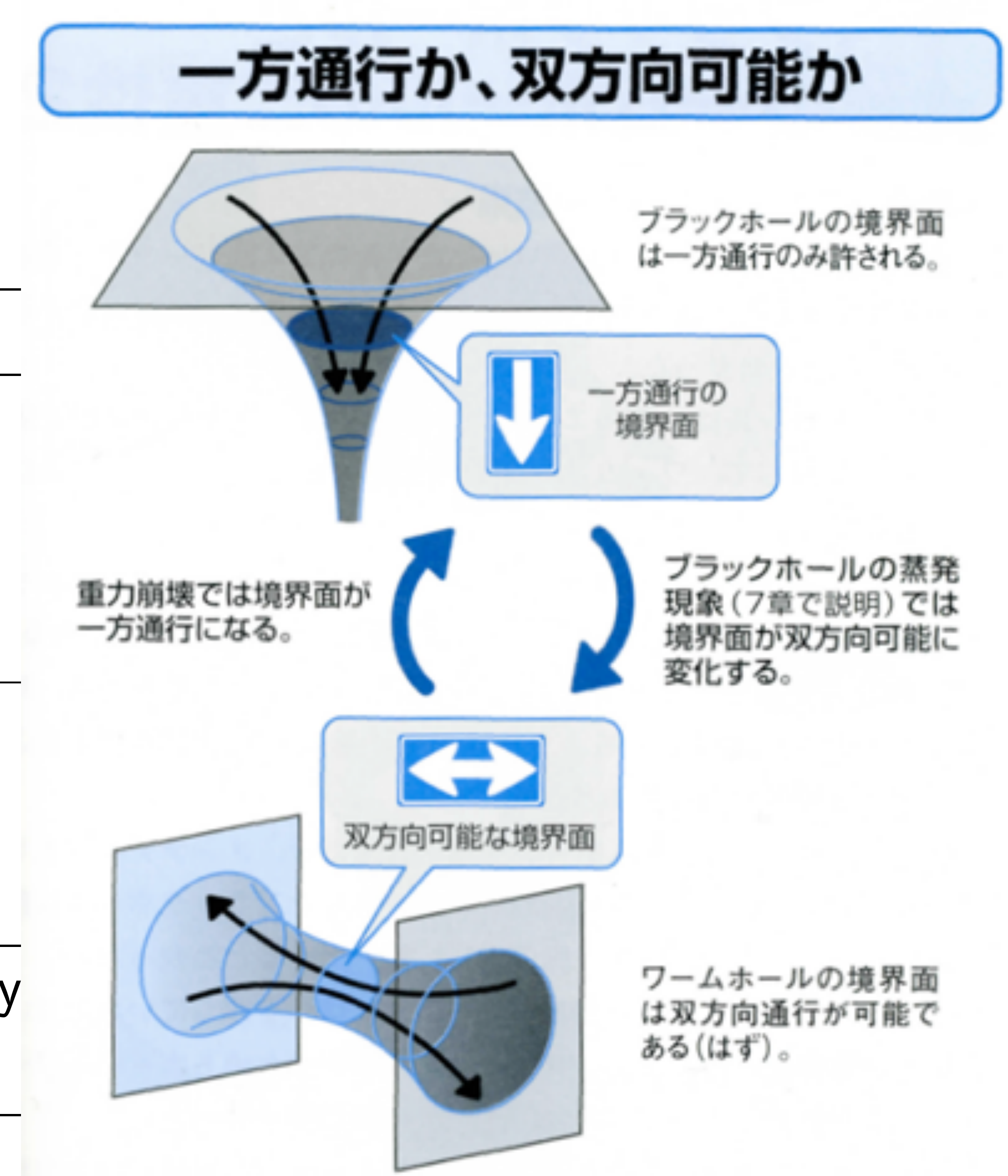
"Ellis (Morris-Thorne) wormhole"

BH and WH are interconvertible ? (New Duality?)

S.A. Hayward, Int. J. Mod. Phys. D 8 (1999) 373

- They are very similar – both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH)
- Only the causal nature of the THs differs, whether THs evolve in plus / minus density.

	Black Hole	Wormhole
Locally defined by	Achronal(spatial/null) outer TH $\Rightarrow$ 1-way traversable	Temporal (timelike) outer THs $\Rightarrow$ 2-way traversable
Einstein eqs.	Positive energy density normal matter (or vacuum)	Negative energy density "exotic" matter
Appearance	occur naturally	Unlikely to occur naturally but constructible ???



Part I Wormhole dynamics in 4-dim GR

PHYSICAL REVIEW D **66**, 044005 (2002)

Fate of the first traversible wormhole: Black-hole collapse or inflationary expansion

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Fate of Morris-Thorne (Ellis) wormhole?

- "Dynamical wormhole" defined by local trapping horizon
- spherically symmetric, both normal/ghost KG field
- apply dual-null formulation in order to seek horizons
- Numerical simulation

ghost/normal Klein-Gordon fields

$$T_{\mu\nu} = T_{\mu\nu}(\psi) + T_{\mu\nu}(\phi) = \underbrace{\left[\psi_{,\mu}\psi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}(\nabla\psi)^2 + V_1(\psi)\right)\right]}_{\text{normal}} + \underbrace{\left[-\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(-\frac{1}{2}(\nabla\phi)^2 + V_2(\phi)\right)\right]}_{\text{ghost}}$$

$$\Box\psi = \frac{dV_1(\psi)}{d\psi}, \qquad \Box\phi = \frac{dV_2(\phi)}{d\phi}.$$

(Hereafter, we set $V_1(\psi) = 0, V_2(\phi) = 0$)

dual-null formulation, spherically symmetric spacetime (4D)

- The spherically symmetric line-element:

$$ds^2 = -2e^{-f}dx^+dx^- + r^2dS^2, \quad \text{where} \quad r = r(x^+, x^-), f = f(x^+, x^-), \cdots$$

- To obtain a system accurate near $\Im^{\pm}$, we introduce the conformal factor $\boxed{\Omega = 1/r}$. We also define first-order variables, the conformally rescaled momenta

expansions	$\vartheta_\pm = 2\partial_\pm r = -2\Omega^{-2}\partial_\pm\Omega$	$(\theta_\pm = 2r^{-1}\partial_\pm r)$	(1)
inaffinities	$\nu_\pm = \partial_\pm f$		(2)
momenta of ϕ	$\wp_\pm = r\partial_\pm\phi = \Omega^{-1}\partial_\pm\phi$		(3)
momenta of ψ	$\pi_\pm = r\partial_\pm\psi = \Omega^{-1}\partial_\pm\psi$		(4)

The set of equations (remember the identity: $\partial_+\partial_- = \partial_-\partial_+$):

$$\partial_\pm\vartheta_\pm = -\nu_\pm\vartheta_\pm - 2\Omega\pi_\pm^2 + 2\Omega\wp_\pm^2, \tag{5}$$

$$\partial_\pm\vartheta_\mp = -\Omega(\vartheta_+\vartheta_-/2 + e^{-f}), \tag{6}$$

$$\partial_\pm\nu_\mp = -\Omega^2(\vartheta_+\vartheta_-/2 + e^{-f} - 2\pi_+\pi_- + 2\wp_+\wp_-), \tag{7}$$

$$\partial_\pm\wp_\mp = -\Omega\vartheta_\mp\wp_\pm/2, \tag{8}$$

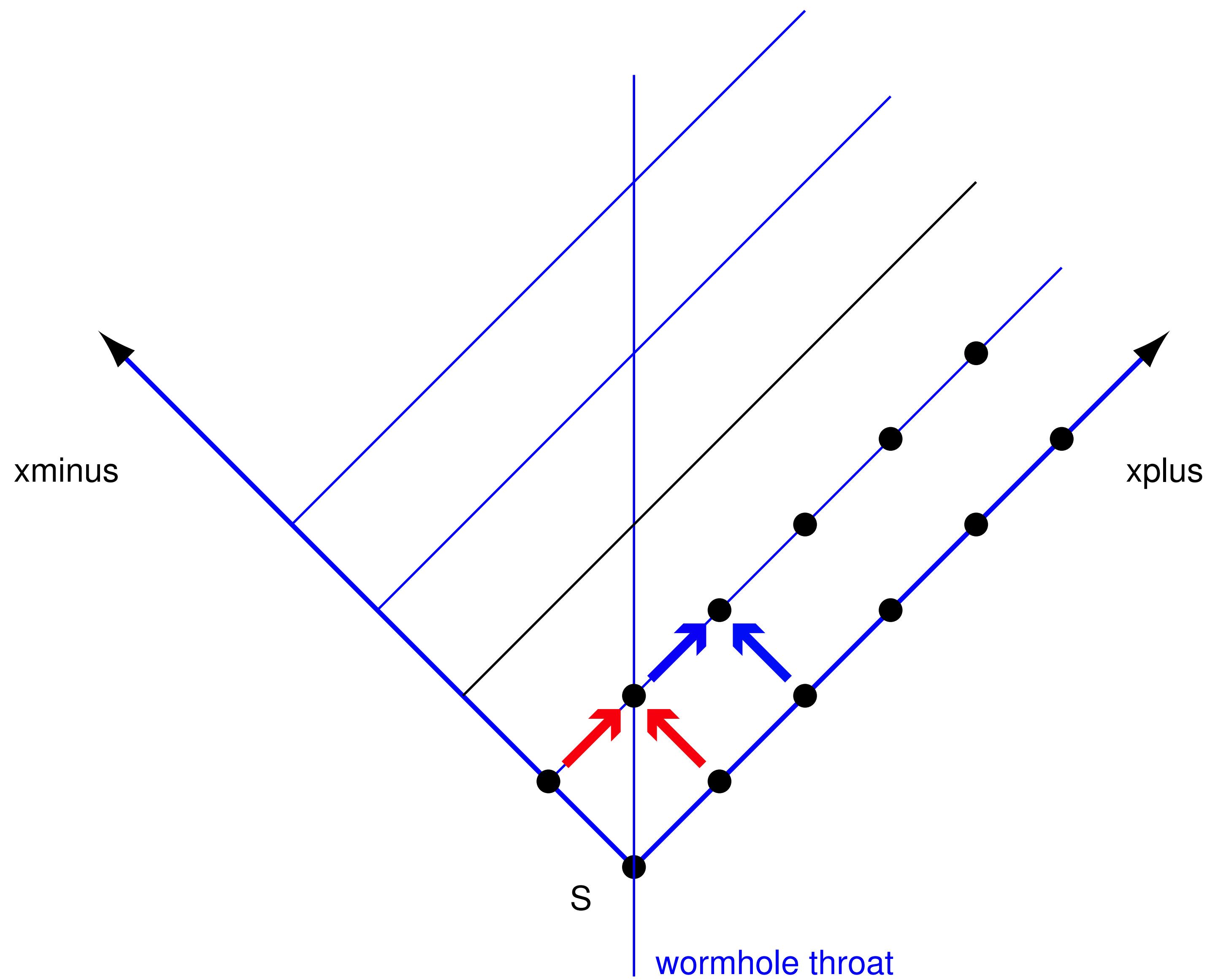
$$\partial_\pm\pi_\mp = -\Omega\vartheta_\mp\pi_\pm/2. \tag{9}$$

Initial data on $x^+ = 0$, $x^- = 0$ slices and on S

Generally, we have to set :

$$(\Omega, f, \vartheta_\pm, \phi, \psi) \quad \text{on } S\text{: } x^+ = x^- = 0$$
$$(\nu_\pm, \wp_\pm, \pi_\pm) \quad \text{on } \Sigma_\pm\text{: } x^\mp = 0,\ x^\pm \geq 0$$

Grid Structure for Numerical Evolution



Ghost pulse input -- Bifurcation of the horizons（4d)

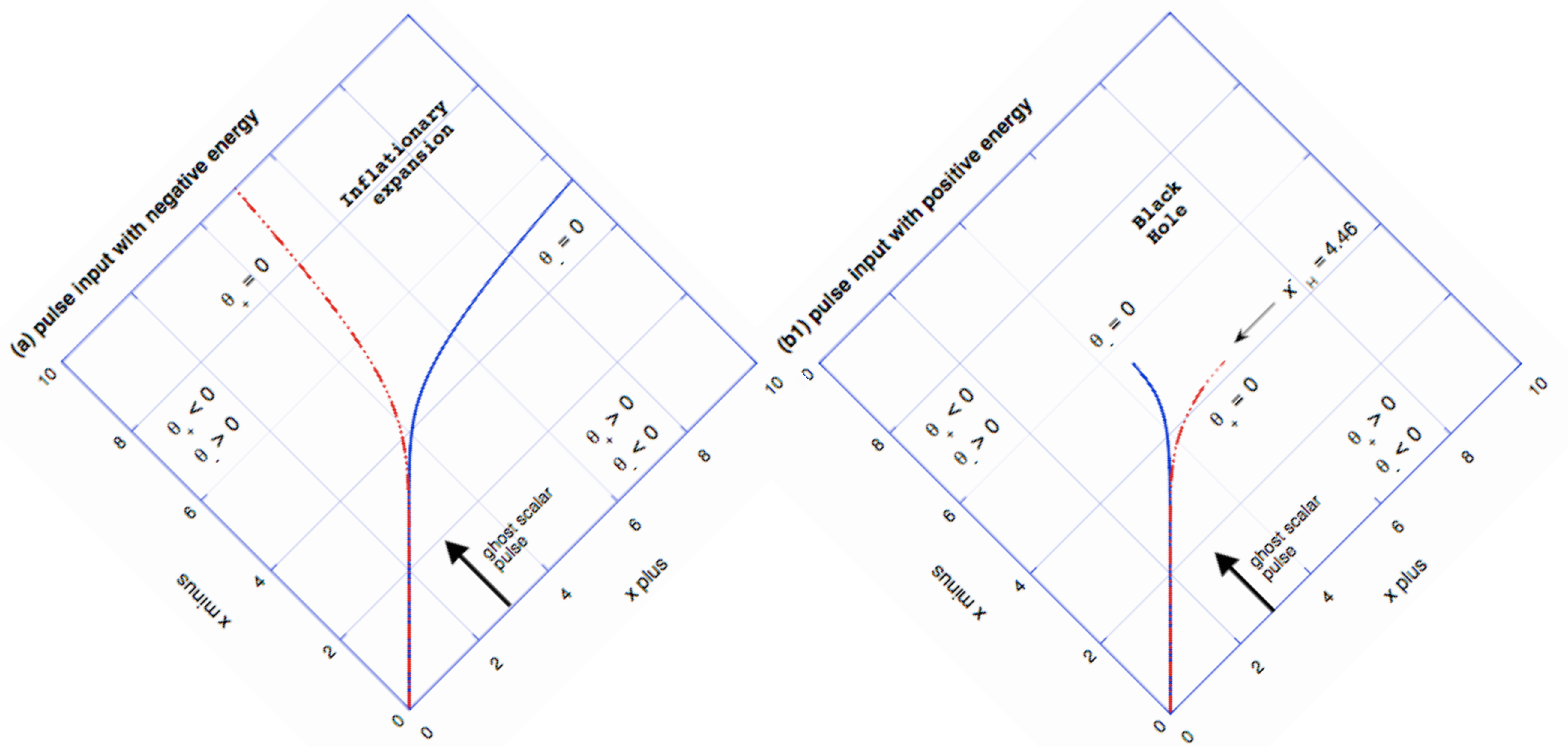
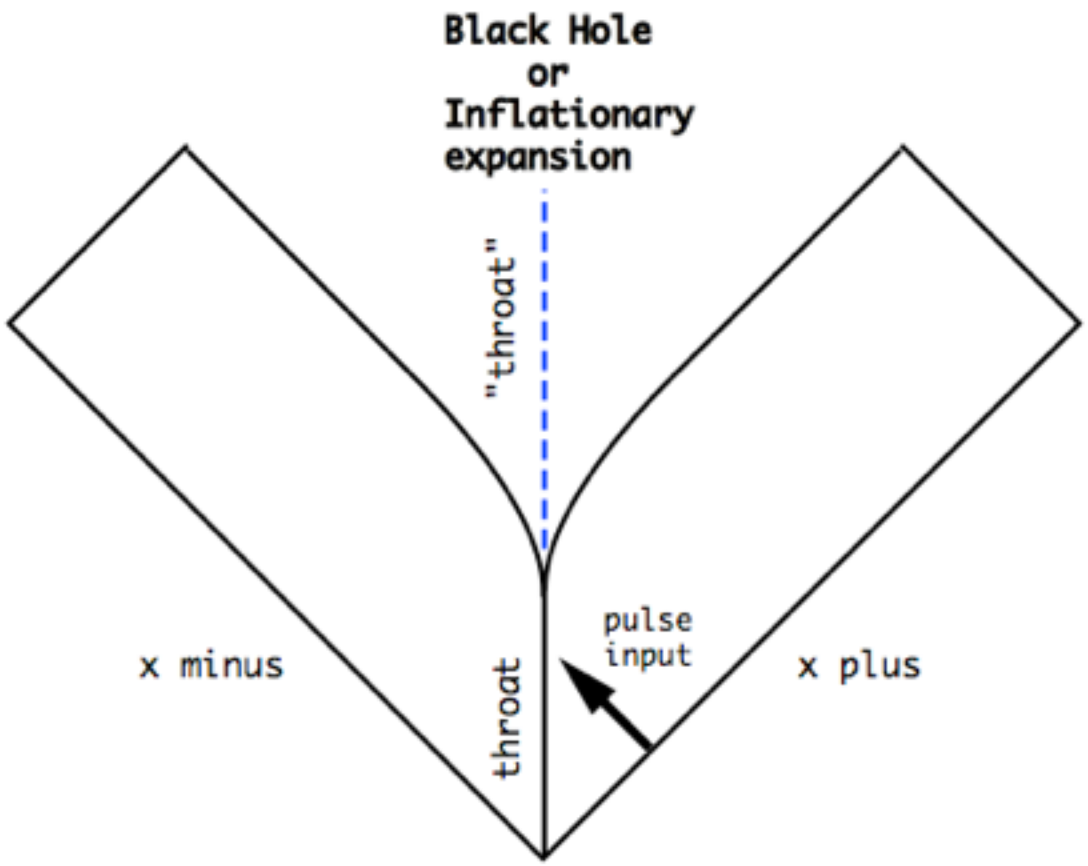


Figure 3: Horizon locations, $\vartheta_{\pm} = 0$, for perturbed wormhole. Fig.(a) is the case we supplement the ghost field, $c_a = 0.1$, and (b1) and (b2) are where we reduce the field, $c_a = -0.1$ and -0.01. Dashed lines and solid lines are $\vartheta_+ = 0$ and $\vartheta_- = 0$ respectively. In all cases, the pulse hits the wormhole throat at $(x^+, x^-) = (3, 3)$. A 45° counterclockwise rotation of the figure corresponds to a partial Penrose diagram.

Bifurcation of the horizons

-- go to a Black Hole or Inflationary expansion



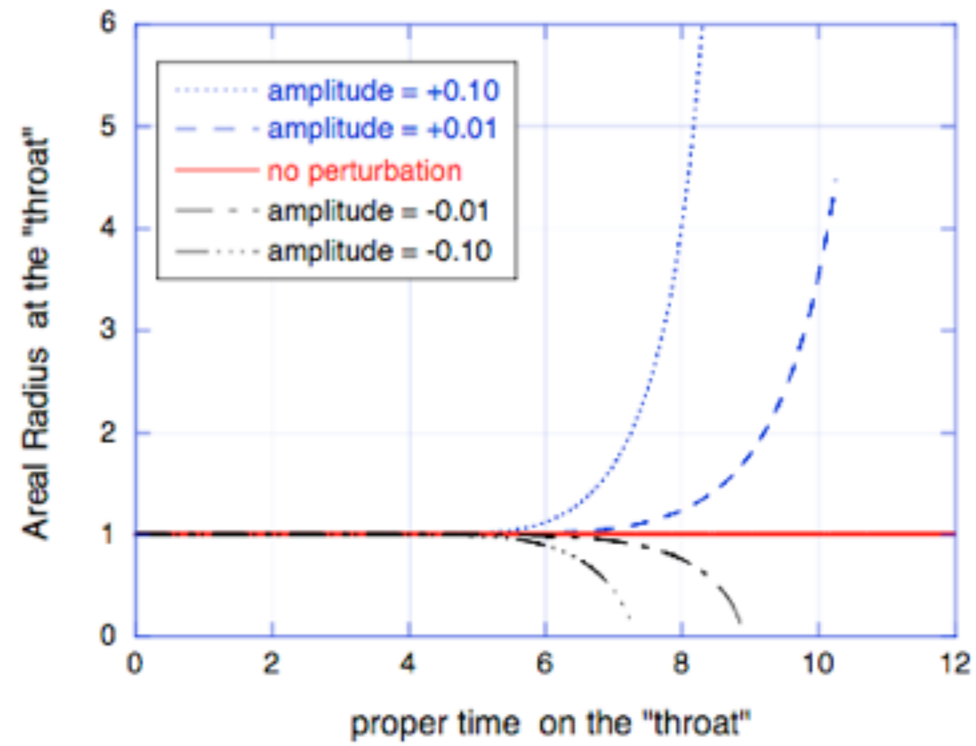
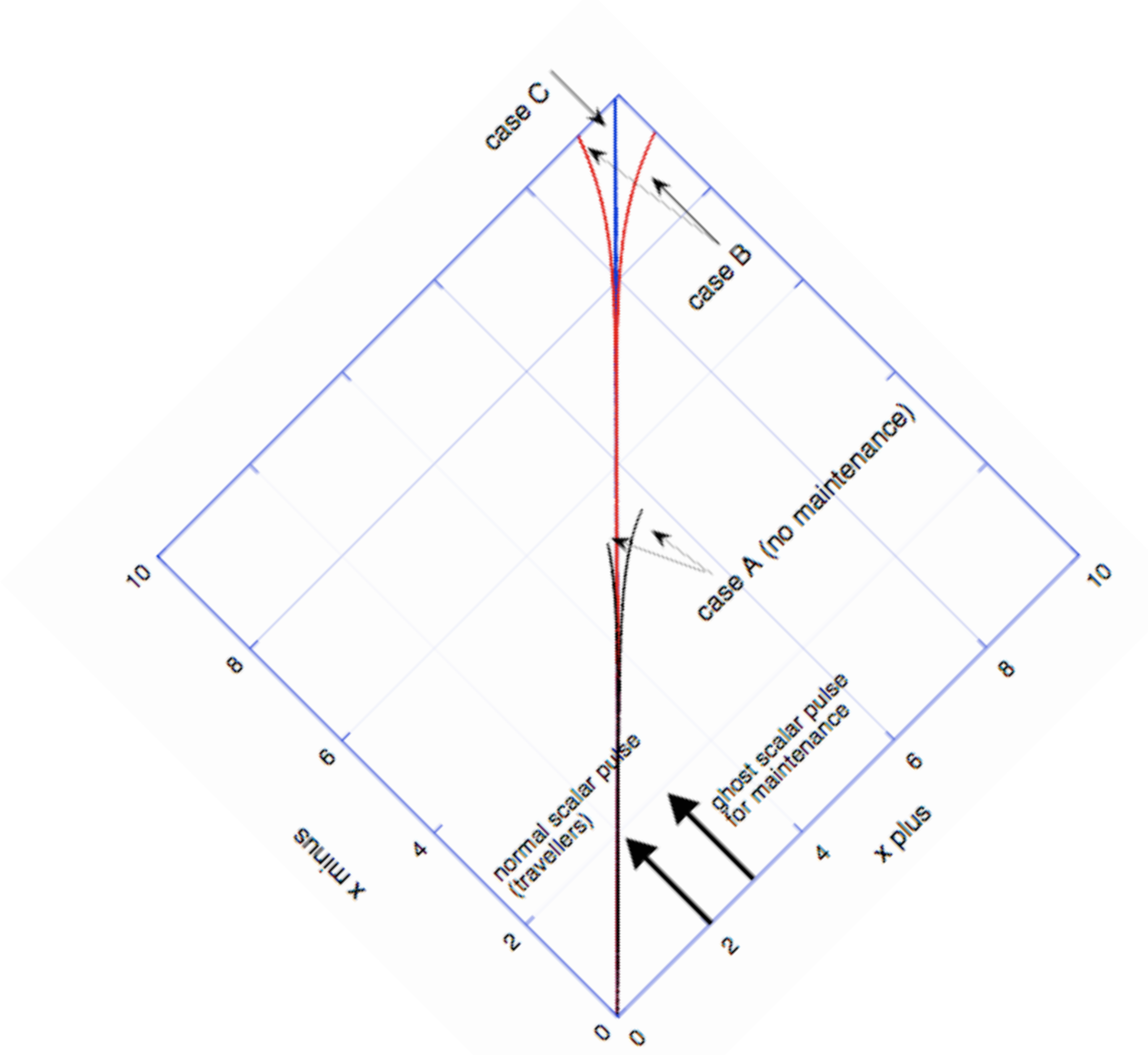


Figure 4: Partial Penrose diagram of the evolved space-time.
Figure 6: Areal radius r of the "throat" $x^+ = x^-$, plotted as a function of proper time. Additional negative energy causes inflationary expansion, while reduced negative energy causes collapse to a black hole and central singularity.

Travel through a Wormhole
-- with Maintenance Operations!



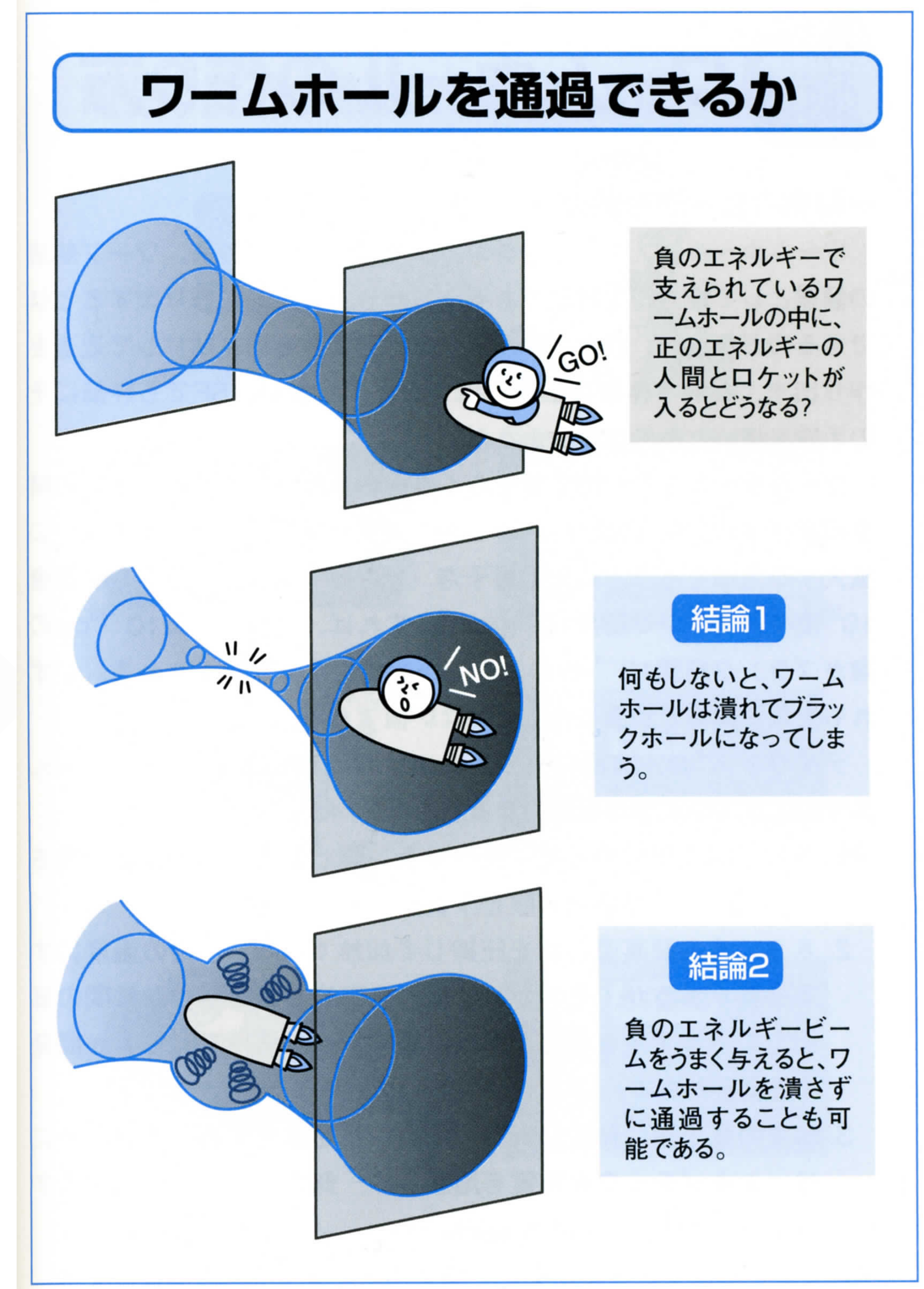


Figure 11: A trial of wormhole maintenance. After a normal scalar pulse, we signalled a ghost scalar pulse to extend the life of wormhole throat. The travellers pulse are commonly expressed with a normal scalar field pulse, $(\tilde{c}_a, \tilde{c}_b, \tilde{c}_c) = (+0.1, 6.0, 2.0)$. Horizon locations $\vartheta_+ = 0$ are plotted for three cases:
(A) no maintenance case (results in a black hole),
(B) with maintenance pulse of $(c_a, c_b, c_c) = (0.02390, 6.0, 3.0)$ (results in an inflationary expansion),
(C) with maintenance pulse of $(c_a, c_b, c_c) = (0.02385, 6.0, 3.0)$ (keep stationary structure upto the end of this range).

Summary of Part I

HS & Hayward, PRD66 (2002) 044005

Dynamics of Ellis (Morris-Thorne) traversible WH

WH is Unstable

(A) with positive energy pulse ---> BH

---> confirms duality conjecture between BH and WH.

(B) with negative energy pulse ---> Inflationary expansion

---> provides a mechanism for enlarging a quantum WH to macroscopic size

(C) can be maintained by sophisticated operations

---> a round-trip is available for our hero/heroine

The basic behaviors has been confirmed by

A Doroshkevich, J Hansen, I Novikov, A Shatskiy, IJMPD 18 (2009) 1665

J A Gonzalez, F S Guzman & O Sarbach, CQG 26 (2009) 015010, 015011

J A Gonzalez, F S Guzman & O Sarbach, PRD80 (2009) 024023

O Sarbach & T Zannias, PRD 81 (2010) 047502

Part II Wormhole Dynamics in higher-dim. GR

(1) Exact Solution : Basic eqns.

Torii & HS, PRD88 (2013) 064027

- general relativity, *n-dimension*

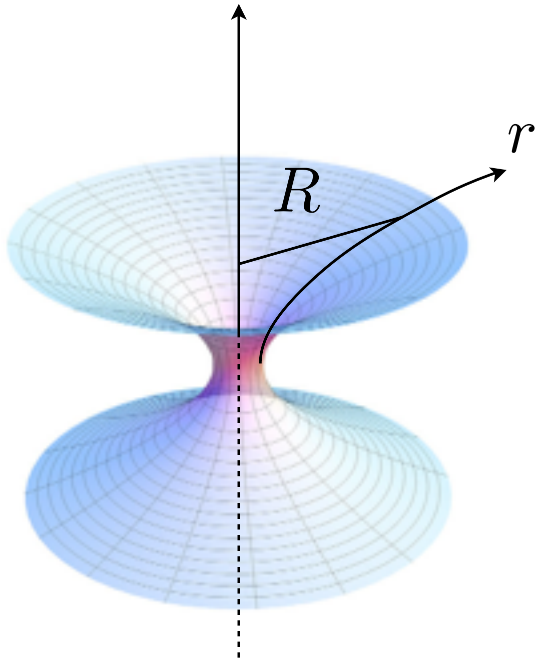
massless scalar field (ghost)

$$S=\int d^nx\sqrt{-g}\left[\frac{1}{2\kappa_n^2}R-\frac{1}{2}\epsilon(\partial\phi)^2-V(\phi)\right],\qquad \epsilon=-1$$

- static, spherical sym., asymptotically flat

$$ds_n^2=-f(r)dt^2+f(r)^{-1}dr^2+R(r)^2h_{ij}dx^idx^j$$

$(k=1)$



- Basic equations

$$(t,t):\quad -\frac{n-2}{2}f^2\left[\frac{2R''}{R}+\frac{f'R'}{fR}+\frac{(n-3)R'^2}{R^2}\right]+\frac{(n-2)(n-3)kf}{2R^2}=\kappa_n^2 f\left[\frac{1}{2}\epsilon f\phi'^2+V(\phi)\right],$$

$$(r,r):\quad \frac{n-2}{2}\frac{R'}{R}\left[\frac{f'}{f}+\frac{(n-3)R'}{R}\right]-\frac{(n-2)(n-3)k}{2fR^2}=\frac{\kappa_n^2}{f}\left[\frac{1}{2}\epsilon f\phi'^2-V(\phi)\right],$$

$$(i,j):\quad \frac{f''}{2}+(n-3)f\left(\frac{R''}{R}+\frac{f'R'}{fR}+\frac{n-4}{2}\frac{R'^2}{R^2}\right)-\frac{(n-3)(n-4)k}{2R^2}=\kappa_n^2\left[\frac{1}{2}\epsilon f\phi'^2+V(\phi)\right],$$

$$(KG):\quad \frac{1}{R^{n-2}}\left(R^{n-2}f\phi'\right)'=-\epsilon\frac{dV}{d\phi}.\quad\Rightarrow\quad \phi'=\frac{C}{fR^{n-2}}$$

C: constant

(1) Exact Solution: Solution

- regularity at the throat ($r = 0$)

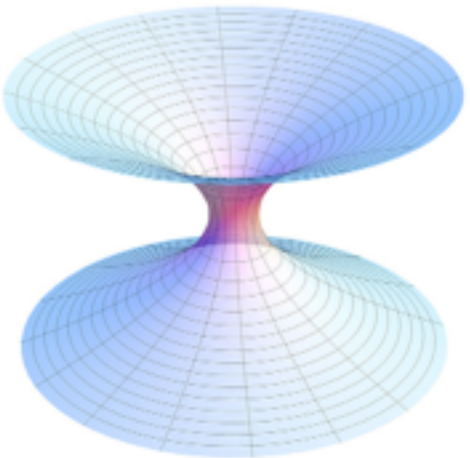
$R = a$ ← throat radius

$R' = 0, \quad f = f_0, \quad f' = 0, \quad \phi = 0$

★ from the scaling rule

$a = 1 \qquad f_0 = 1$

Basics eqns. → $\kappa_n^2 C^2 = (n-2)(n-3)a^{2(n-3)}$



- Exact solution

$$f \equiv 1$$

$$r(R) = -mB_z\left[-m, \frac{1}{2}\right] - \frac{\sqrt{\pi}\Gamma[1-m]}{\Gamma[m(n-4)]}$$

$$\phi = \frac{\sqrt{(n-2)(n-3)}}{\kappa_n} a^{n-3} \int \frac{1}{R(r)^{n-2}} dr$$

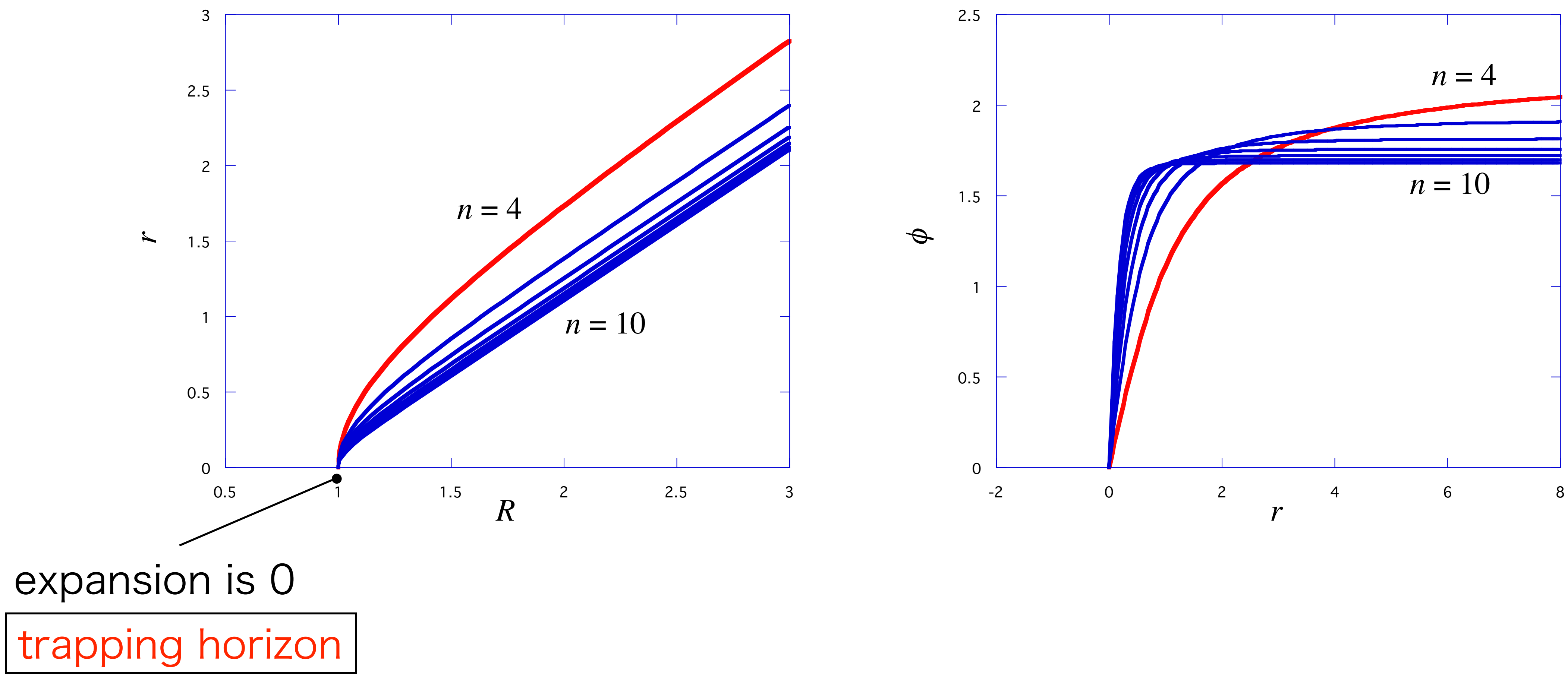
$$m = \frac{1}{2(n-3)}, \quad z = R^m \quad B_z(p,q) := \int_0^z t^{p-1}(1-t)^{q-1} dt$$

Incomplete Beta func.

★ in another metric form: V. Dzhunushaliev+, 2013

(1) Exact Solution : Configurations

- configurations



★ large curvature near the throat.

★ scalar field goes steep if n is large.

★ In the $n \to \infty$ limit

$$R = r + 1 \qquad \phi = 0 \;\; (r = 0) \;\; \frac{\pi}{2} \;\; (r > 0)$$

(2) Linear Stability: Master eqn.

Torii & HS, PRD88 (2013) 064027

- metric

$$ds_n^2 = -f(t,r)e^{-2\delta(t,r)}dt^2 + f(t,r)^{-1}dr^2 + R(t,r)^2 h_{ij}dx^i dx^j$$

- linear perturbation

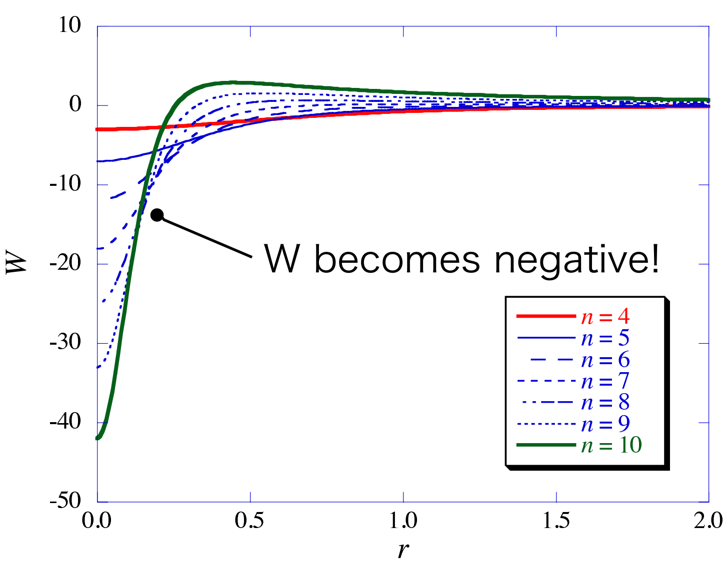
static solution

$$f = f_0(r) + f_1(r)e^{i\omega t}, \quad R = R_0(r) + R_1(r)e^{i\omega t},$$
$$\delta = \delta_0(r) + \delta_1(r)e^{i\omega t}, \quad \phi = \phi_0(r) + \phi_1(r)e^{i\omega t}.$$

- master equation

$$-\Psi_1'' + W(r)\Psi_1 = \omega^2\Psi_1,$$

$$W(r) = -\frac{1}{4R_0^2}\left[\frac{3(n-2)^2}{R_0^{2(n-3)}} - (n-4)(n-6)\right].$$



★ potential W

$$\Psi_1 = \mathcal{D}_+\psi_1 \quad \mathcal{D}_+ = \frac{d}{dr} - \frac{\bar{\psi}_1'}{\bar{\psi}_1} \quad \psi_1 = R_0^{\frac{n-2}{2}}\left(\phi_1 - \frac{\phi_0'}{R_0'}R_1\right),$$

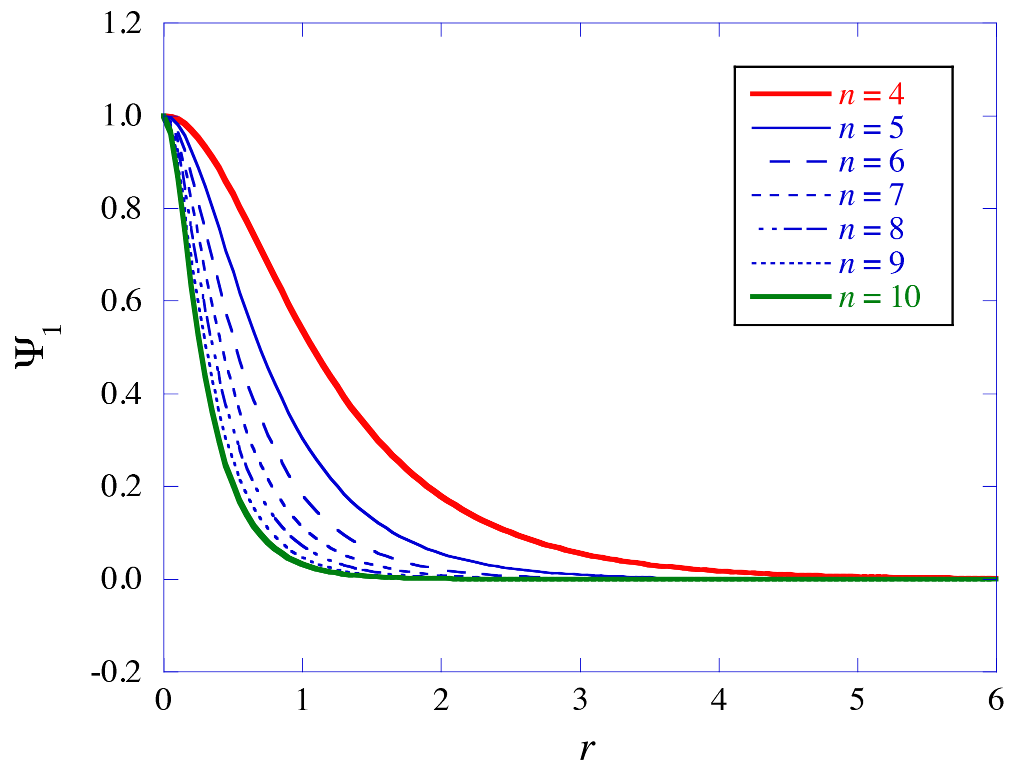
★ Ψ_1 : Gauge invariant in spherical sym.

(2) Linear Stability: Unstable!

- exist negative mode

n	ω^2
4	-1.39705243371511
5	-2.98495893027790
6	-4.68662054299460
7	-6.46258414126318
8	-8.28975936306259
9	-10.1535530451867
10	-12.0442650147438
11	-13.9552091676647
20	-31.5751101285105
50	-91.3457759137153
100	-191.283017729717

eigenvalues of negative mode



eigenfunction of negative mode

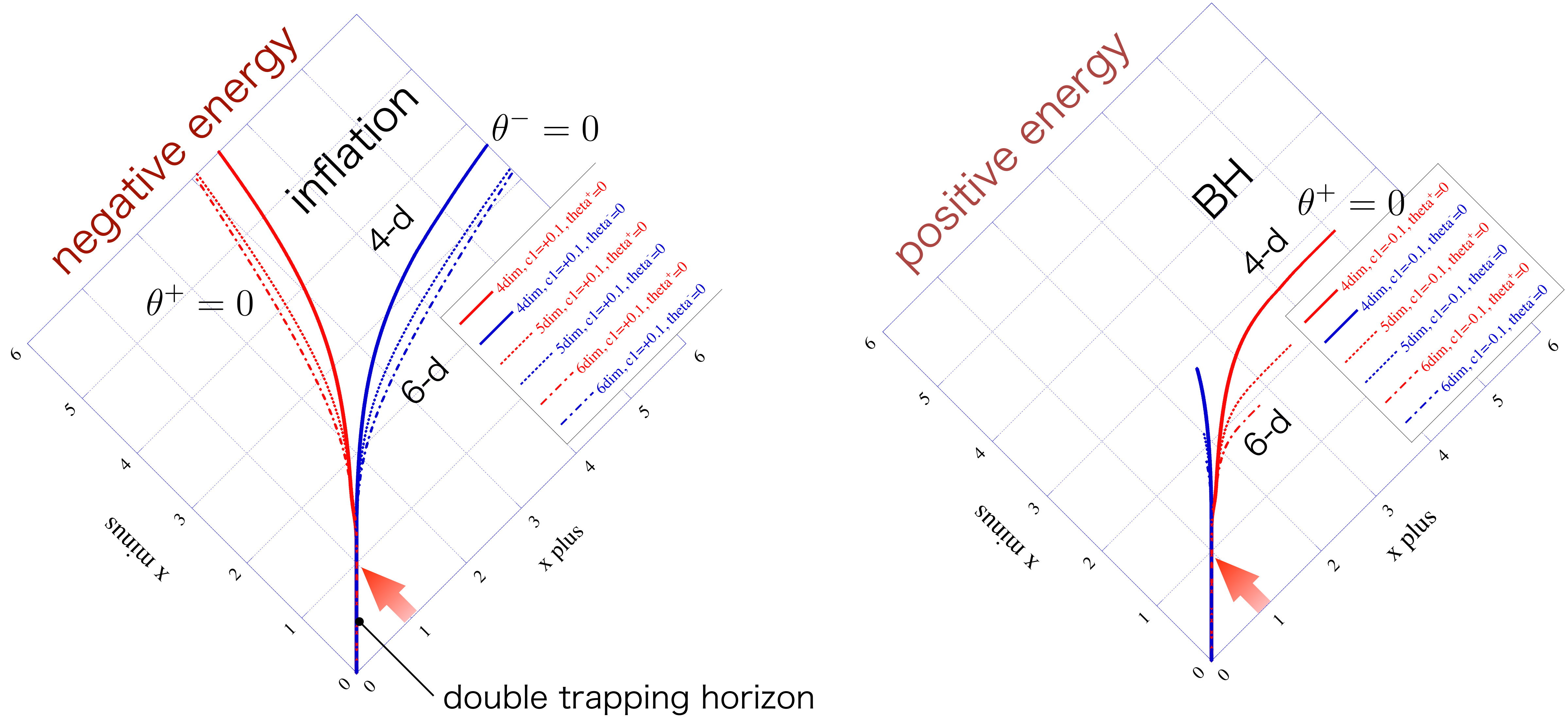
★ In all dimensions, we found negative modes.

→ Ellis's wormhole is unstable

★ Higher dimension, instability appears in short time scale

(3) Dynamical Stability: Unstable!

HS & Torii, in preparation



- The throat horizons (double trapping horizons) bifurcates , and they propagates as null.
- with negative energy pulse —> throat inflates
- with positive energy pulse —> turns to be black hole

(3) Dynamical Stability: ∃ minimum BH mass

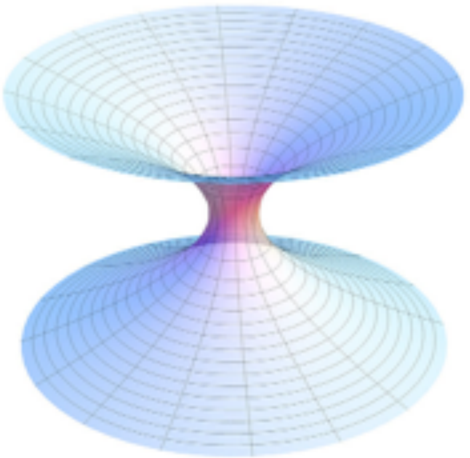
HS & Torii, in preparation

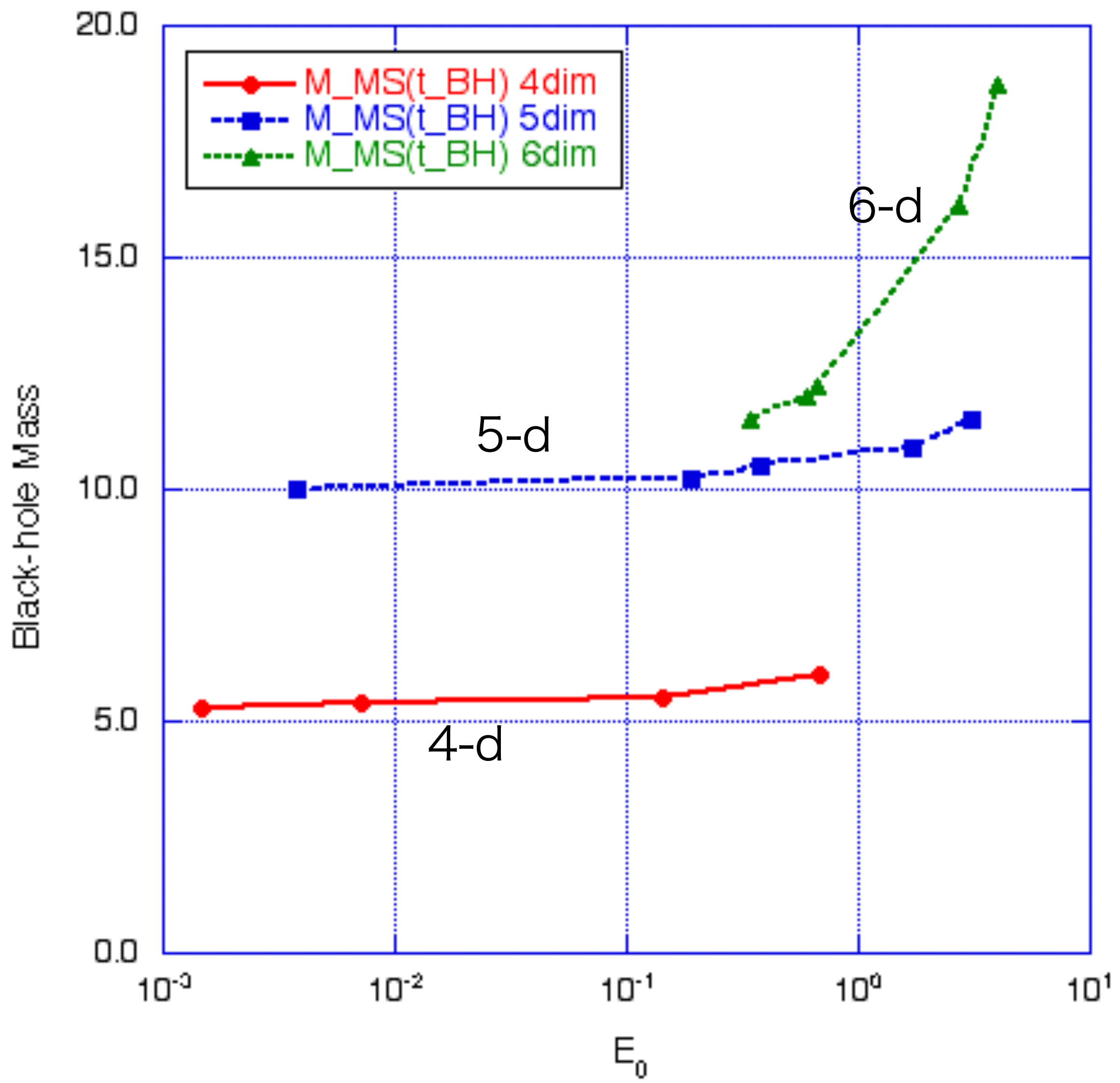
- BH mass (Misner-Sharp mass)

$$E_n = \frac{(n-2)A_{n-2}}{2\kappa_n^2}\Omega\left[-\frac{1}{\Omega^2}\tilde{\Lambda} + \left(k + \frac{2}{(n-2)^2}e^f\vartheta_+\vartheta_-\right)\right]$$

(Maeda & Nozawa, 2008)

→ existence of minimum mass





Part III Wormhole Dynamics in Gauss-Bonnet gravity

HS & Torii, in preparation

Gauss-Bonnet gravity

$$S = \int_{\mathcal{M}} d^{N+1}x\sqrt{-g}\Big[\frac{1}{2\kappa^2}\{\alpha_1\mathcal{R} + \alpha_2\left(\mathcal{R}^2 - 4\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta} + \mathcal{R}_{\alpha\beta\gamma\delta}\mathcal{R}^{\alpha\beta\gamma\delta}\right)\} + \mathcal{L}_{\rm matter}\Big]$$

- has GR correction terms from String Theory.
- has two solution branches (GR/non-GR).
- is expected to have singularity avoidance feature.
  (but has never been demonstrated.)

- new topic in numerical relativity.

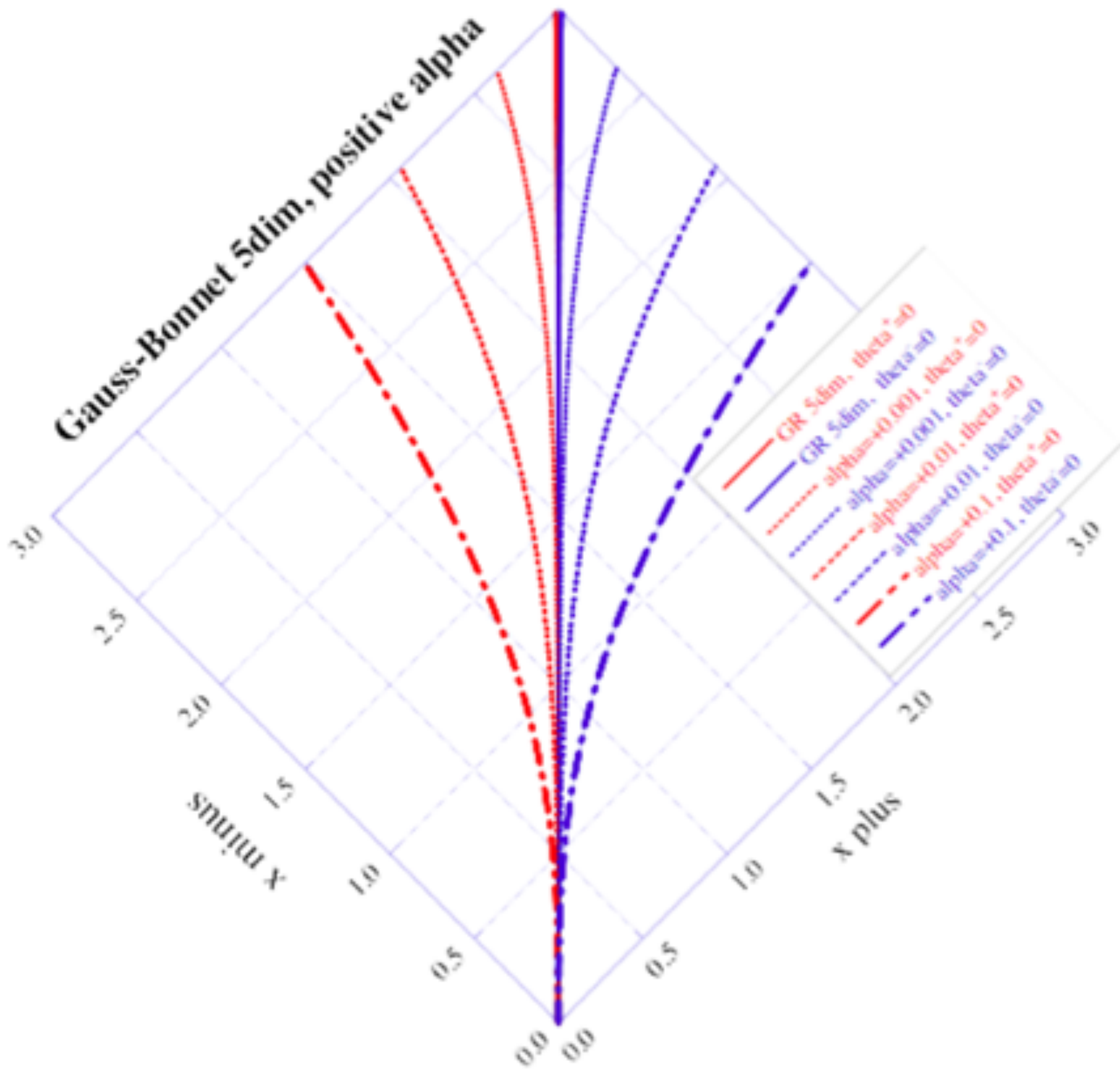
  Golod & Piran, PRD 85 (2012) 104015
  Deppe, Leonard, Taves, Kunstatter, & Mann, PRD 86 (2012) 104011
  Izaurieta & Rodriguez, CQG 30 (2013) 155009

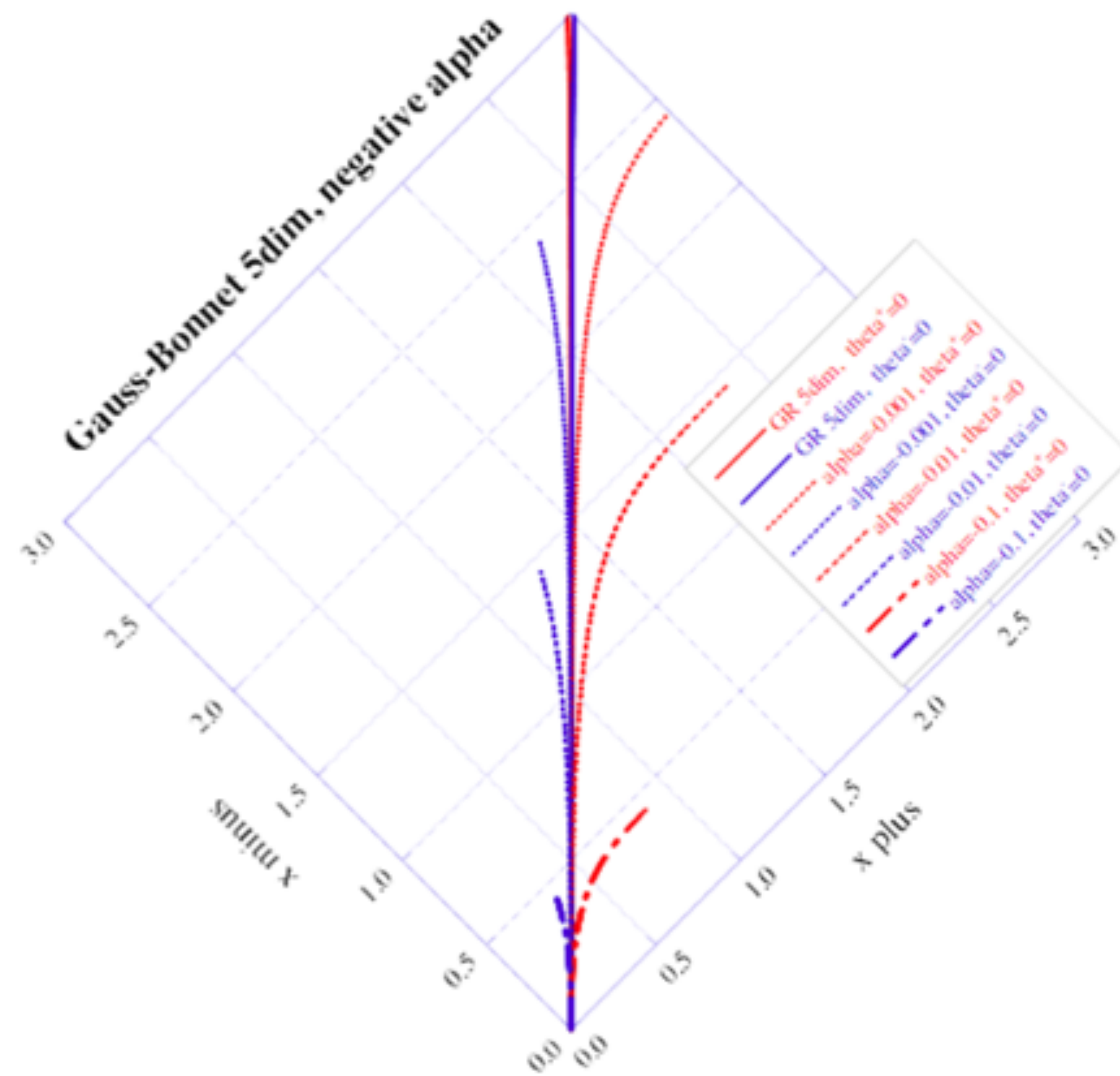
Part III Wormhole Dynamics in Gauss-Bonnet gravity

HS & Torii, in preparation

positive GB term accelerates WH expansion

negative GB term accelerates BH formation





$$S = \int_{\mathcal{M}} d^{N+1}x\sqrt{-g}\Big[\frac{1}{2\kappa^2}\{\alpha_1 \mathcal{R} + \alpha_2 \mathcal{L}_{\rm GB}\} + \mathcal{L}_{\rm matter}\Big]$$

where $\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$

Summary

Ellis (Morris-Thorne) traversable wormhole is unstable both linearly & dynamically, in any dimension.

Wormhole will change its style to either a blackhole or an expanding throat depending on the violation of the energy balance.

by J.A.Wheeler

Similar wormhole in Gauss-Bonnet gravity is also unstable.

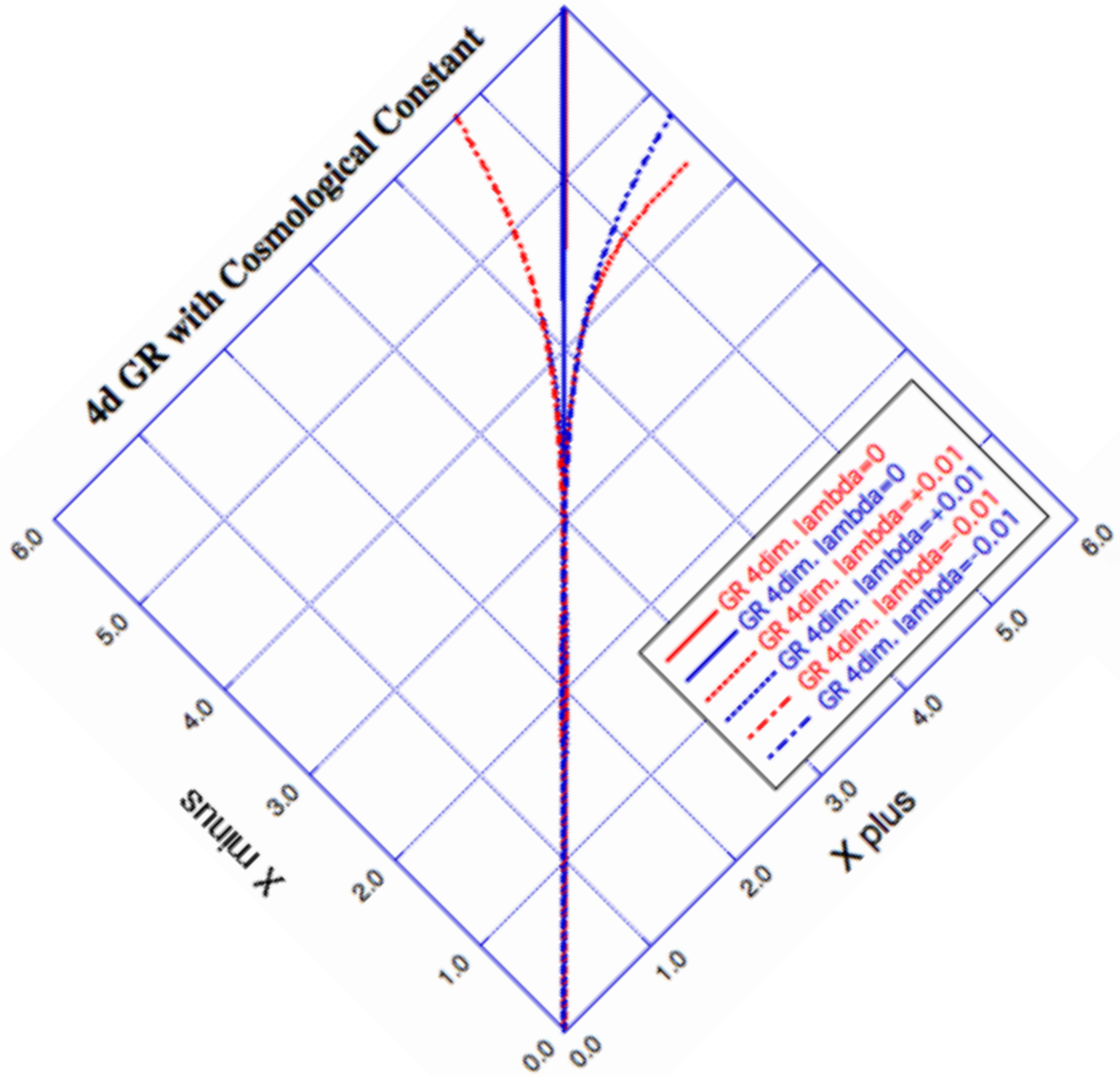
(A) with negative GB term ---> Black hole

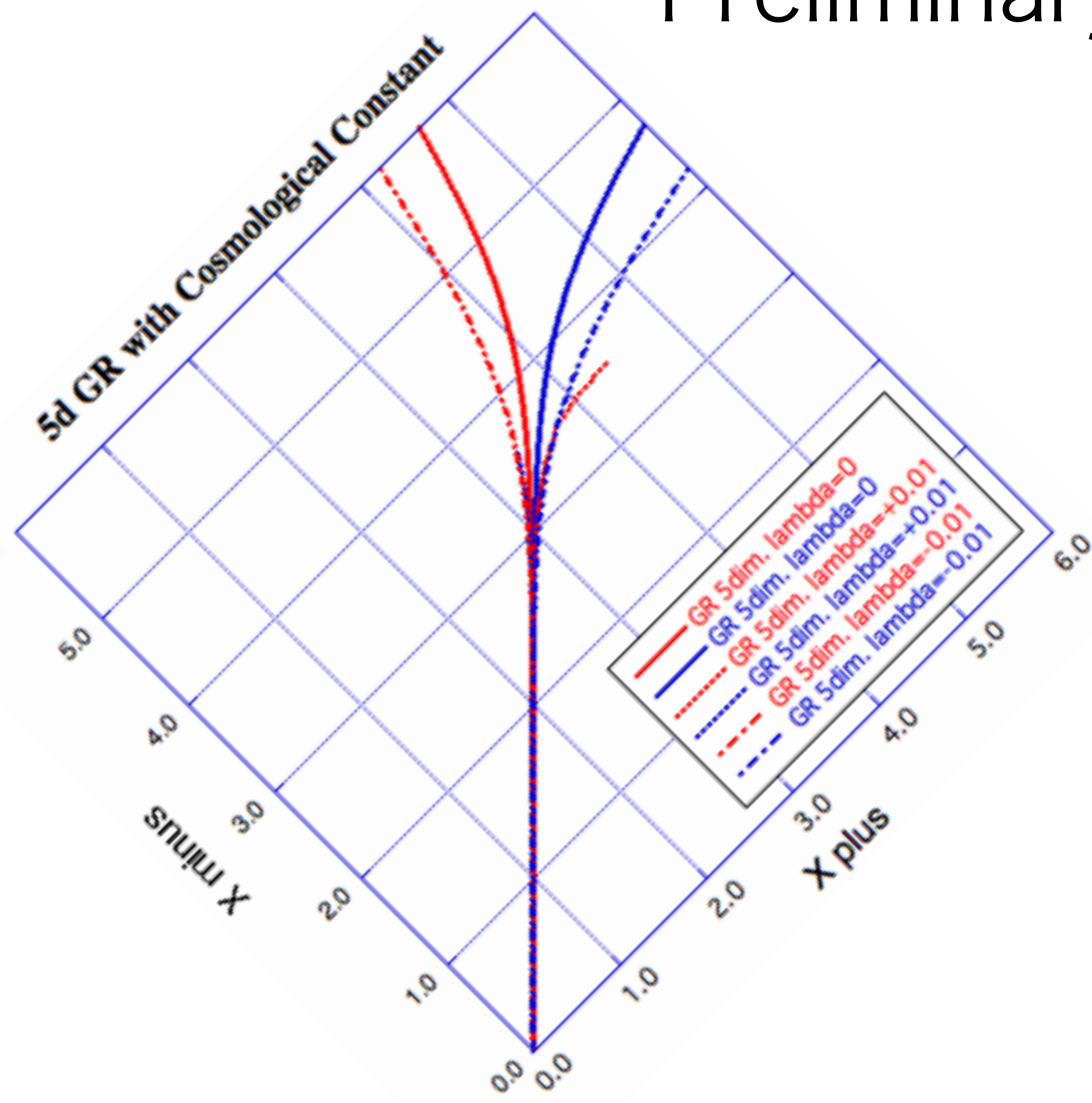
(B) with positive GB term ---> Inflationary expansion

[In cosmology, positive GB gravity is used for avoiding singularity]

Wormhole with Cosmological Constant

Preliminary





positive Λ → BH formation

negative Λ → expanding throat