P29

Singularity Formation in n-dim Gauss-Bonnet gravity

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1. Motivation

Dynamics in Gauss-Bonnet gravity?

- Action

$$S=\int_{\mathcal{M}} d^{N+1}x\sqrt{-g}\Big[\frac{1}{2\kappa^2}\{\alpha_1\mathcal{R}+\alpha_2\mathcal{L}_{\rm GB}\}+\mathcal{L}_{\rm matter}\Big]$$

where $\mathcal{L}_{GB}=\mathcal{R}^2-4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$

- Field equation

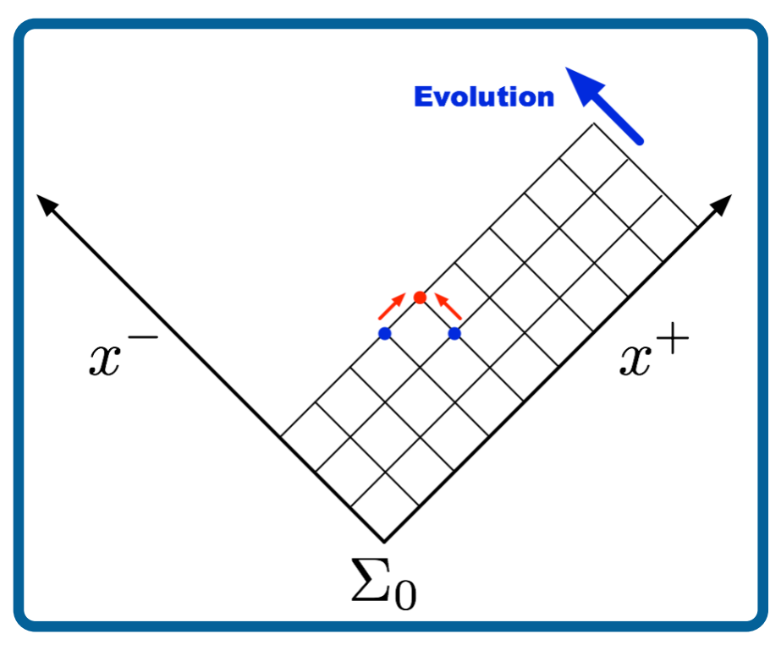
$$\alpha_1 G_{\mu\nu}+\alpha_2 H_{\mu\nu}+g_{\mu\nu}\Lambda=\kappa^2 T_{\mu\nu}$$

where $H_{\mu\nu}=2[\mathcal{R}\mathcal{R}_{\mu\nu}-2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu}-2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta}+\mathcal{R}_{\mu}{}^{\alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}]-\frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB}$

- **has GR correction terms from String Theory**
- **has two solution branches (GR/non-GR).**
- **is expected to have singularity avoidance feature.**
  (but has never been demonstrated.)

- **new topic in numerical relativity.**
  S Golod & T Piran, PRD 85 (2012) 104015
  N Deppe+, PRD 86 (2012) 104011
  F Izaurieta & E Rodriguez, 1207.1496
- **much attentions in WH community**
  H Maeda & M Nozawa, PRD 78 (2008) 024005
  P Kanti, B Kleihaus & J Kunz, PRL 107 (2011) 271101
  P Kanti, B Kleihaus & J Kunz, PRD 85 (2012) 044007

Field Eqs.



Field Equations (1)

Formulation for evolution [dual null]

Metric n-dimensional, dual-null coordinate, $2+(n-2)$ decomposition

$$ds^2=-2e^{-f(x^+,x^-)}dx^+dx^-+r^2(x^+,x^-)\gamma_{ij}dx^idx^j \tag{1}$$

Variables

$\Omega=\frac{1}{r}$	Conformal factor
$\vartheta_\pm=(n-2)\partial_\pm r$	expansion
f	lapse function
$\nu_\pm=\partial_\pm f$	inaffinity (shift)
ψ	scalar field (normal)
$\pi_\pm=r\partial_\pm\psi$	scalar momentum
ϕ	scalar field (ghost)
$p_\pm=r\partial_\pm\phi$	scalar momentum

Parameters

n	dimension
k	curvature
Λ	cosmological constant

For simplicity, we define

$$\begin{aligned}
\tilde\alpha &= (n-3)(n-4)\alpha_2, && (2)\\
A &= \alpha_1+2\tilde\alpha\Omega^2 Z, && (3)\\
W &= \frac{2e^f}{(n-2)^2}\vartheta_+\vartheta_-, && (4)\\
Z &= k+W, && (5)\\
\eta &= \Omega^2\frac{(n-2)(n-3)}{2}e^{-f}Z, && (6)
\end{aligned}$$

Field Equations (2)

matter variables

normal field $\psi(u,v)$ and/or ghost field $\phi(u,v)$

$$\begin{aligned}
T_{\mu\nu} &= T_{\mu\nu}(\psi)+T_{\mu\nu}(\phi)\\
&= \left[\psi_{,\mu}\psi_{,\nu}-g_{\mu\nu}\left(\frac{1}{2}(\nabla\psi)^2+V_1(\psi)\right)\right]+\left[-\phi_{,\mu}\phi_{,\nu}-g_{\mu\nu}\left(-\frac{1}{2}(\nabla\phi)^2+V_2(\phi)\right)\right]
\end{aligned}$$

this derives Klein-Gordon equations

$$\Box\psi=\frac{dV_1}{d\psi},\qquad \Box\phi=\frac{dV_2}{d\phi}.$$

Scalar field variables

$$\begin{aligned}
\pi_\pm &\equiv r\partial_\pm\psi=\frac{1}{\Omega}\partial_\pm\psi\\
p_\pm &\equiv r\partial_\pm\phi=\frac{1}{\Omega}\partial_\pm\phi
\end{aligned}$$

Klein-Gordon eqs.

$$\begin{aligned}
\Box\phi &= -\frac{e^f}{r}\left(2r\phi_{uv}+(n-2)r_u\phi_v+(n-2)r_v\phi_u\right)\\
&= -2e^f\phi_{uv}-e^f\Omega^2\left(\vartheta_-p_++\vartheta_+p_-\right)
\end{aligned}$$

Energy-momentum tensor

$$\begin{aligned}
T_{++} &= \Omega^2(\pi_+^2-p_+^2)\\
T_{--} &= \Omega^2(\pi_-^2-p_-^2)\\
T_{+-} &= -e^{-f}\left(V_1(\psi)+V_2(\phi)\right)\\
T_{zz} &= e^f(\pi_+\pi_--p_+p_-)-\frac{1}{\Omega^2}\left(V_1(\psi)-V_2(\phi)\right)
\end{aligned}$$

Field Equations (3)

evolution equations (1)

Equations for x^+ direction

$$\begin{aligned}
\partial_+\Omega &= -\frac{1}{n-2}\vartheta_+\Omega^2 && (7)\\
\partial_+\vartheta_+ &= -\vartheta_+\nu_+-\frac{1}{\Omega A}\kappa^2T_{++}=-\vartheta_+\nu_+-\frac{1}{A}\kappa^2\Omega(\pi_+^2-p_+^2) && (8)\\
\partial_+\vartheta_- &= \frac{1}{A}\frac{e^{-f}}{\Omega}\left[-\alpha_1\Omega^2\frac{(n-2)(n-3)}{2}Z+\Lambda+\kappa^2(V_1+V_2)\right]-\frac{\tilde\alpha}{A}\Omega^3e^{-f}\frac{(n-2)(n-5)}{2}\left[Z^2+W\right] && (9)\\
\partial_+f &= \nu_+ && (10)\\
\partial_+\nu_+ &= \text{no evolution eq. exists}\\
\partial_+\nu_- &= \frac{\alpha_1}{A}Ze^{-f}\Omega^2\frac{(n-3)}{2}\left\{-\frac{\alpha_1}{A}2(n-3)+n-4\right\}\\
&\quad+\frac{1}{A}\Omega^2e^{-f}\kappa^2(\pi_+\pi_--p_+p_-)+\frac{1}{A}e^{-f}\left\{\frac{\alpha_1}{A}\frac{2(n-3)}{(n-2)}-1\right\}\{\Lambda+\kappa^2(V_1+V_2)\}\\
&\quad-\frac{\tilde\alpha}{A}e^{-f}\Omega^2(n-5)\times\left[\frac{\alpha_1}{A}\Omega^2(n-3)\left\{k^2+2WZ+2Z^2\right\}+\frac{\tilde\alpha}{A}\Omega^4 2(n-5)\left\{k^2+2WZ\right\}Z\right]\\
&\quad+\frac{\tilde\alpha}{A}e^{-f}\Omega^2(n-5)\times\left[\frac{1}{2}\Omega^2\left\{(n-2)k^2+2WZ-4Z^2\right\}+\frac{1}{A}\frac{4}{n-2}Z\left\{\Lambda+\kappa^2(V_1+V_2)\right\}\right]\\
&\quad-\frac{\tilde\alpha}{A}e^f\Omega^2\frac{4}{(n-2)^2}\left\{\nu_+\vartheta_+(\partial_-\vartheta_-)+\nu_-\vartheta_-(\partial_+\vartheta_+)+(\partial_+\vartheta_+)(\partial_-\vartheta_-)+\nu_+\nu_-\vartheta_+\vartheta_--(\partial_-\vartheta_+)^2\right\} && (11)\\
\partial_+\psi &= \Omega\pi_+ && (12)\\
\partial_+\phi &= \Omega p_+ && (13)\\
\partial_+\pi_+ &= \text{no evolution eq. exists}\\
\partial_+\pi_- &= \left(\frac{1}{n-2}-\frac{1}{2}\right)\Omega\vartheta_+\pi_--\frac{1}{2}\Omega\vartheta_-\pi_+-\frac{1}{2e^f\Omega}\frac{dV_1}{d\psi} && (14)\\
\partial_+p_+ &= \text{no evolution eq. exists}\\
\partial_+p_- &= \left(\frac{1}{n-2}-\frac{1}{2}\right)\Omega\vartheta_+p_--\frac{1}{2}\Omega\vartheta_-p_+-\frac{1}{2e^f\Omega}\frac{dV_2}{d\phi} && (15)
\end{aligned}$$

Field Equations (4)

evolution equations (2)

Equations for x^- direction

$$\begin{aligned}
\partial_-\Omega &= -\frac{1}{n-2}\vartheta_-\Omega^2 && (16)\\
\partial_-\vartheta_+ &= (9) && (17)\\
\partial_-\vartheta_- &= -\vartheta_-\nu_--\frac{1}{\Omega A}\kappa^2T_{--}=-\vartheta_-\nu_--\frac{1}{A}\Omega\kappa^2(\pi_-^2-p_-^2) && (18)\\
\partial_-f &= \nu_- && (19)\\
\partial_-\nu_+ &= (11) && (20)\\
\partial_-\nu_- &= \text{no evolution eq. exists}\\
\partial_-\psi &= \Omega\pi_- && (21)\\
\partial_-\phi &= \Omega p_- && (22)\\
\partial_-\pi_+ &= -\frac{1}{2}\Omega\vartheta_+\pi_-+\left(\frac{1}{n-2}-\frac{1}{2}\right)\Omega\vartheta_-\pi_+-\frac{1}{2e^f\Omega}\frac{dV_1}{d\psi} && (23)\\
\partial_-\pi_- &= \text{no evolution eq. exists}\\
\partial_-p_+ &= -\frac{1}{2}\Omega\vartheta_+p_-+\left(\frac{1}{n-2}-\frac{1}{2}\right)\Omega\vartheta_-p_+-\frac{1}{2e^f\Omega}\frac{dV_2}{d\phi} && (24)\\
\partial_-p_- &= \text{no evolution eq. exists}
\end{aligned}$$

This constitutes the first-order dual-null form, suitable for numerical coding.

Black Hole, Expanding Universe, and Gravitational Wave
--- 100 years of General Relativity

H.Shinkai (2015 Kobunsha)
[Korean ver. (2016 Kachi Pub.)]

Outline & Summary

We numerically investigated how the dynamics depend on the dimensionality and how the higher-order curvature terms affect to singularity formation in two models:
(i) colliding scalar pulses in planar space-time, and (ii) perturbed wormhole in spherical symmetric space-time.
Our numerical code uses dual-null formulation, and we compare the dynamics in 5, 6 and 7-dimensional General Relativity and Gauss-Bonnet (GB) gravity.

(1) For scalar wave collisions, we observe that curvarure evolutions (Kretschmann invariant) are milder in the presence of GB term and/or in higher-dimensional space-time.

(2) For wormhole dynamics, we observe that the perturbed throat will be easily enhance in the presence of GB term.

Both suggest that the thresholds for the singularity formation become higher in higher dimension and/or in presence of GB terms, although it is not evitable.

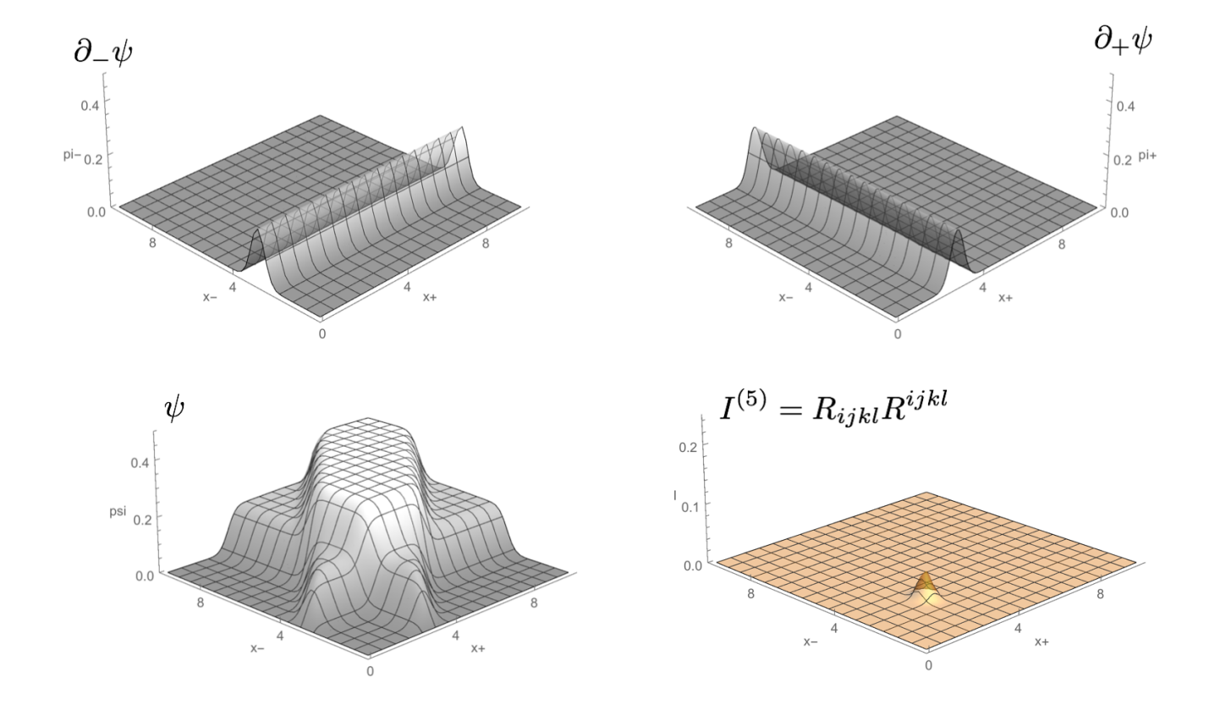
Colliding Scalar Waves

Colliding Scalar Waves

GR 5d: small amplitude waves

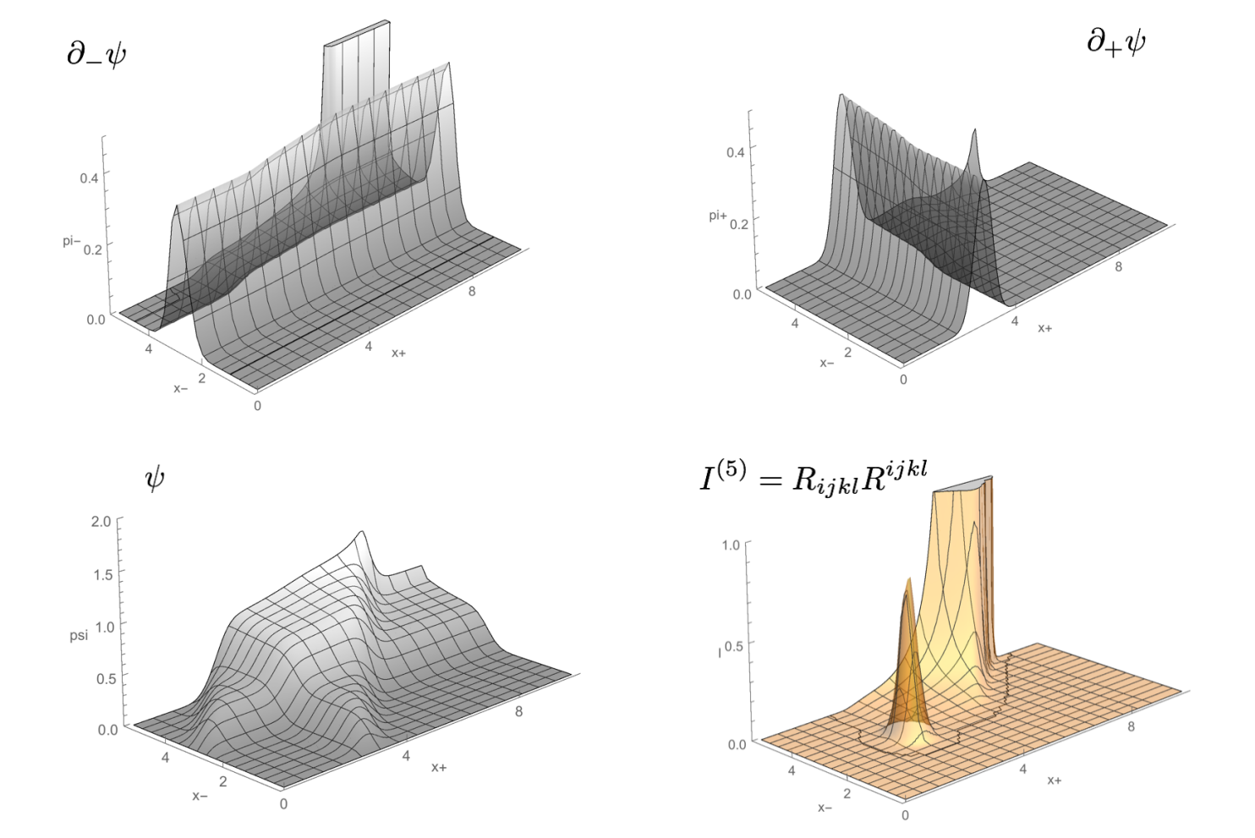
flat background, normal scalar field

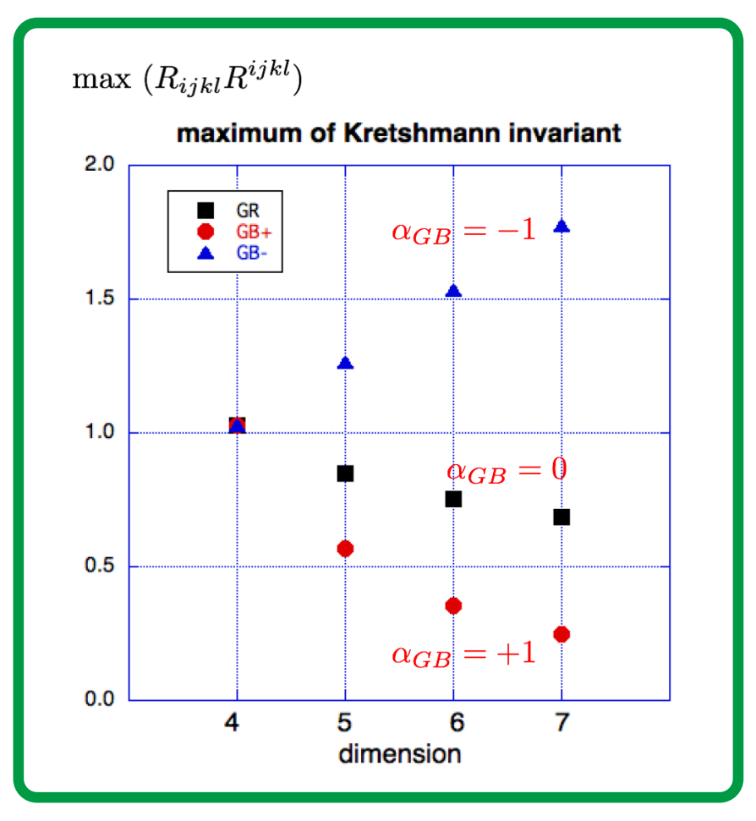
Initial data:
$\psi=0,\ \pi_+=a\exp(-b(z-c)^2)$ on $x_-=0$ surface, where $z=x^+/\sqrt{2}$
$\psi=0,\ \pi_-=a\exp(-b(z-c)^2)$ on $x_+=0$ surface, where $z=x^-/\sqrt{2}$

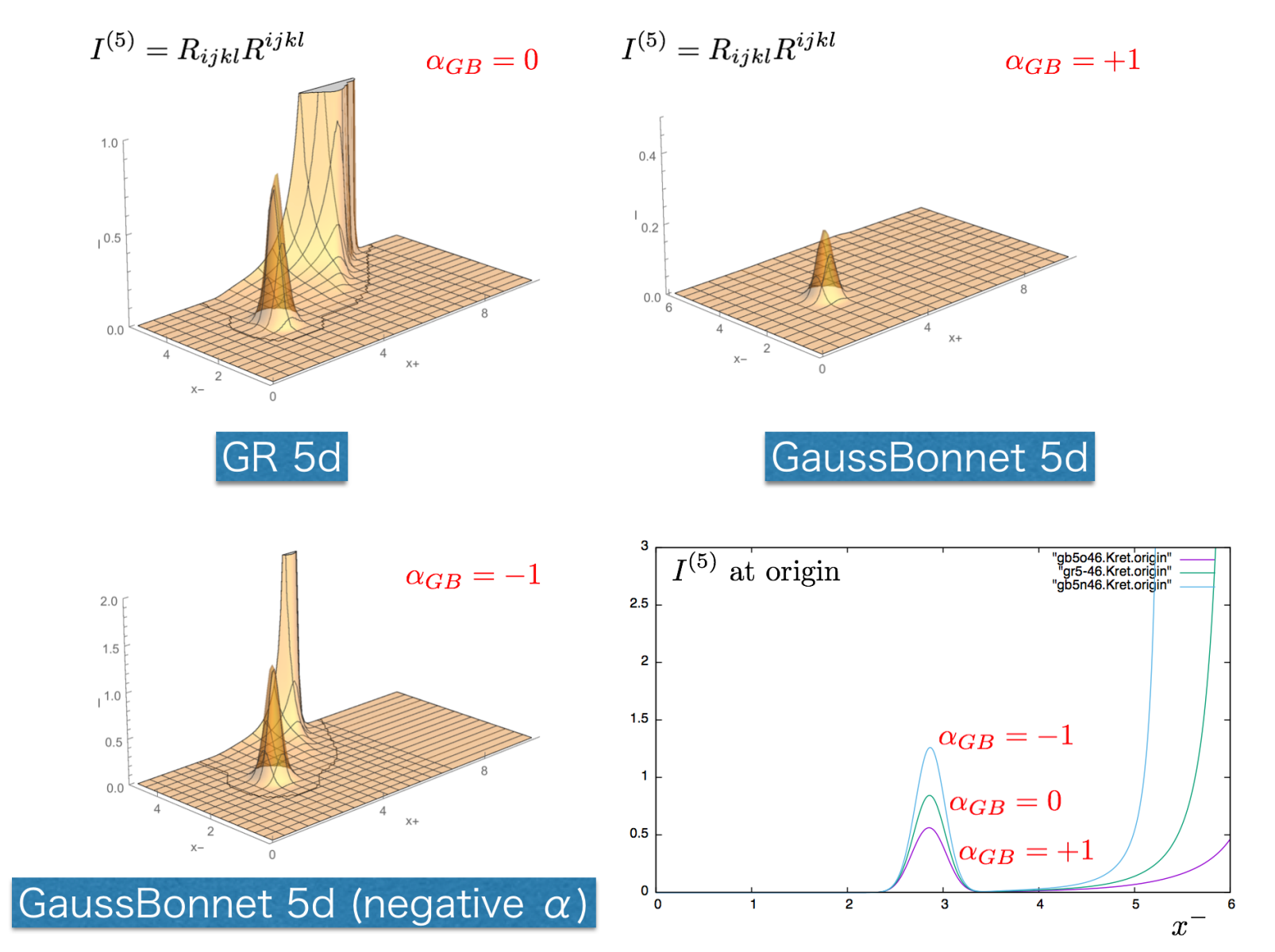


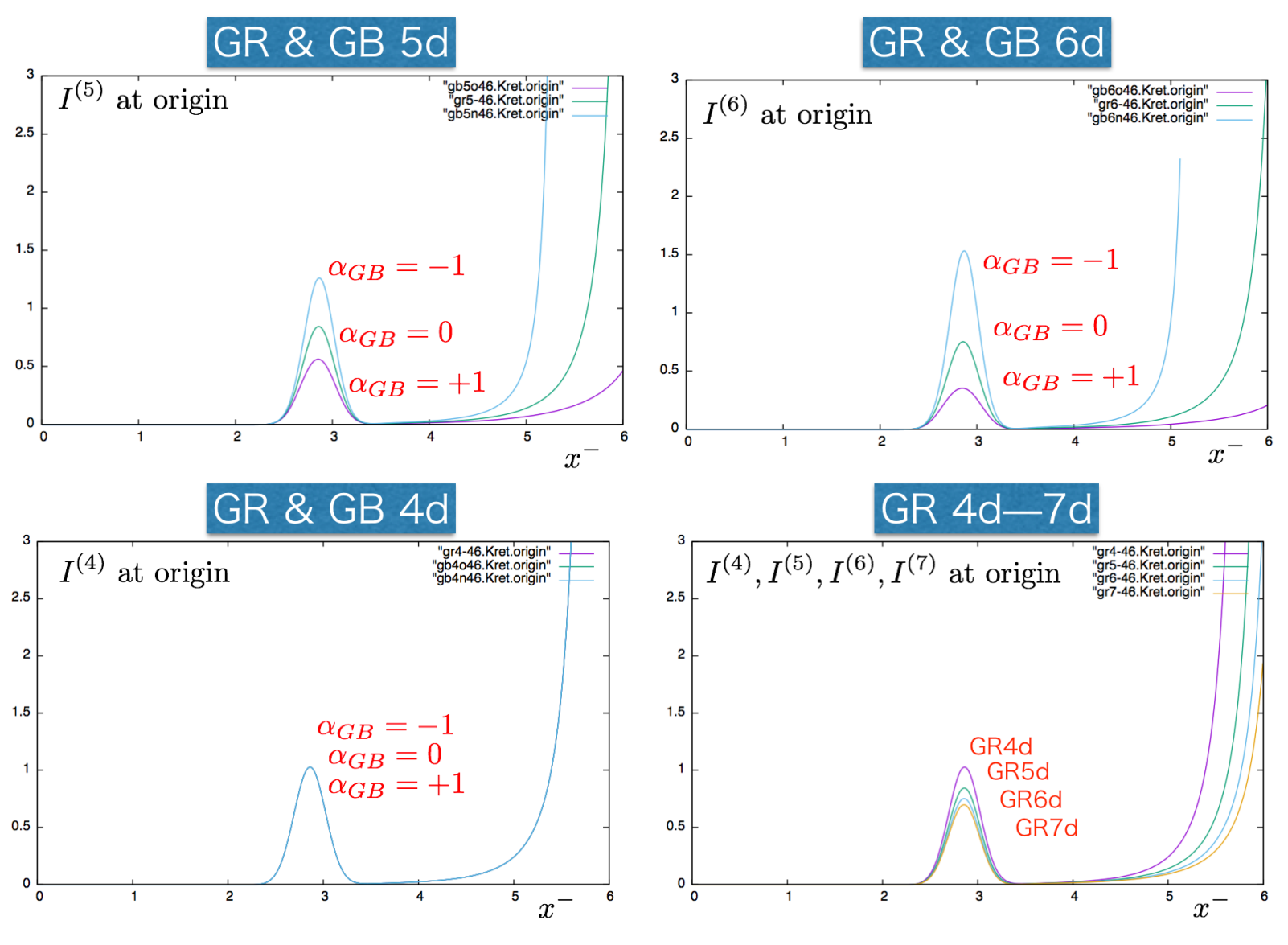
Colliding Scalar Waves

GR 5d: large amplitude waves









Wormhole Evolutions

1. Motivation

Why Wormhole?

They increase our understanding of gravity when the usual energy conditions are not satisfied, due to quantum effects (Casimir effect, Hawking radiation) or alternative gravity theories, brane-world models etc.

They are very similar to black holes --both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH).

Wormhole = Hypersurface foliated by marginally trapped surfaces

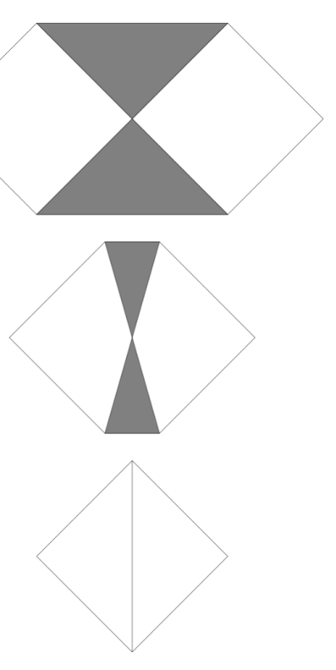
BH & WH are interconvertible?

S.A. Hayward, Int. J. Mod. Phys. D 8 (1999) 373

They are very similar -- both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH)

Only the causal nature of the THs differs, whether THs evolve in plus / minus density which is given locally.

	Black Hole	Wormhole
Locally defined by	Achronal (spatial/null) outer TH ⇝ 1-way traversable	Temporal (timelike) outer THs ⇝ 2-way traversable
Einstein eqs.	Positive energy density normal matter (or vacuum)	Negative energy density "exotic" matter
Appear-ance	occur naturally	Unlikely to occur naturally. but constructible??



initial data

- Static condition

$$\begin{aligned}
(\partial_++\partial_-)\Omega=0 &\implies \vartheta_++\vartheta_-=0\\
(\partial_++\partial_-)\psi=0 &\implies \pi_++\pi_-=0\\
(\partial_++\partial_-)\phi=0 &\implies p_++p_-=0\\
\left.\begin{aligned}(\partial_++\partial_-)\vartheta_+=0\\(\partial_++\partial_-)\vartheta_-=0\end{aligned}\right\} &\implies \vartheta_+\nu_++\frac{1}{A}\Omega\kappa^2(\pi_+^2-p_+^2)=\vartheta_-\nu_-+\frac{1}{A}\Omega\kappa^2(\pi_-^2-p_-^2)
\end{aligned}$$

- Solve x^+ and x^- equations with the starting condition at the throat

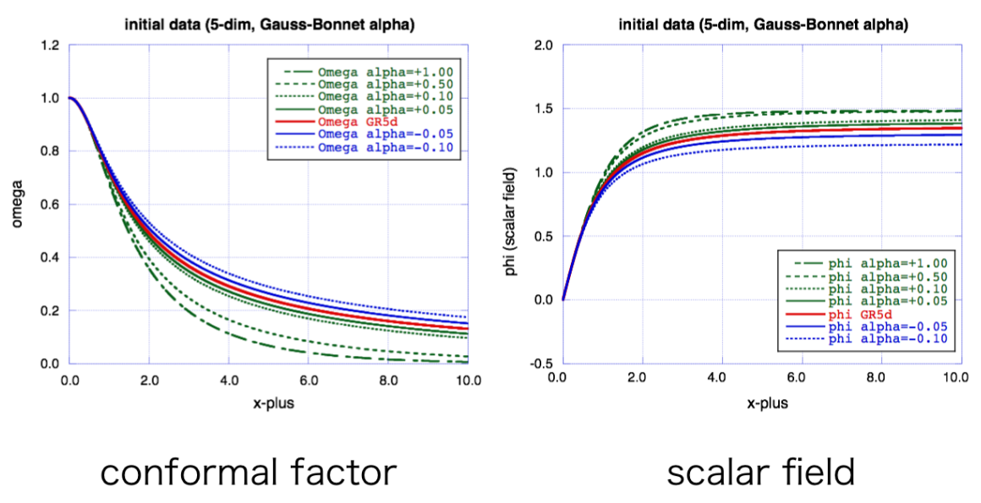
$$\begin{aligned}
&\vartheta_+=\vartheta_-(=0)\\
&\nu_+=\nu_-(=0)\\
&-\kappa^2\Omega(\pi_+^2-p_+^2)e^f=-\frac{1}{\Omega}\left[-\alpha_1\Omega^2\frac{(n-2)(n-3)}{2}k+\Lambda+\kappa^2(V_1+V_2)\right]+\tilde\alpha\Omega^3\frac{(n-2)(n-5)}{2}k^2
\end{aligned}$$

If we assume only ghost field ϕ, then

$$p_+=-p_-=\sqrt{\frac{1}{\kappa^2e^f}\left[\alpha_1\frac{(n-2)(n-3)}{2}k-\frac{1}{\Omega^2}(\Lambda+\kappa^2V_2)+\tilde\alpha\Omega^2\frac{(n-2)(n-5)}{2}k^2\right]}$$

- add perturbation

$$p_+(x^+=x,x^-=0)=p_+(\text{solution})+a\exp[-100(x-0.5)^2]$$

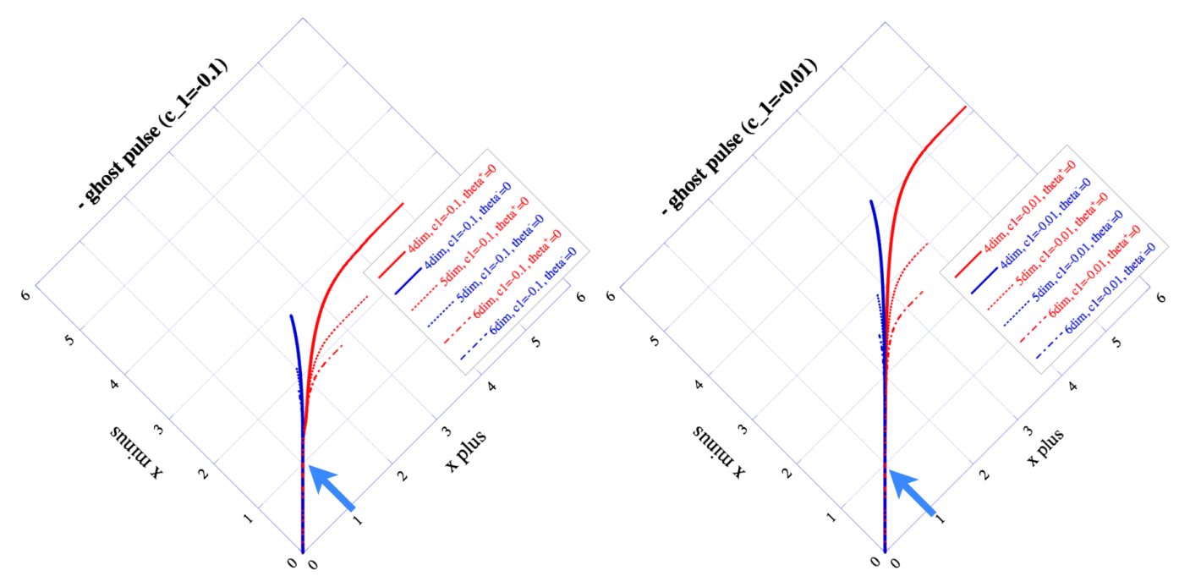


conformal factor　　scalar field

in GR

4d 5d 6d GR

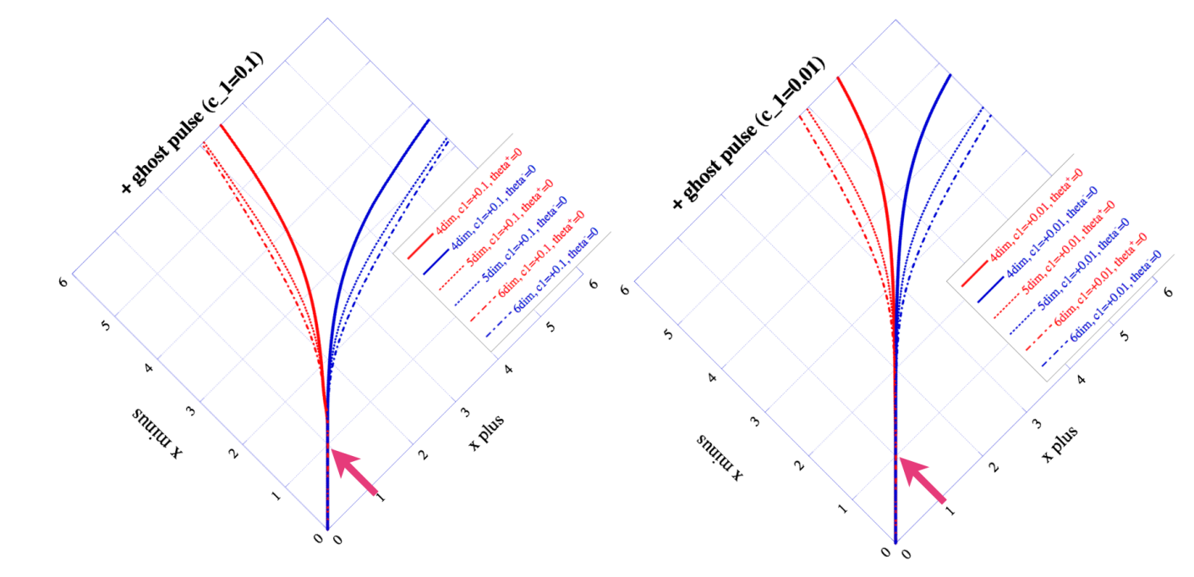
ghost pulse (negative amp.) input



positive energy input --> BH formation

4d 5d 6d GR

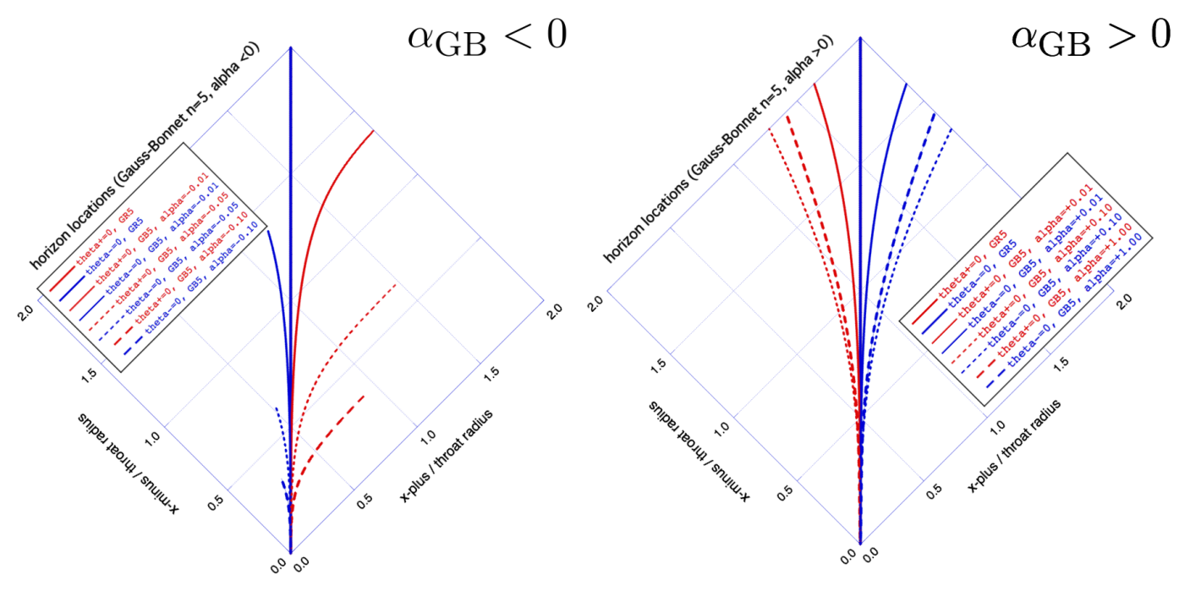
ghost pulse (positive amp.) input



negative energy input --> throat inflates

in GB

5d GR vs Gauss-Bonnet　instability appears

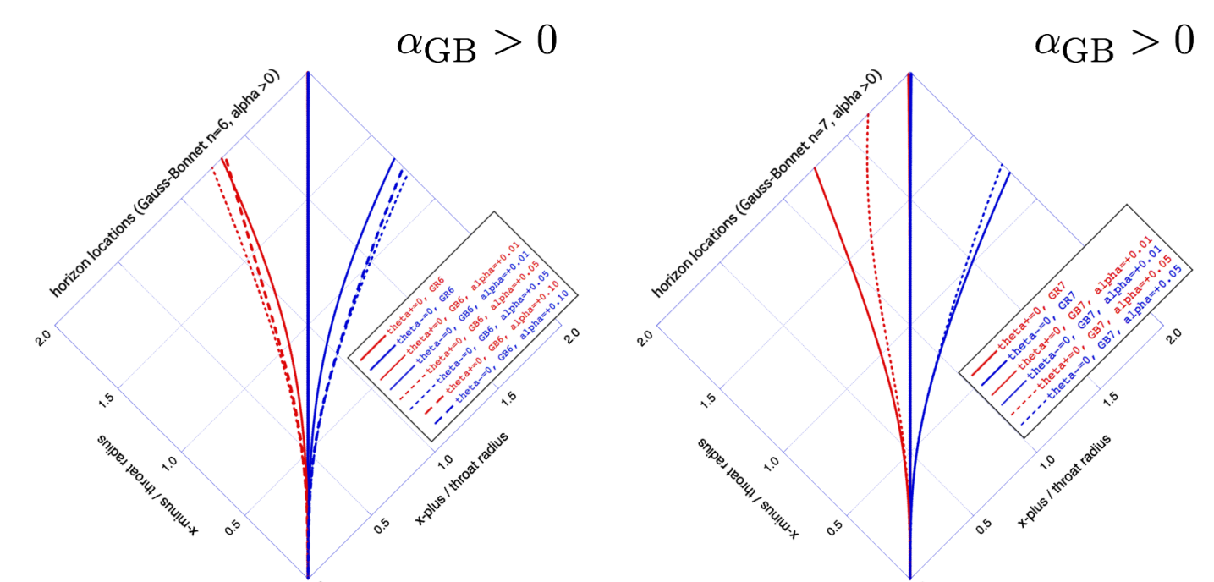


BH formation　　throat inflates

$$S=\int_{\mathcal{M}} d^{N+1}x\sqrt{-g}\Big[\frac{1}{2\kappa^2}\{\alpha_1\mathcal{R}+\alpha_2\mathcal{L}_{\rm GB}\}+\mathcal{L}_{\rm matter}\Big]$$

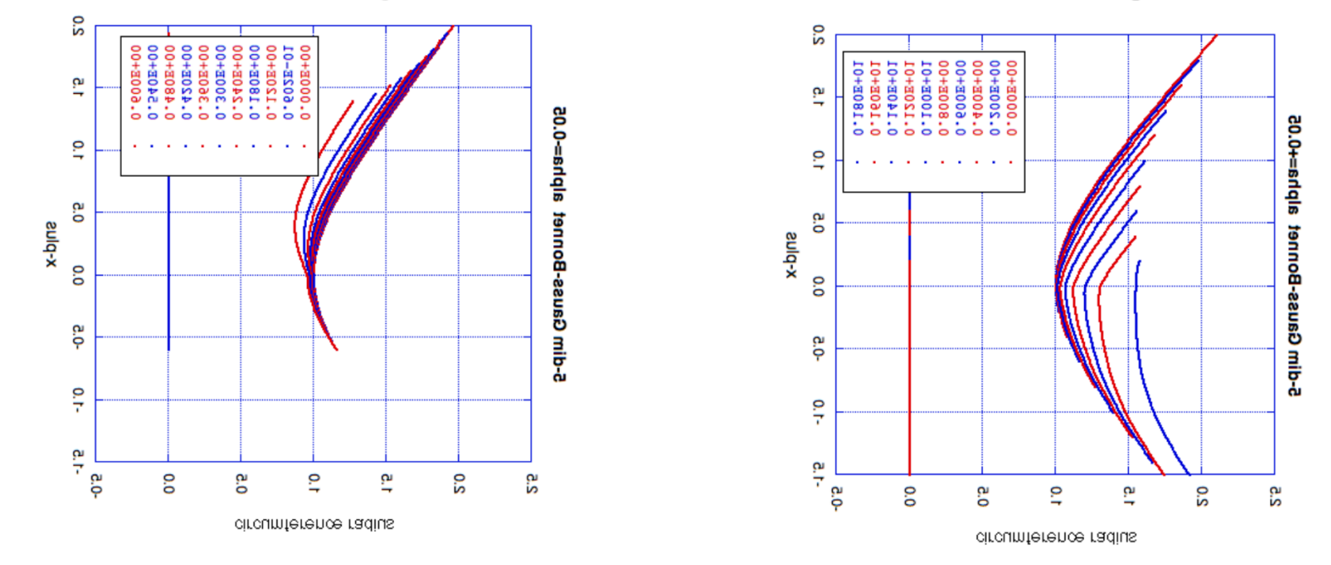
where $\mathcal{L}_{GB}=\mathcal{R}^2-4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$

6d 7d Gauss-Bonnet　instability appears easily



6 dim.　　7 dim.

wormhole configurations (5dim. GaussBonnet)



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