一般相対論の数値計算手法

近畿大セミナー
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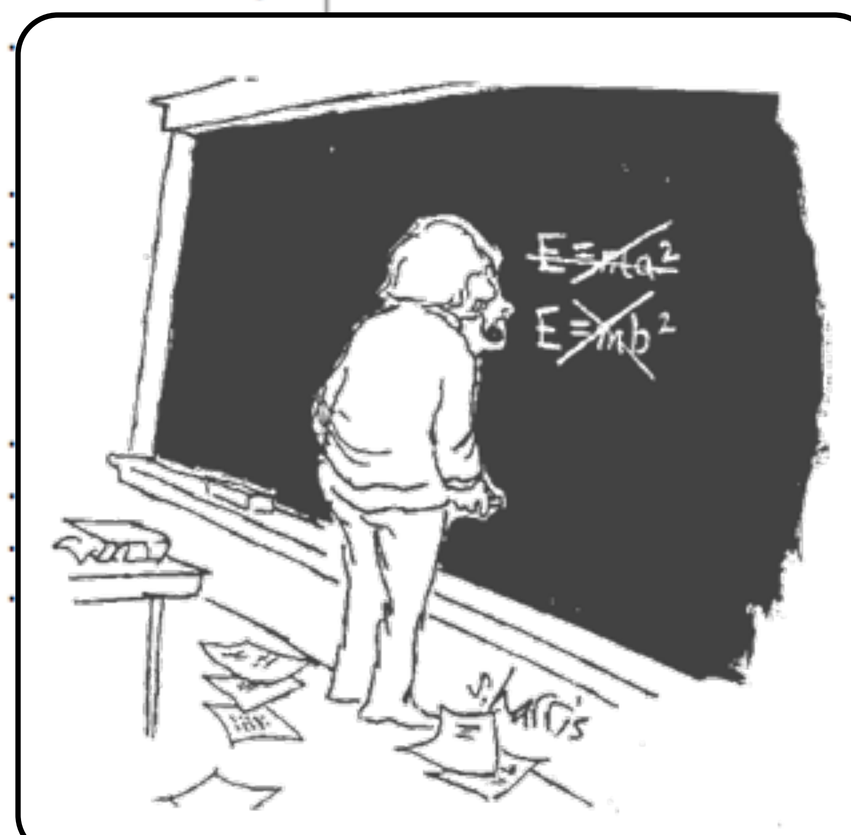


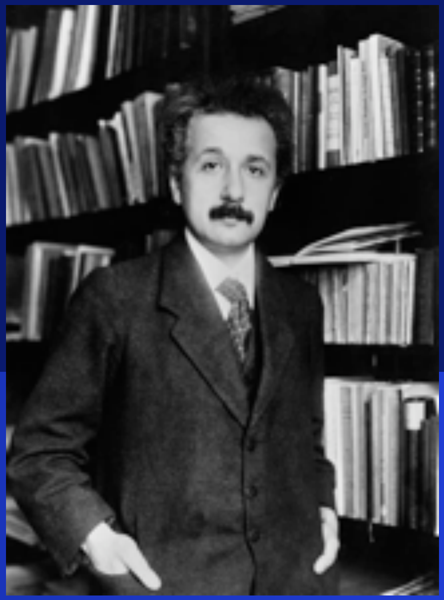
1.　Introduction

1.1　　一般相対性理論の概略と研究テーマ

1.2　　なぜ数値相対論？

1.3　　数値相対論の方法論の概略

一般相対性理論



重力場の方程式（1916）

空間の曲がりがモノの運動を決める ⇔モノがあると空間が曲がる

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

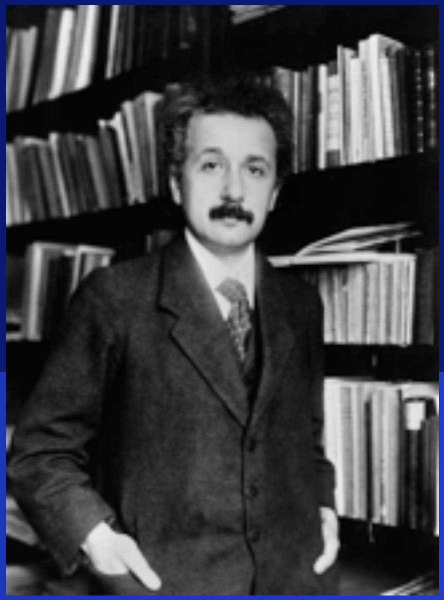
アインシュタイン曲率テンソル
＜空間の歪み＞

エネルギー運動量テンソル
＜モノの分布＞

$$\begin{aligned}
\Gamma^{\alpha}_{\mu\nu} &\equiv \frac{1}{2} g^{\alpha\beta}(\partial_\nu g_{\beta\mu} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}) \\
R^{\mu}{}_{\nu\alpha\beta} &\equiv \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\nu\alpha} \\
R_{ab} \equiv R^{\mu}{}_{a\mu b} &\equiv \partial_\mu \Gamma^{\mu}_{ab} - \partial_b \Gamma^{\mu}_{a\mu} + \Gamma^{\mu}{}_{\nu\mu}\Gamma^{\nu}{}_{ab} - \Gamma^{\mu}{}_{\nu b}\Gamma^{\nu}{}_{a\mu} \\
R &= g^{ab} R_{ab}
\end{aligned}$$

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$$

一般相対性理論



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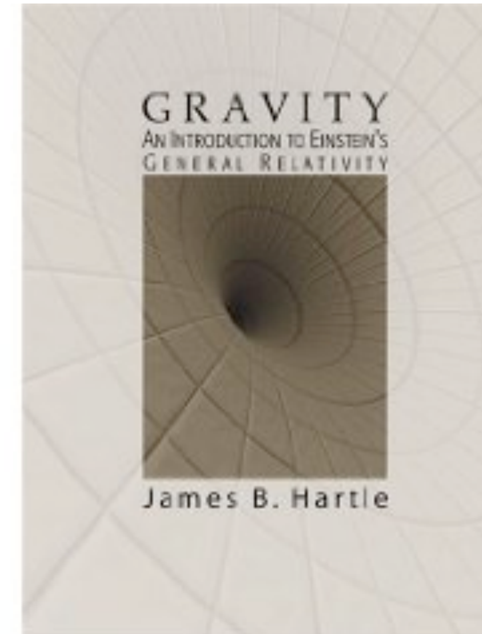
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

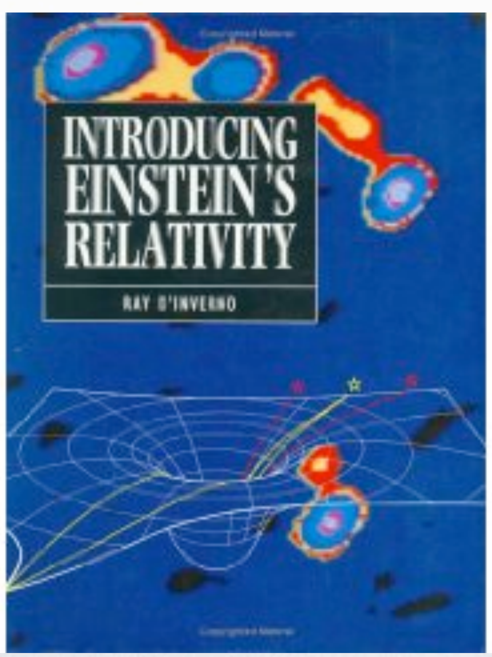
⇒定常的な宇宙モデルをつくるために，方程式を修正 (1917)

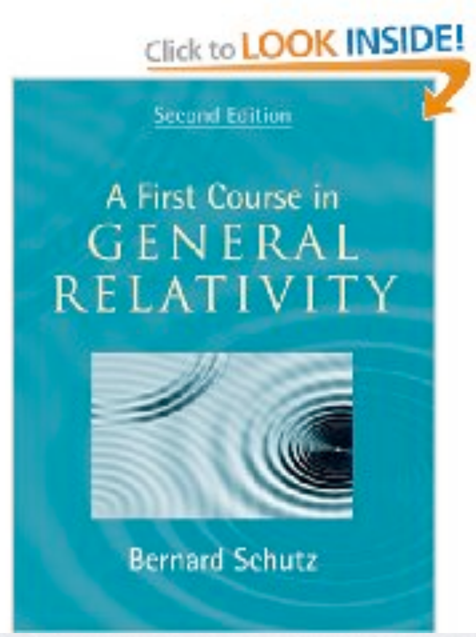
（宇宙項，cosmological constant）

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

相対論の教科書（入門書レベル）

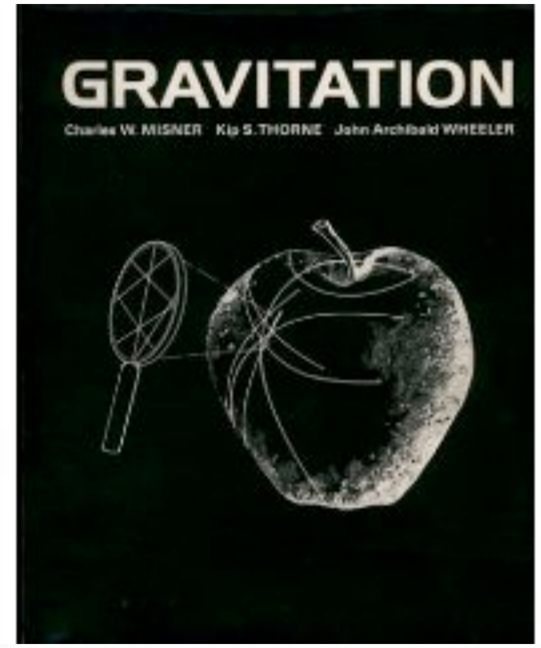


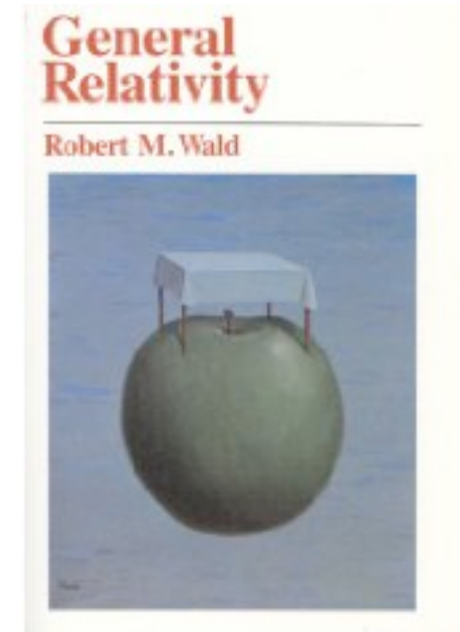


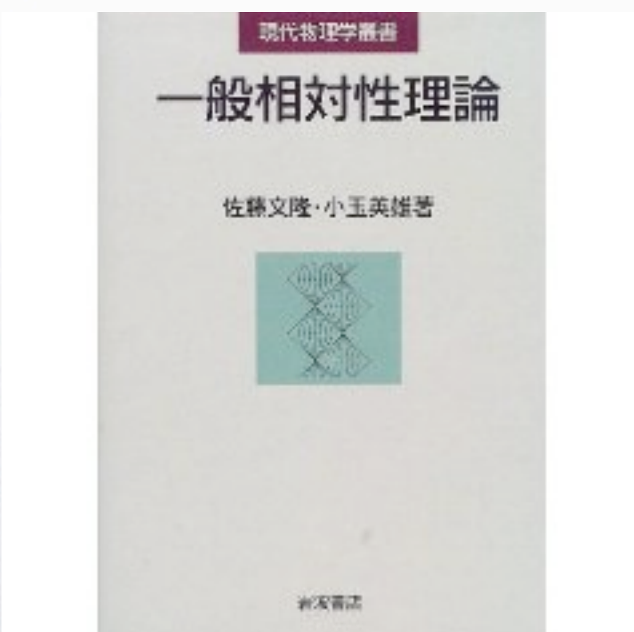


★弱い重力場での検証
★ブラックホール
★コンパクト星
★重力波
★宇宙論

相対論の教科書（本格的に学ぶ）

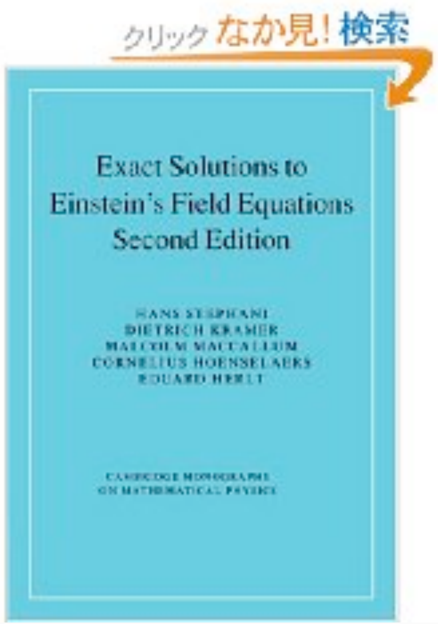


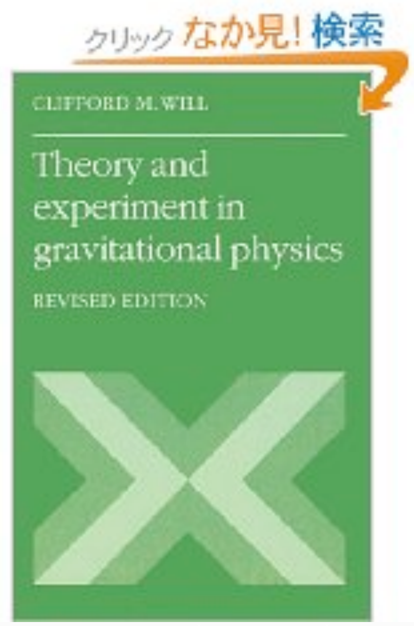


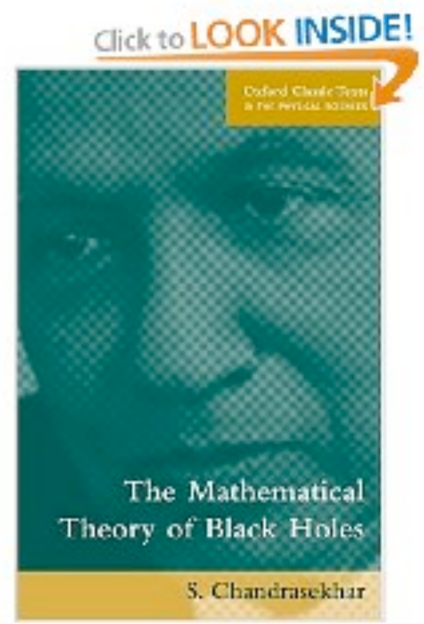


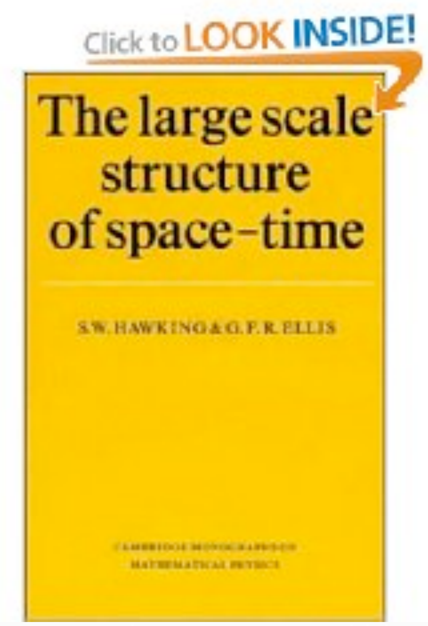
- ★時空の対称性，一様時空
- ★ブラックホール，因果構造
- ★特異点，大域構造
- ★時空の動力学
- ★スピノール
- ★量子効果

相対論の教科書（トピックを学ぶ）

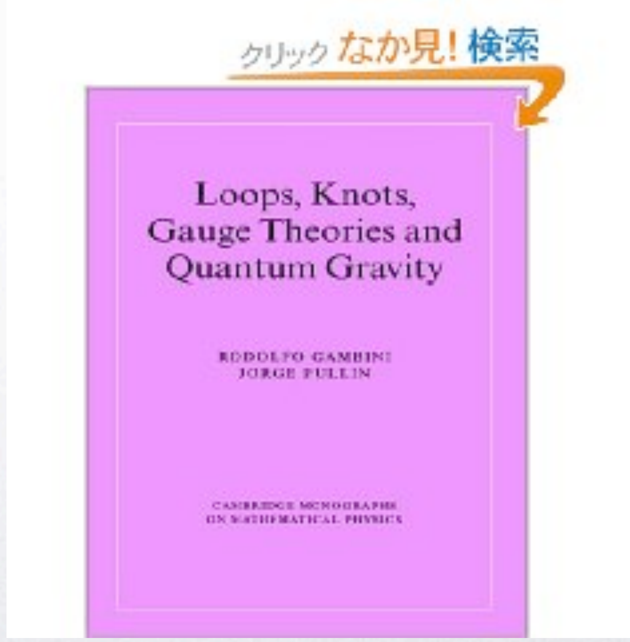






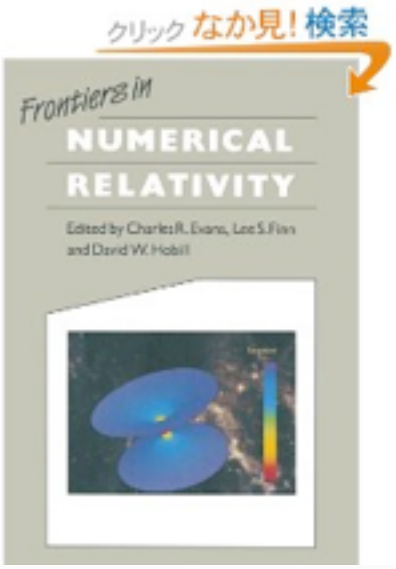


- ★厳密解
- ★重力理論の検証
- ★BH摂動
- ★特異点定理
- ★ループ重力．．．．．

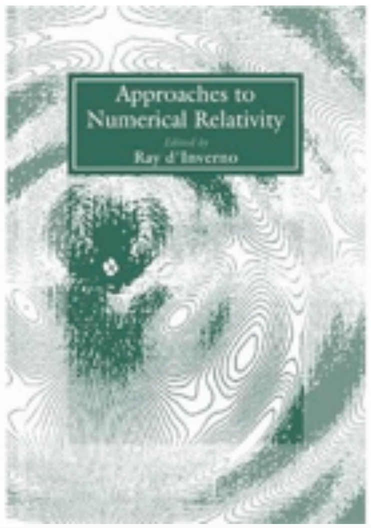


数値相対論の教科書

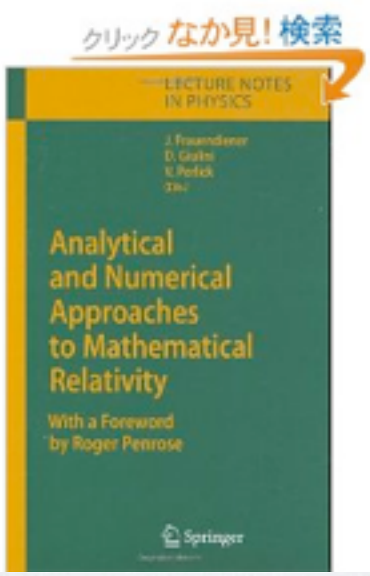
1989



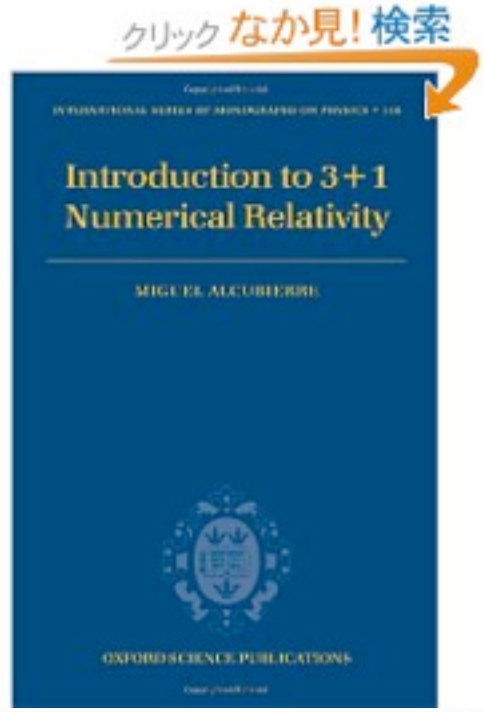
1992



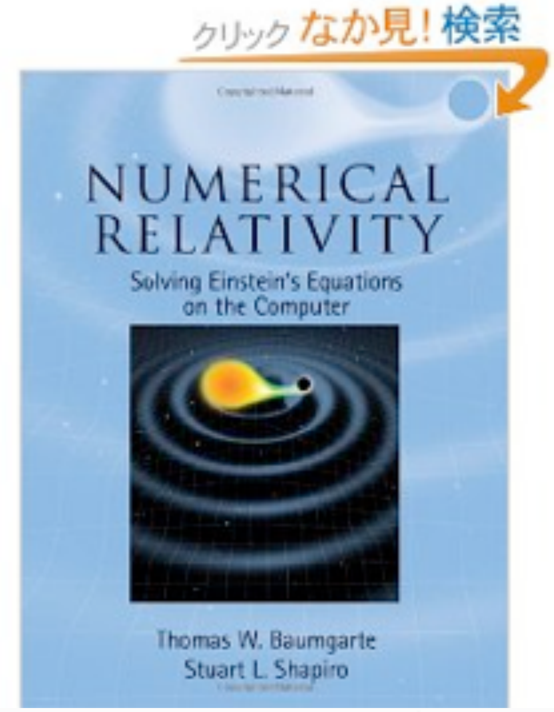
1998



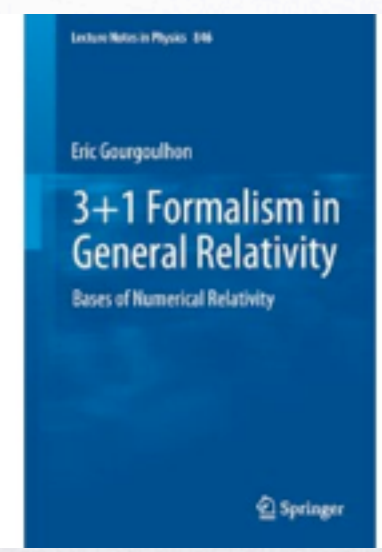
2006



2008



2010



2012

一般相対論の研究分野

http://relativity.livingreviews.org/

MAX-PLANCK-GESELLSCHAFT

Living Reviews in Relativity

Max Planck Institute for Gravitational Physics
Albert Einstein Institute

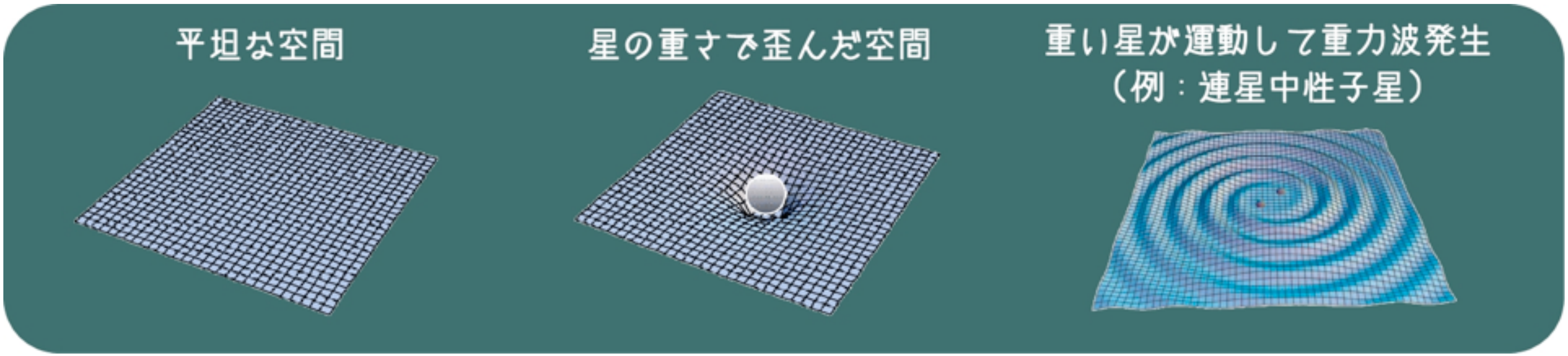
一般相対論研究者向けに，レビュー論文を更新しているサイト　2011/12/1現在.

宇宙物理現象
科学史
弦理論
数学的な側面
数値シミュレーション
重力波
量子重力
宇宙論
実験的検証

1	重力波 (Gravitational Waves)	12 本+12
2	数値シミュレーション (Numerical Relativity)	10 本+9
3	数学的な側面 (Mathematical Relativity)	7 本+5
4	量子重力 (Quantum General Relativity)	11 本+4
5	実験的検証 (Experimental Foundations of Gravitation)	10 本+5
6	宇宙物理現象 (Relativity in Astrophysics)	9 本+5
7	弦理論 (String Theory and Gravitation)	4 本+3
8	宇宙論 (Physical Cosmology)	5 本+2
9	科学史 (History of Relativity)	2 本+3

重力波とは

gravitational wave



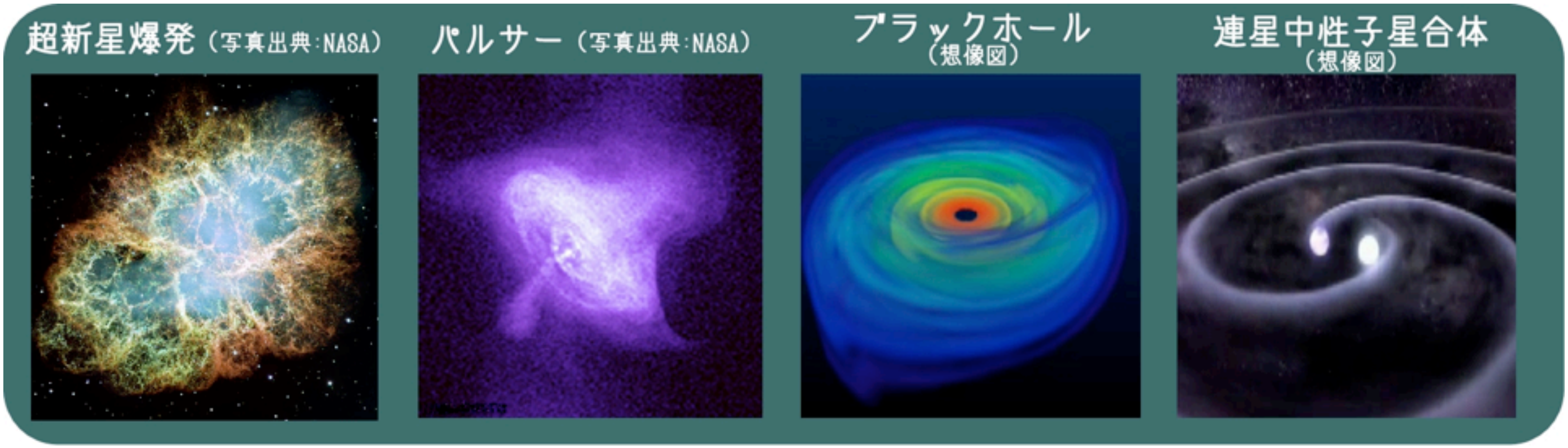
http://gwcenter.icrr.u-tokyo.ac.jp/

★ 一般相対性理論では，強い重力があると，時空が歪む

★ 強い重力源が激しい運動をすると，時空の歪みが波となって伝わる.

重力波の波源

sources of gravitational wave



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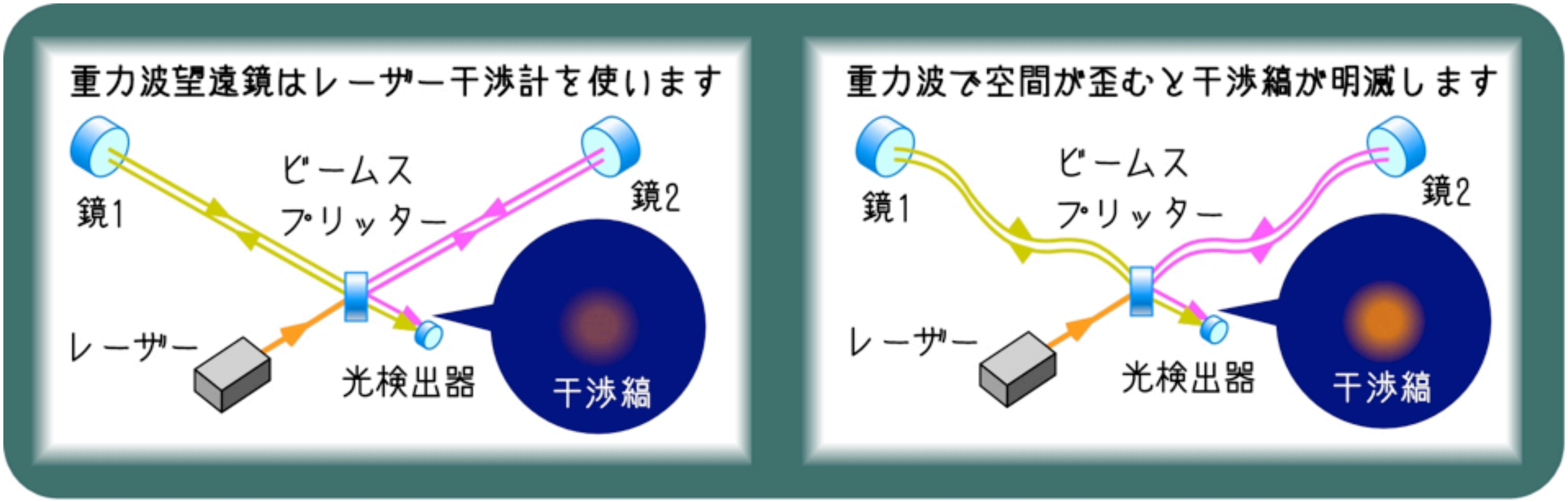
重力波の検出方法

detection method of gravitational wave



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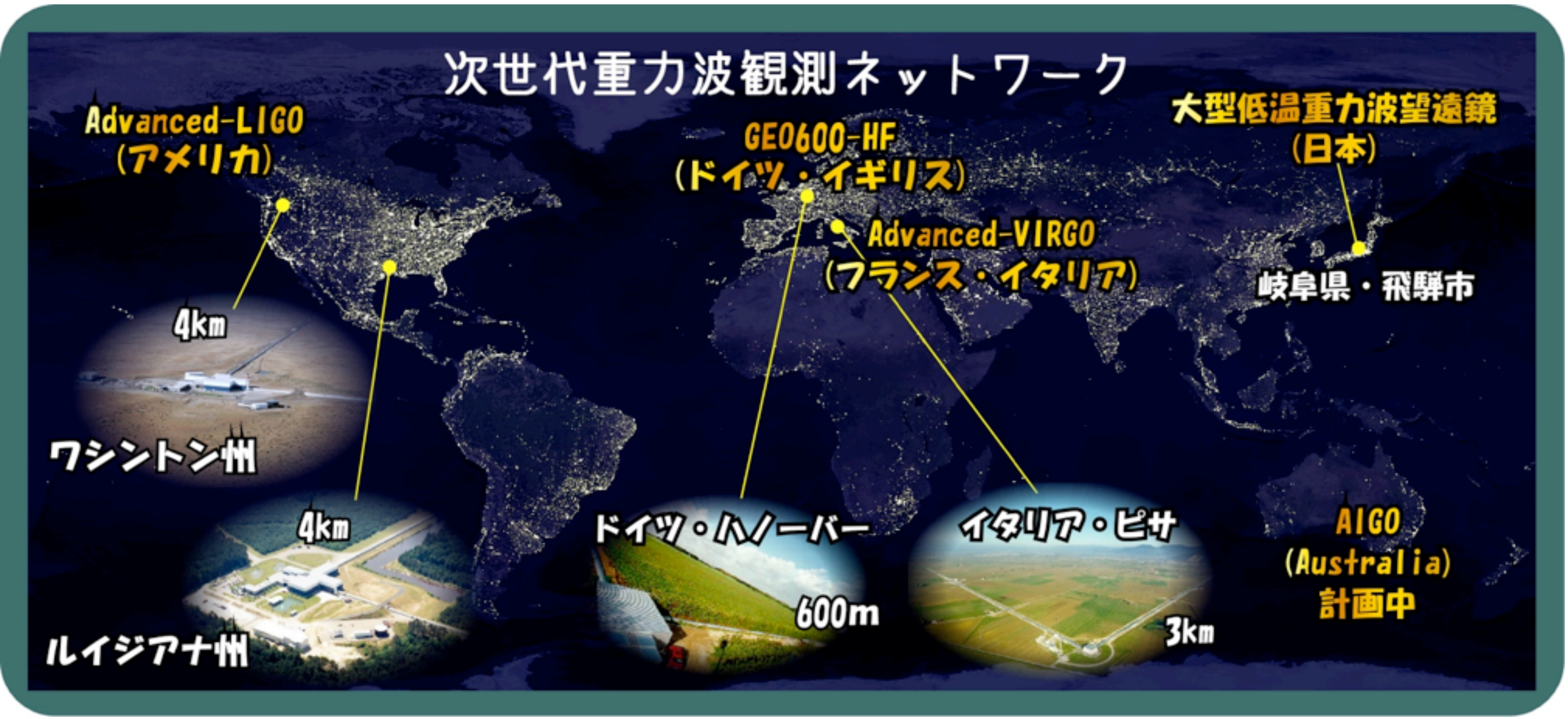
重力波の検出方法



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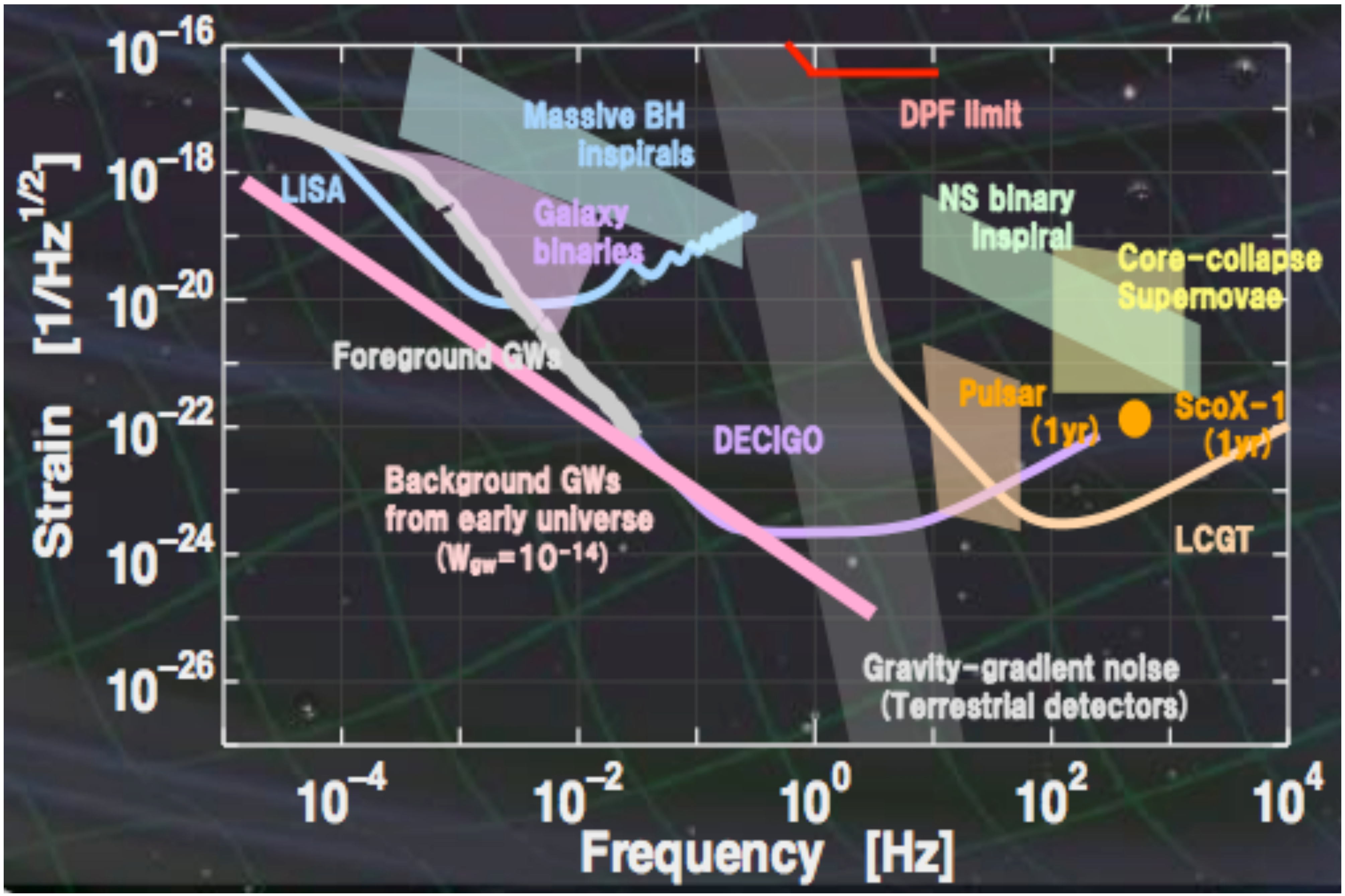
重力波干渉計の感度

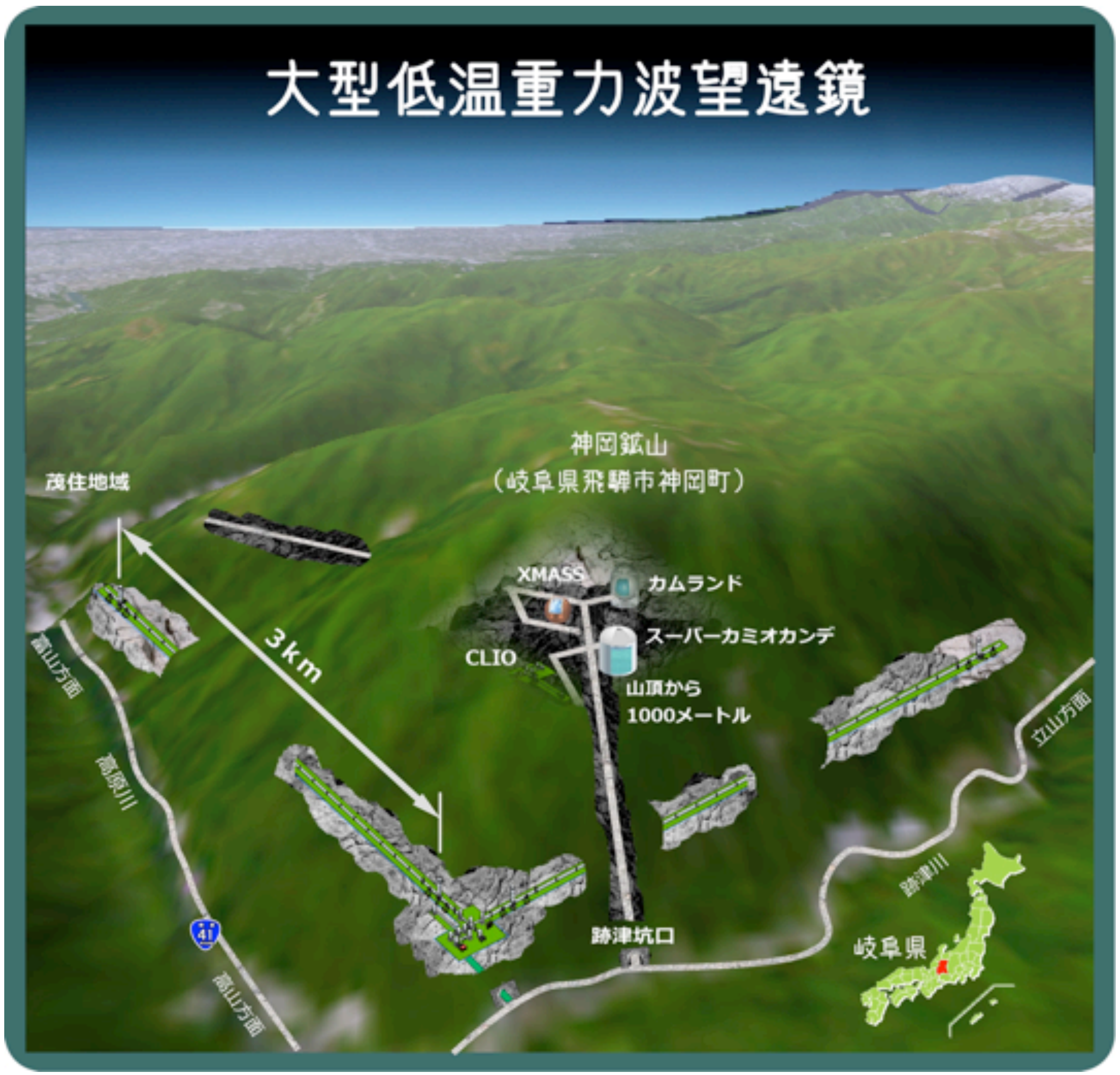
detection method of gravitational wave



by Masaki Ando

LCGT 建設進む日本の重力波干渉計

Large-scale Cryogenic Gravitational-wave Telescope



望遠鏡の大きさ：基線長 3km

望遠鏡を神岡鉱山内に建設

地面振動が小さい岐阜県飛騨市にある神岡鉱山

鏡をマイナス250度（20ケルビン）まで冷却

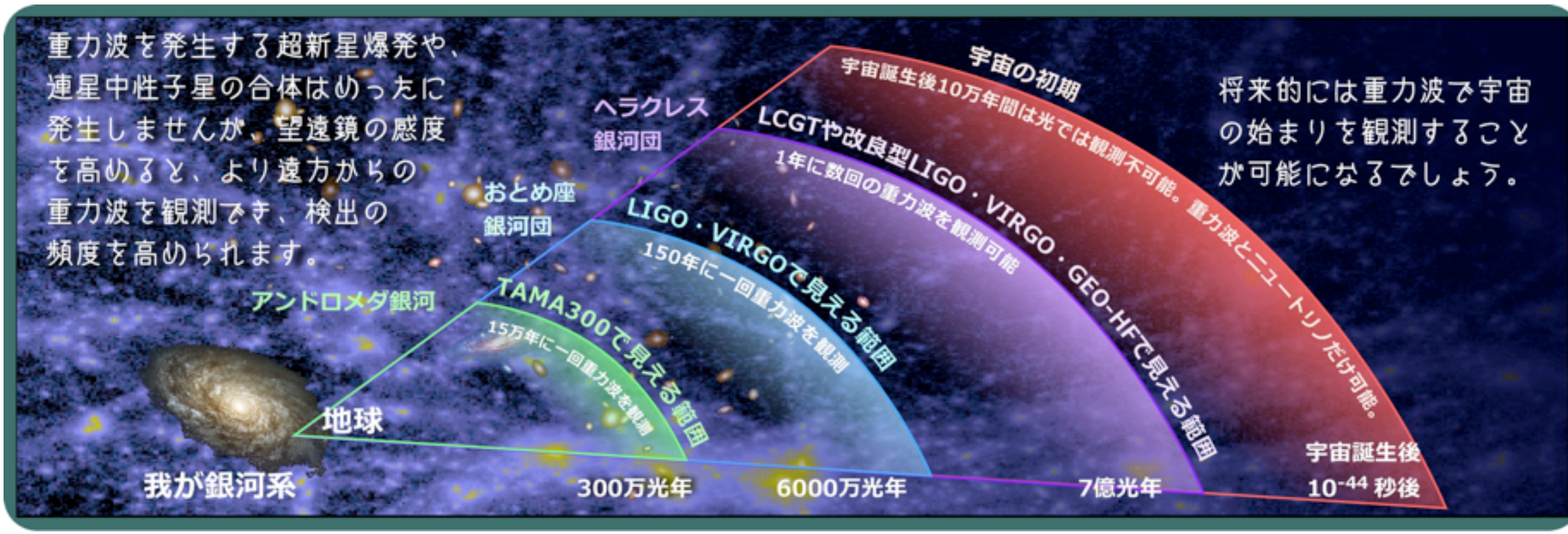
熱雑音を小さくするため

鏡の材質としてサファイアを用いる

光学特性に優れ、低温に冷却すると熱伝導や
機械的損失が少なくなる

http://tamago.mtk.nao.ac.jp/spacetime/lcgt_j.html

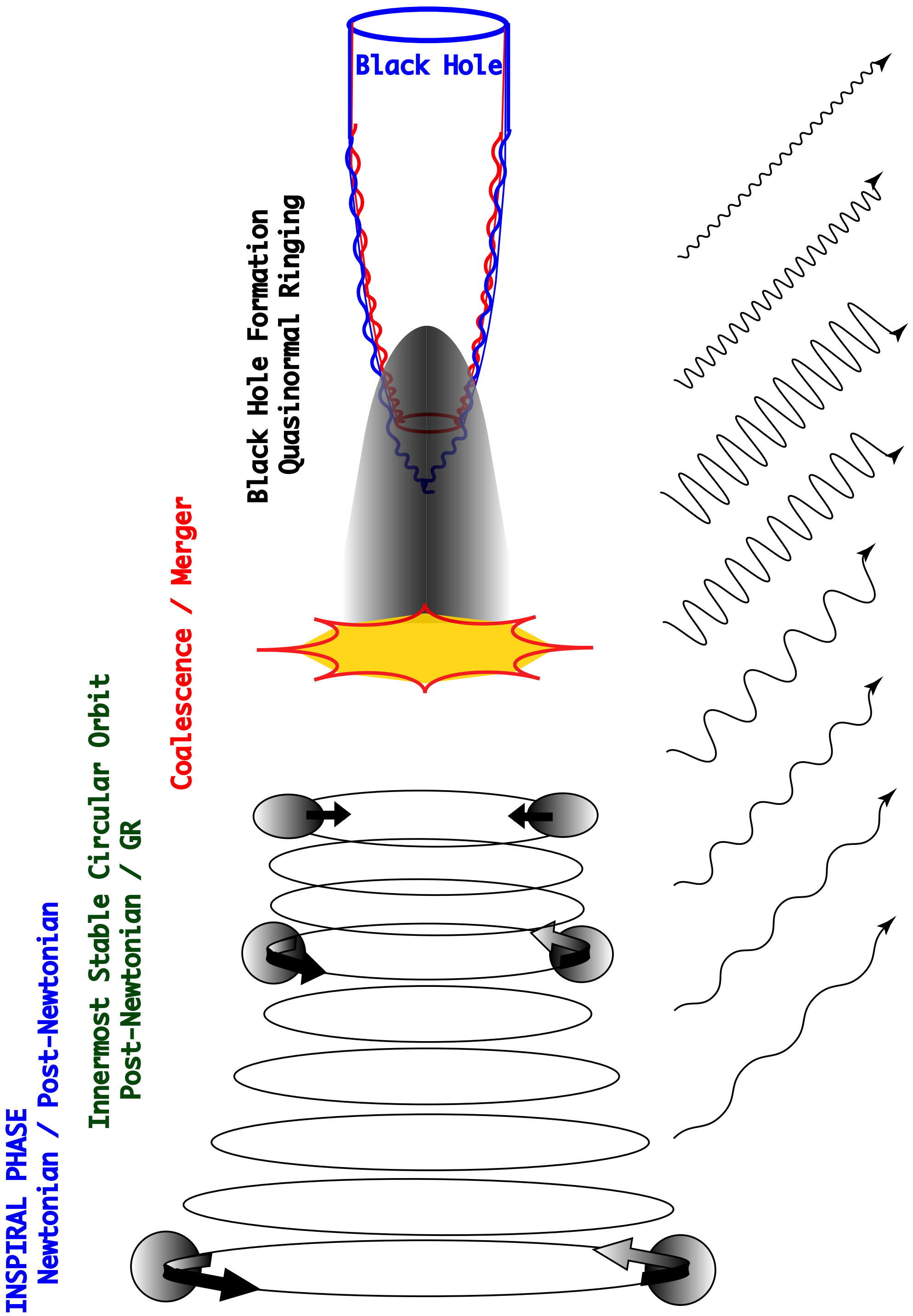
重力波でどこまで宇宙が見えるか

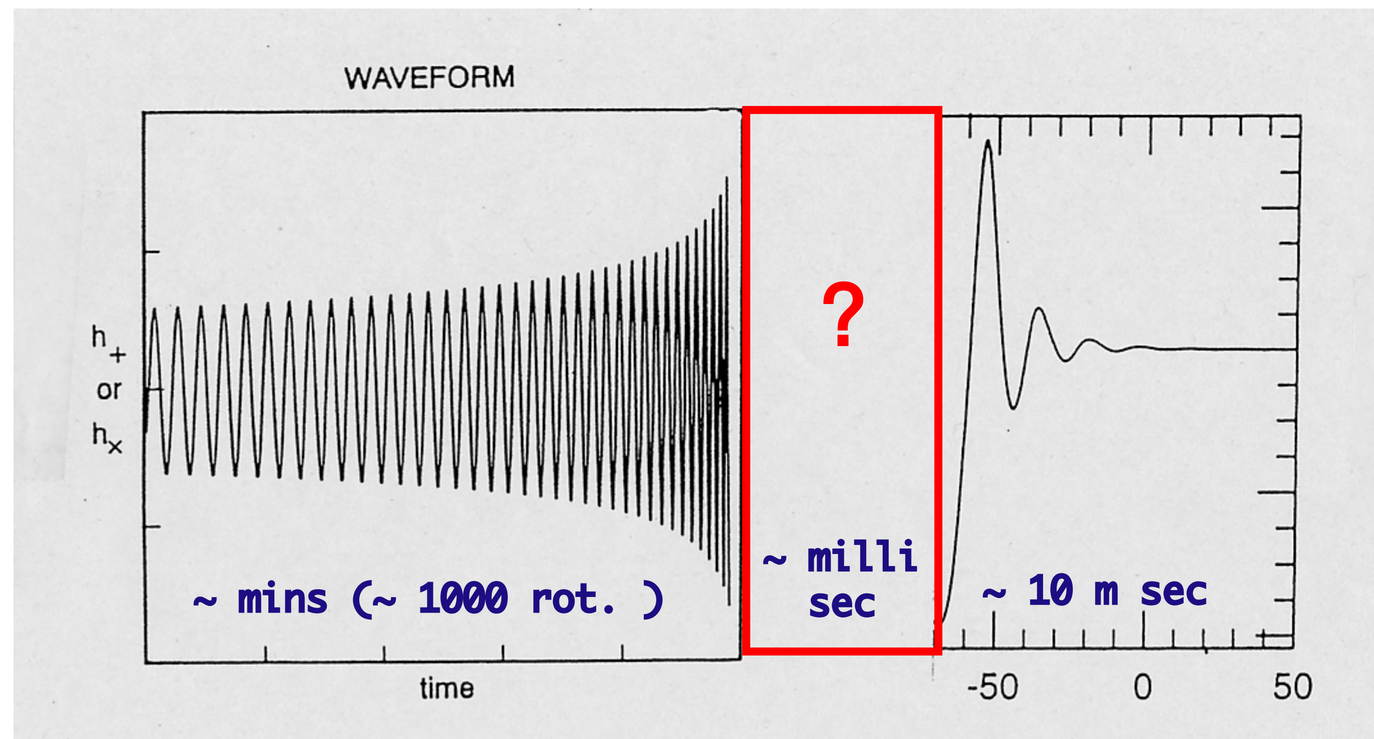


http://gwcenter.icrr.u-tokyo.ac.jp/

重力波の検出には波形の予測が不可欠

最も有望な重力波源は,
連星中性子星や連星BHの合体





INSPIRAL → COALESCE → BLACKHOLE FORMATION

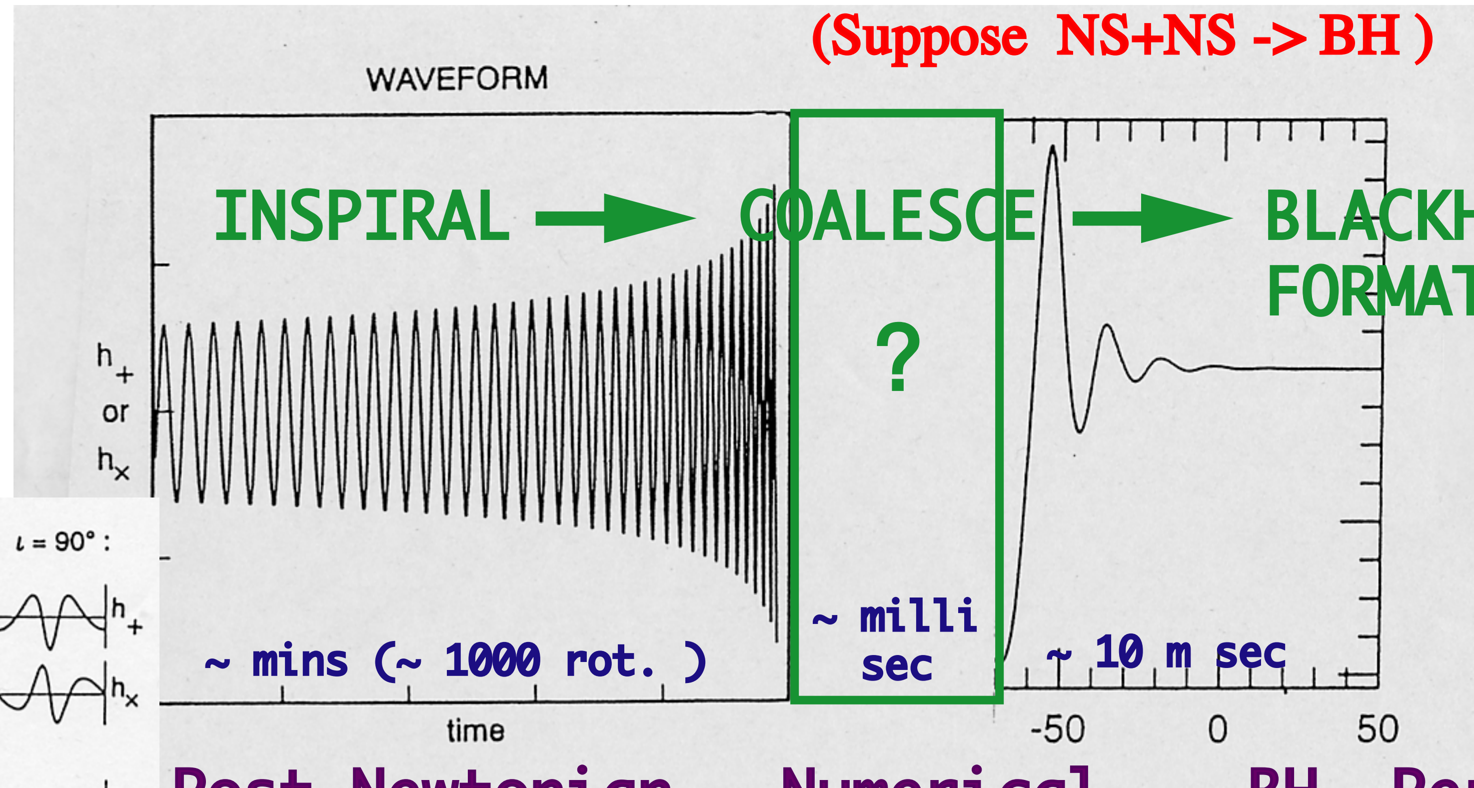
Innermost Stable
Circular Orbit?

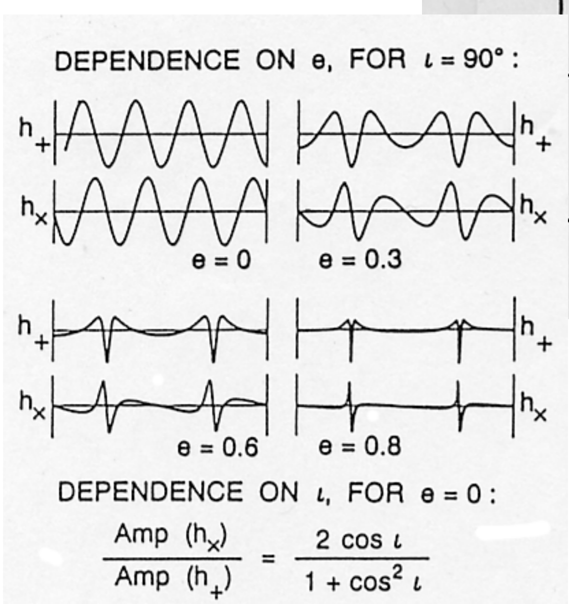
Post Newtonian Approx.
→ Numerical Relativity
→ BH. Perturbation

重力波の検出で何がわかるのか？

What can we learn from gravitational waveform?

(Suppose NS+NS -> BH)





Post Newtonian Approx. | Numerical Relativity | BH. Perturbation

ISCO freq => EoS of NS,
waveform => Formation of BH or NS,
BH mass,
BH angular momentum,...

"chirps" df/dt => chirp mass, $M_c = (M_1 M_2)^{3/5} / (M_1+M_2)^{1/5}$
amplitude up => M_c, distance
amplitude $h_+/h_\times$ => inclination
waveform => eccentricity
moduration => spin, ...

statistics => cosmological parameters

Einstein方程式の厳密解は4000以上

The Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Chandrasekhar says ...

"Einstein equations are easy to solve. Look at the *Exact Solutions* book. There are more than 400 solutions. "

Exact Solutions book says ...

1st Edition (1980): "... checked 2000 references, ..., there are now over 100 papers on exact solutions every year, ..."

2nd Edition (2003): "... we looked at 4000 new papers published during 1980-1999, ... "

D. Kramer, et al, Exact Solutions to Einstein's Field Equations, (Cambridge, 1980)

H. Stephani, et al, Exact Solutions to Einstein's Field Equations, (Cambridge, 2003)

ただし，ダイナミクスを表現する解は，ほとんど未知

1.2　なぜ数値相対論？

Why don't we solve it using computers?

- dynamical behavior, no symmetry in space, ...
- strong gravitational field, gravitational wave! ...
- any dimension, any theories, ...

数値的に解かねばならないから！

Numerical Relativity

= Solve the Einstein equations numerically.

= Necessary for unveiling the nature of strong gravity.

For example:

- gravitational waves from colliding black holes, neutron stars, supernovae, ...
- relativistic phenomena like cosmology, active galactic nuclei, ...
- mathematical feedback to singularity, exact solutions, chaotic behavior, ...
- laboratory for gravitational theories, higher-dimensional models, ...

The most robust way to study the strong gravitational field. Great.

数値相対論のどこが難しい？

The Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

What are the difficulties?

- for 10-component metric, highly nonlinear partial differential equations.
  mixed with 4 elliptic eqs and 6 dynamical eqs if we apply 3+1 decomposition.
- completely free to choose cooordinates, gauge conditions, and even for decomposition of the space-time.
- has singularity in its nature.

簡単に解けない！

How to solve it?

1.3 数値相対論の方法論概略

Numerical Relativity – open issues

どのように時空をとらえるか

0. How to foliate space-time

Cauchy $(3+1)$, Hyperboloidal $(3+1)$, characteristic $(2+2)$, or combined?

$\Rightarrow$ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data

Theoretical:	Proper formulation for solving constraints? How to prepare realistic initial data? Effects of background gravitational waves? Connection to the post-Newtonian approximation?
Numerical:	Techniques for solving coupled elliptic equations? Appropriate boundary conditions?

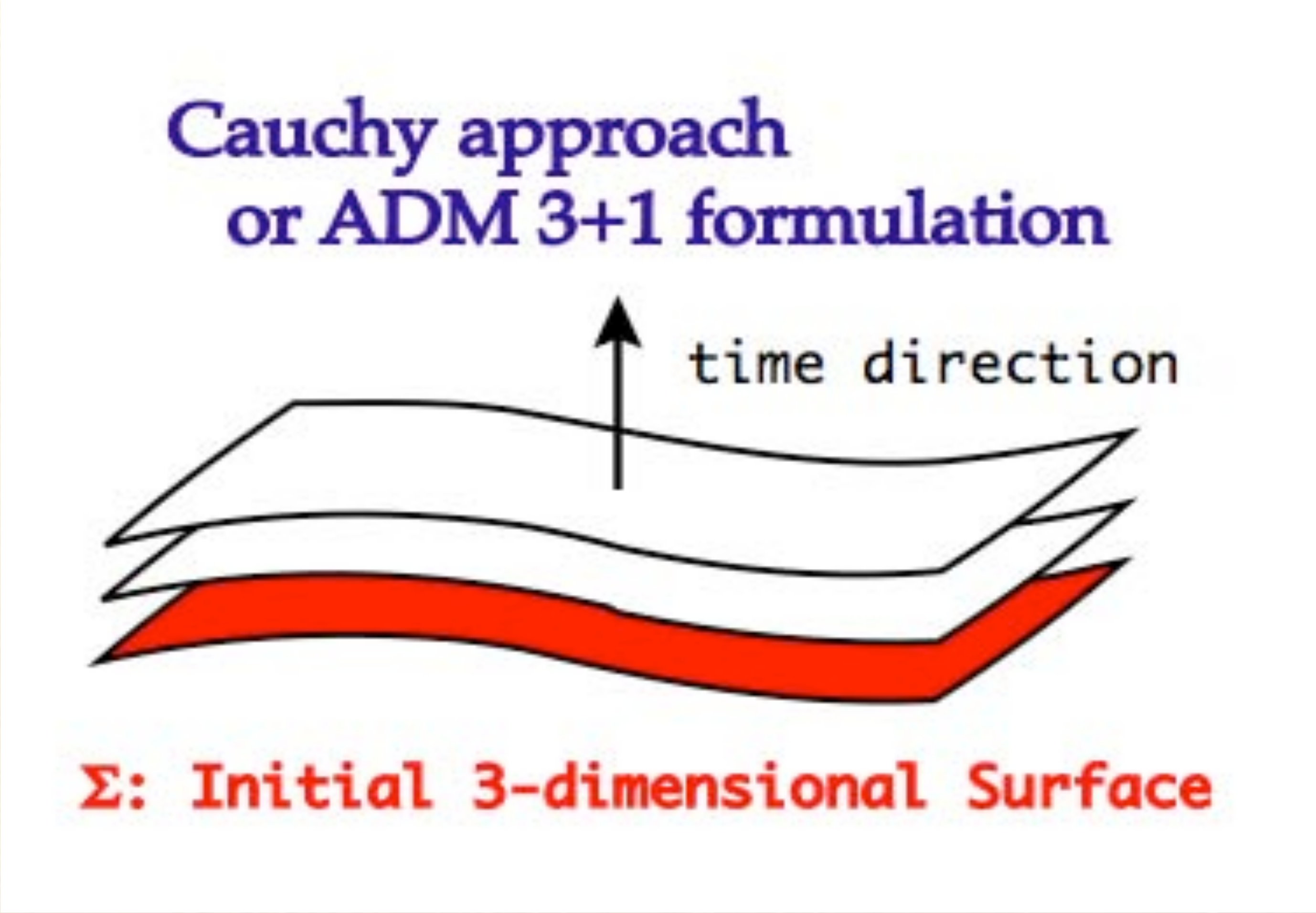
2. How to evolve the data

Theoretical:	Free evolution or constrained evolution? Proper formulation for the evolution equations? Suitable slicing conditions (gauge conditions)?
Numerical:	Techniques for solving the evolution equations? Appropriate boundary treatments? Singularity excision techniques? Matter and shock surface treatments? Parallelization of the code?

3. How to extract the physical information

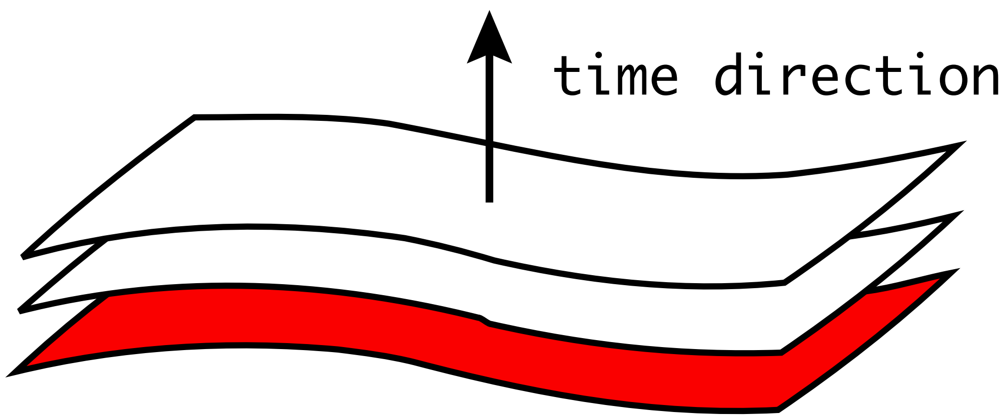
Theoretical:	Gravitational wave extraction? Connection to other approximations?
Numerical:	Identification of black hole horizons? Visualization of simulations?

"3+1" formulation



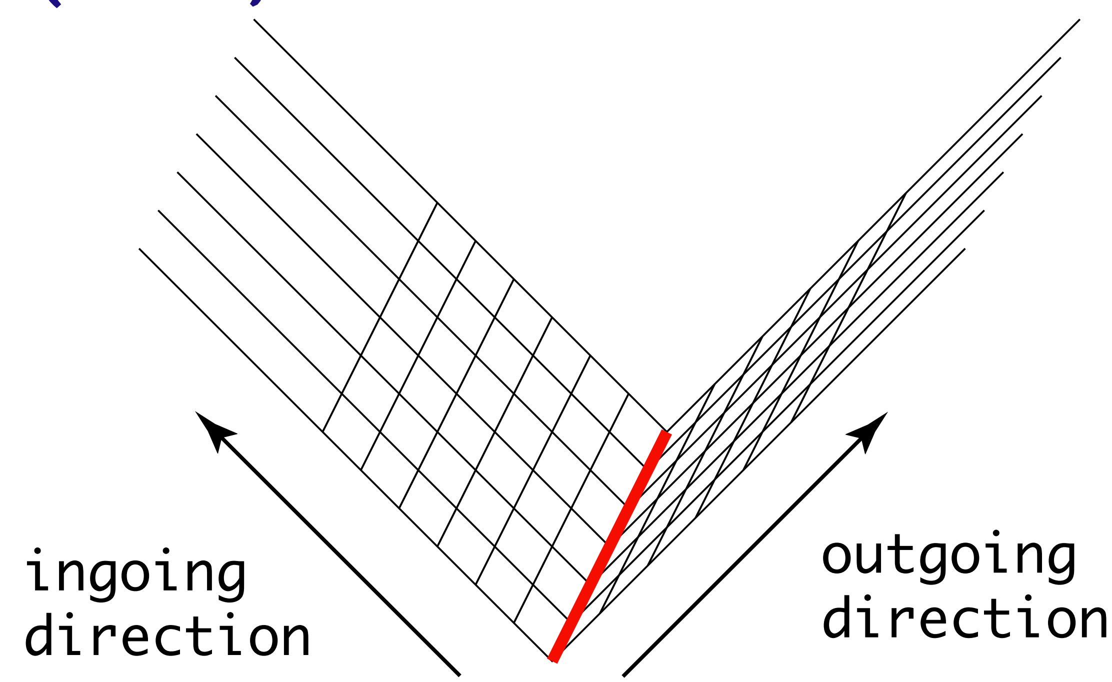
First Question: How to foliate space-time?

Cauchy approach
or ADM 3+1 formulation



Σ: Initial 3-dimensional Surface

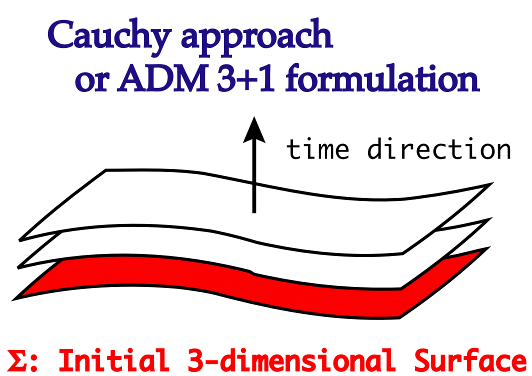
Characteristic approach
(if null, dual-null 2+2 formulation)

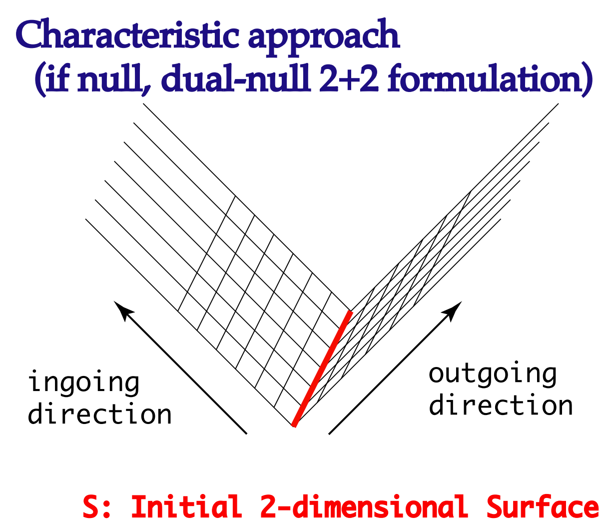


S: Initial 2-dimensional Surface

3+1 versus 2+2

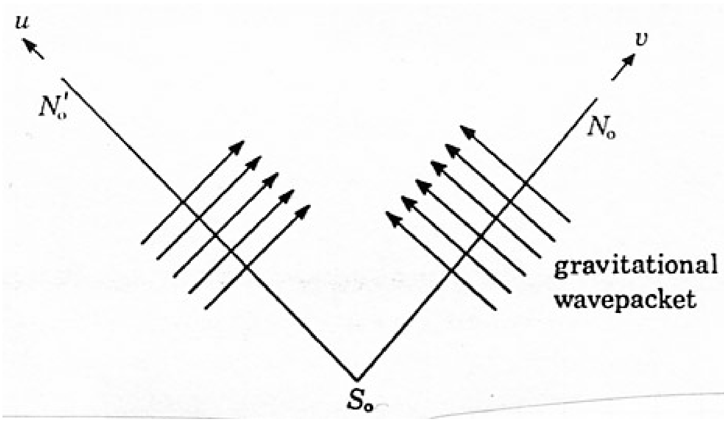
	Cauchy (3+1) evolution	Characteristic (2+2) evolution
pioneers	ADM (1961), York-Smarr (1978)	Bondi *et al* (1962), Sachs (1962), Penrose (1963)
variables	easy to understand the concept of time evolution	has geometrical meanings 1 complex function related to 2 GW polarization modes
foliation	has Hamilton structure	allows implementation of Penrose's space-time compactification
initial data	need to solve constraints	no constraints
evolution	PDEs need to avoid constraint violation	ODEs with consistent conditions propagation eqs along the light rays
singularity	need to avoid by some method	can truncate the grid
disadvantages	can not cover space-time globally	difficulty in treating caustics hard to treat matter





Numerical Relativity in Dual-Null Foliation

J.M. Stewart, H.Friedrich, Proc. R. Soc. Lond. A 384, 427 (1982)
R.W. Corkill, J.M. Stewart, Proc. R. Soc. Lond. A 386, 373 (1983)



Colliding Plane-wave

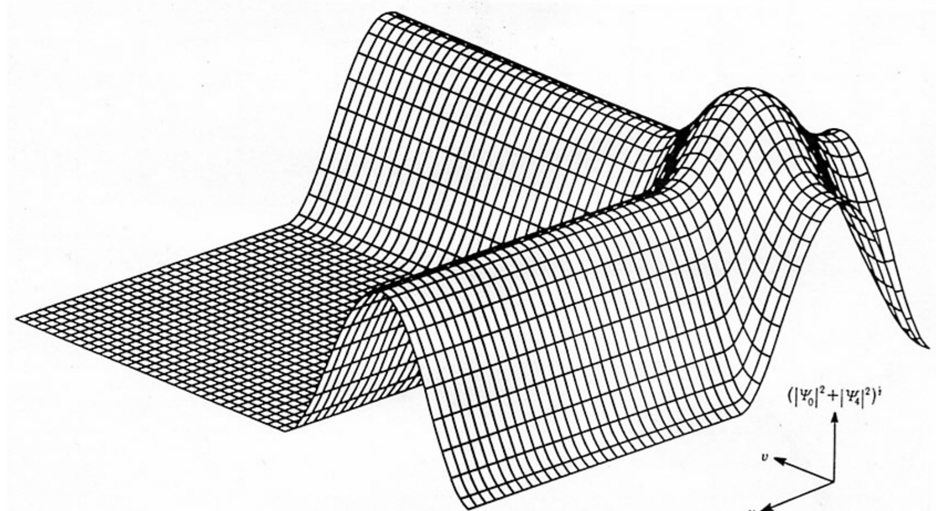


Figure 2. Colliding plane gravitational waves in a linearized theory.

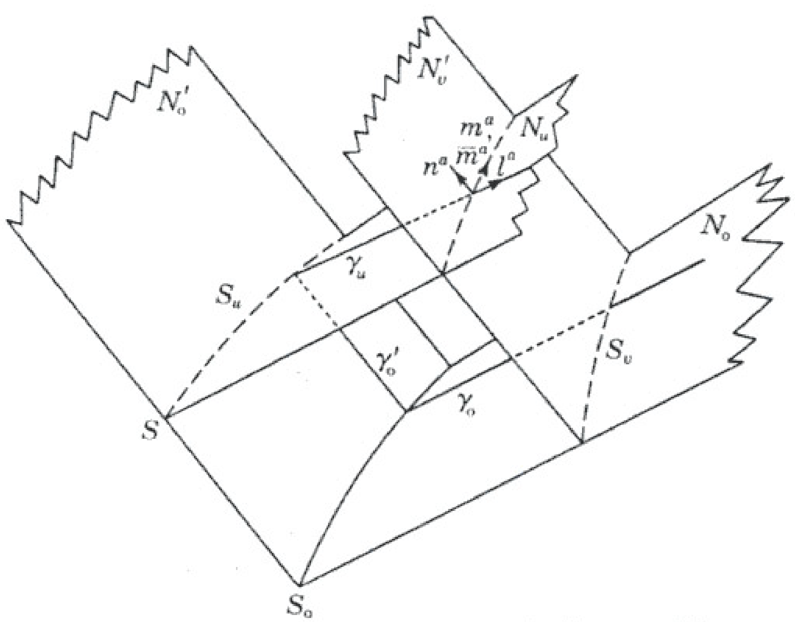


FIGURE 3. The coordinate system on M. (x^A) $(A = 2, 3)$ are arbitrary coordinates on S_o. The coordinate u is an arbitrary parameter along the generators γ'_o of N'_o that vanishes on S_o. S_u is the 2-surface u = constant. We define u in M by requiring u = constant on N_u, the unique null hypersurface (other than N'_0) through S_u. The coordinate v is similarly defined. We propagate (x^A) on to N'_o by requiring x^A = constant along the γ'_o, and into M by requiring x^A = constant along the generators γ_u of N_u.

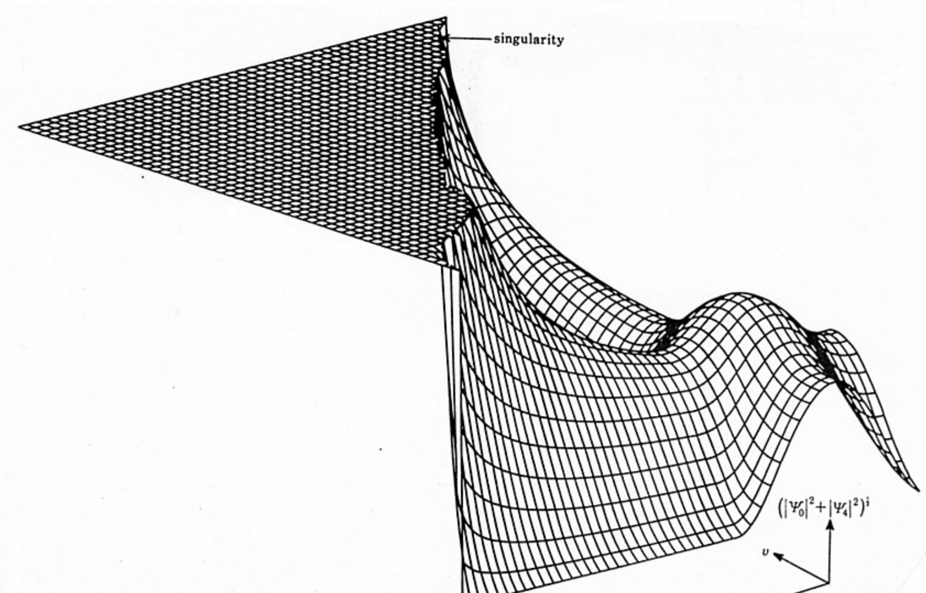


Figure 3. Colliding plane gravitational waves in the full nonlinear theory.

1.3　数値相対論の方法論概略

Numerical Relativity – open issues

どのように時空をとらえるか

0. How to foliate space-time

Cauchy $(3+1)$, Hyperboloidal $(3+1)$, characteristic $(2+2)$, or combined?

$\Rightarrow$ if the foliation is $(3+1)$, then $\cdots$

1. How to prepare the initial data

初期条件をどう設定するか

Theoretical:	Proper formulation for solving constraints? How to prepare realistic initial data? Effects of background gravitational waves? Connection to the post-Newtonian approximation?
Numerical:	Techniques for solving coupled elliptic equations? Appropriate boundary conditions?

2. How to evolve the data

どのように時間発展させるか

Theoretical:	Free evolution or constrained evolution? Proper formulation for the evolution equations? Suitable slicing conditions (gauge conditions)?
Numerical:	Techniques for solving the evolution equations? Appropriate boundary treatments? Singularity excision techniques? Matter and shock surface treatments? Parallelization of the code?

3. How to extract the physical information

物理情報をどう取り出すか

Theoretical:	Gravitational wave extraction? Connection to other approximations?
Numerical:	Identification of black hole horizons? Visualization of simulations?