APCTP Winter School, January 25-26, 2008

# Formulation Problem in Numerical Relativity

Hisaaki Shinkai (Osaka Institute of Technology, Japan)

신카이 히사아키

1. Introduction
2. The Standard Approach to Numerical Relativity
   - ADM/BSSN/hyperbolic formulations
3. Robust system for Constraint Violation
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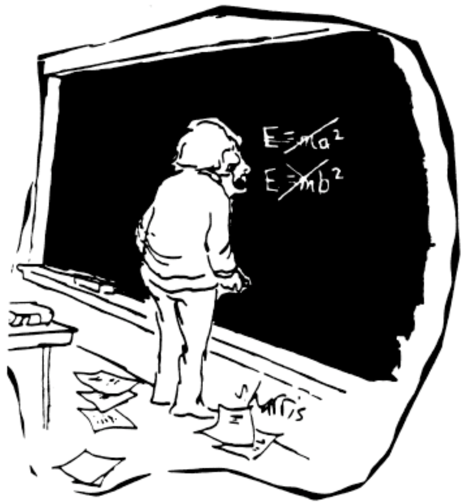
   - Adjusted BSSN system
4. Outlook



http://www.is.oit.ac.jp/~shinkai/

# Procedure of the Standard Numerical Relativity

- 3+1 (ADM) formulation

- Preparation of the Initial Data
  - Assume the background metric
  - Solve the constraint equations

Need to solve elliptic PDEs
-- Conformal approach
-- Thin-Sandwich approach

- Time Evolution

do time=1, time_end
  - Specify the slicing condition
  - Evolve the variables
  - Check the accuracy
  - Extract physical quantities

end do

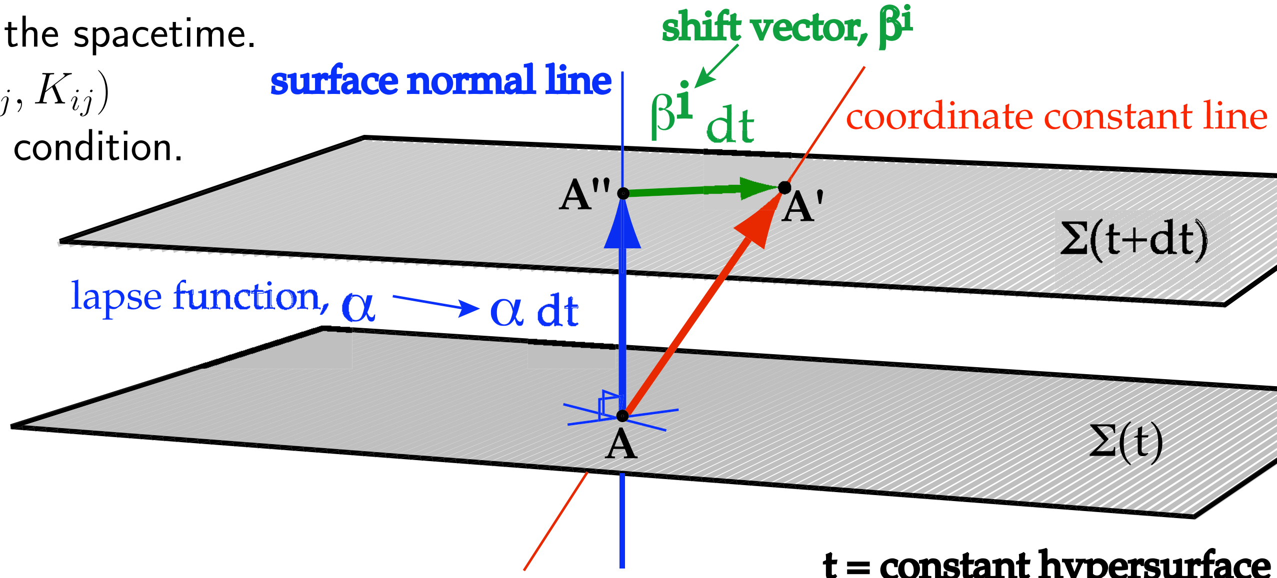
singularity avoidance,
simplify the system,
GW extraction, ...

Robust formulation ?
-- modified ADM / BSSN
-- hyperbolization
-- asymptotically constrained

**Formulation Problem**

## strategy 0 The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

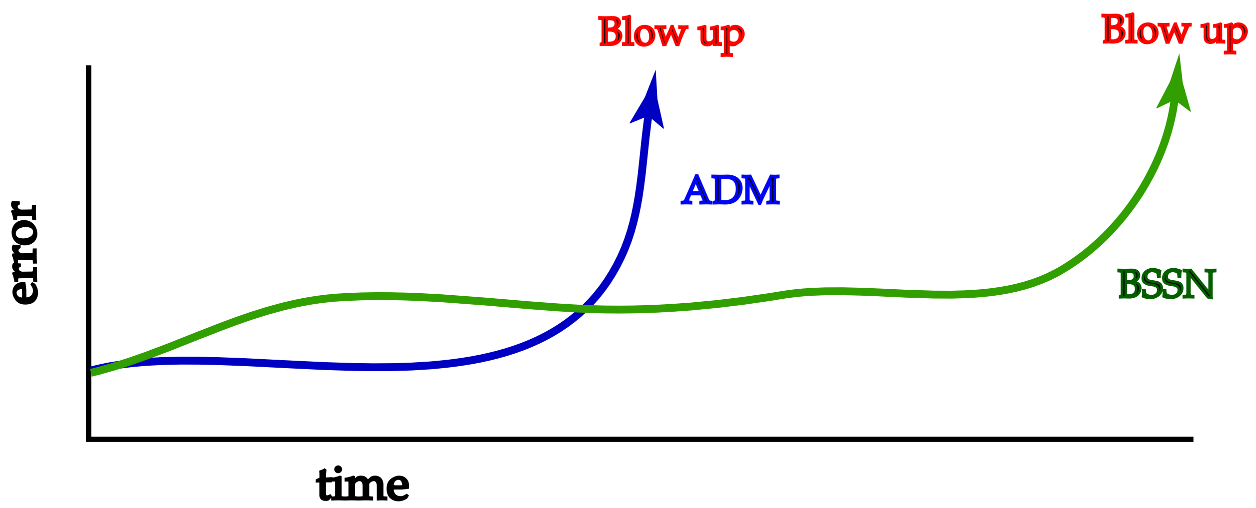
3+1 decomposition of the spacetime.
Evolve 12 variables $(\gamma_{ij}, K_{ij})$
with a choice of gauge condition.




| | Maxwell eqs. | ADM Einstein eq. |
|---|---|---|
| constraints | div $\mathbf{E} = 4\pi\rho$<br>div $\mathbf{B} = 0$ | $^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$<br>$D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$ |
| evolution eqs. | $\frac{1}{c}\partial_t\mathbf{E} = \mathrm{rot}\ \mathbf{B} - \frac{4\pi}{c}\mathbf{j}$<br>$\frac{1}{c}\partial_t\mathbf{B} = -\mathrm{rot}\ \mathbf{E}$ | $\partial_t\gamma_{ij} = -2NK_{ij} + D_jN_i + D_iN_j,$<br>$\partial_t K_{ij} = N(\,^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN$<br>$+ (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$<br>$- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$ |

Best formulation of the Einstein eqs. for long-term stable & accurate simulation?

- Many (too many) trials and errors, hard to find a definit recipe.




Mathematically equivalent formulations, but differ in its stability!

strategy 0: Arnowitt-Deser-Misner (ADM) formulation
strategy 1: Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation
strategy 2: Hyperbolic formulations
strategy 3: "Asymptotically constrained" against a violation of constraints

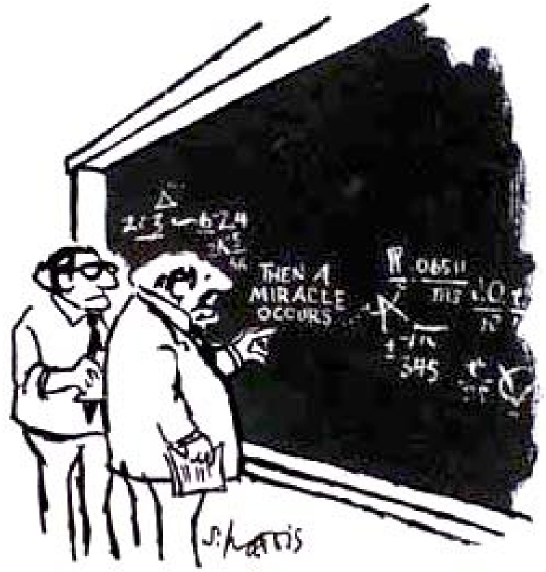
By adding constraints in RHS, we can kill error-growing modes
$\Rightarrow$ How can we understand the features systematically?

# Goals of the Lecture

What is the guiding principle for selecting evolution equations for simulations in GR?

Why many groups use the BSSN equations?

Are there an alternative formulation better than the BSSN?




"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

# Summary up to here (1st half)

[Keyword 1] Formulation Problem

Although mathematically equivalent, different set of equations shows different numerical stability.

[Keyword 2] ADM formulation

The starting formulation (Historically & Numerically).
Successes in 90s, but not for binary BH-BH/NS-NS problems.

[Keyword 3] BSSN formulation

New variables and gauge fixing to ADM, shows better stability.
The reason why it is better was not known at first.
Many simulation groups uses BSSN. Technical tips are accumulated.

[Keyword 4] hyperbolic formulations

Mathematical classification of PDE shows "well-posedness", but its meaning is limited.
Many versions of hyperbolic Einstein equations are available.
Some group try to show the advantage of BSSN using "hyperbolicity".
But are they really helpful in numerics?

**Some known fact (technical):**

- Trace-out $A_{ij}$ at every time step helps the stability.

  Alcubierre, et al, [PRD 62 (2000) 044034]

- "The essential improvement is in the process of replacing terms by the momentum constraints",

  Alcubierre, et al, [PRD 62 (2000) 124011]

- $\tilde{\Gamma}^i$ is replaced by $-\partial_j \tilde{\gamma}^{ij}$ where it is not differentiated,

  Campanelli, et al, [PRL96 (2006) 111101; PRD 73 (2006) 061501R]

- $\tilde{\Gamma}^i$-equation has been modified as suggested in Yo-Baumgarte-Shapiro [PRD 66 (2002) 084026]

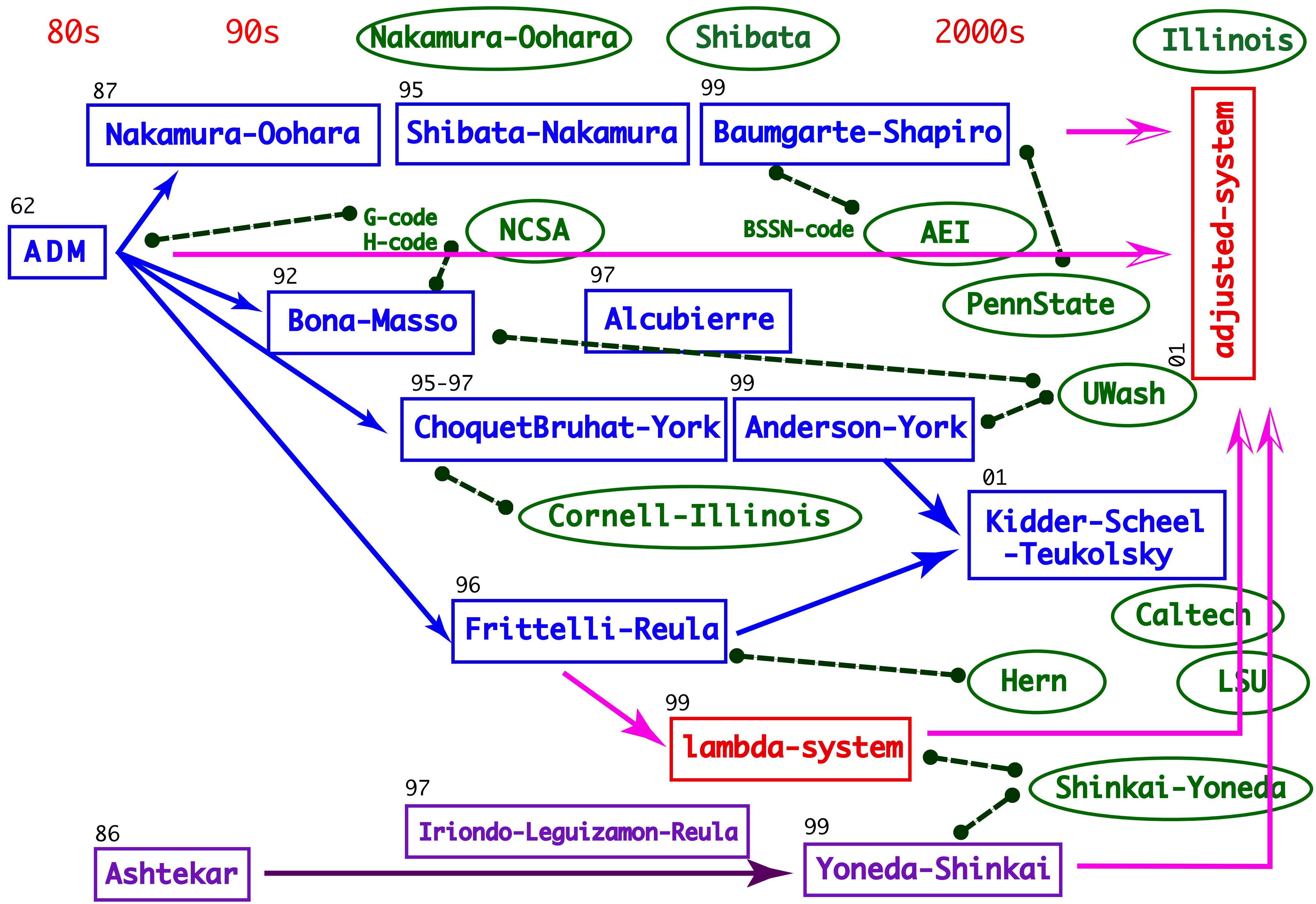
  Baker et al, [PRL96 (2006) 111102; PRD73 (2006) 104002]

**Some guesses:**

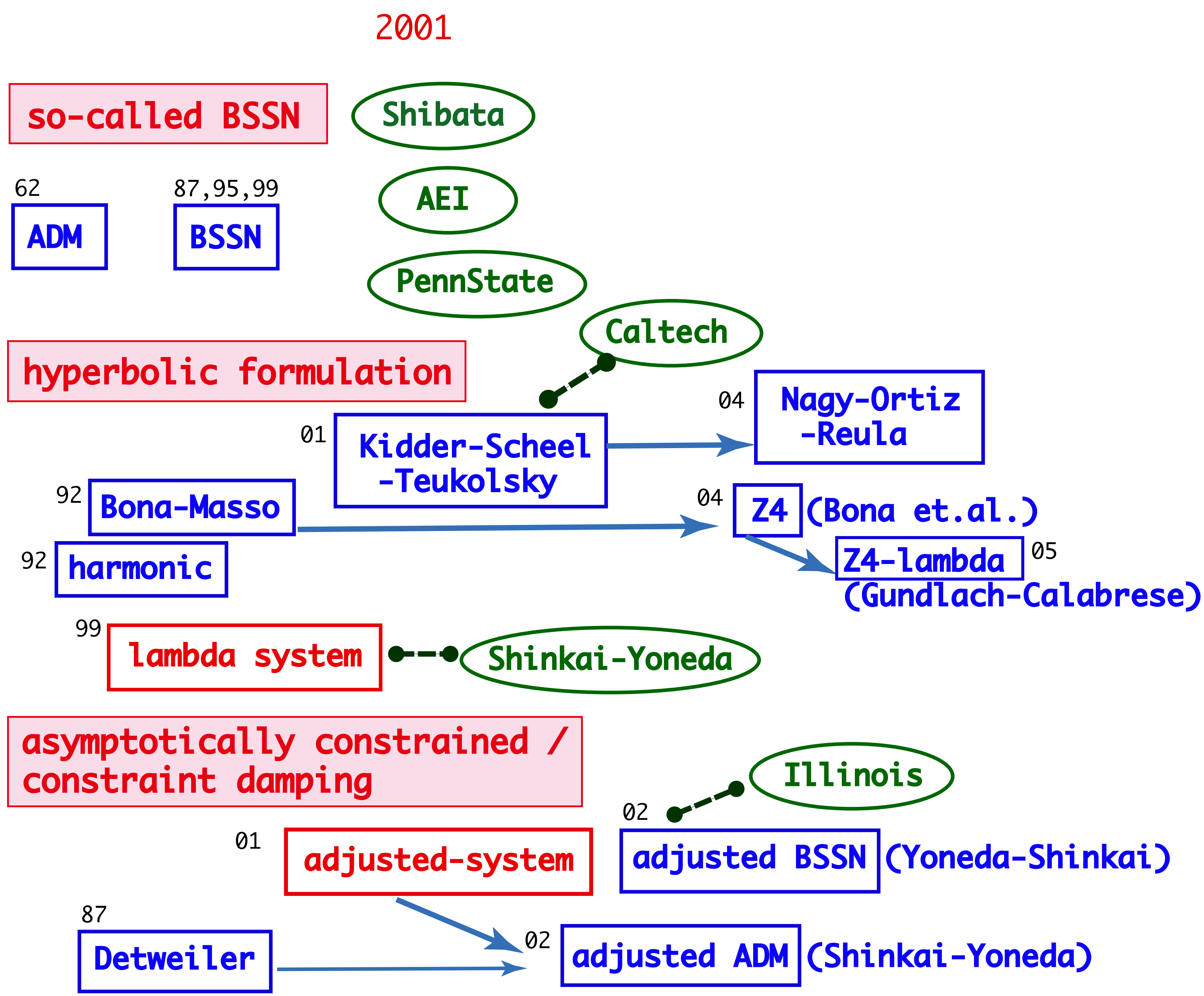
- BSSN has a wider range of parameters that give us stable evolutions in von Neumann's stability analysis. Miller, [gr-qc/0008017]

- The eigenvalues of BSSN *evolution equations* has fewer "zero eigenvalues" than those of ADM, and they conjectured that the instability can be caused by "zero eigenvalues" that violate "gauge mode".

  M. Alcubierre, et al, [PRD 62 (2000) 124011]



80s
90s
2000s
Nakamura-Oohara
Shibata
Illinois
87
Nakamura-Oohara
95
Shibata-Nakamura
99
Baumgarte-Shapiro
62
ADM
G-code
H-code
NCSA
BSSN-code
AEI
adjusted-system
01
92
Bona-Masso
97
Alcubierre
PennState
UWash
95-97
ChoquetBruhat-York
99
Anderson-York
Cornell-Illinois
01
Kidder-Scheel
-Teukolsky
Caltech
96
Frittelli-Reula
Hern
LSU
99
lambda-system
Shinkai-Yoneda
97
Iriondo-Leguizamon-Reula
86
Ashtekar
99
Yoneda-Shinkai


2001
so-called BSSN
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62
ADM
87,95,99
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hyperbolic formulation
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Z4 (Bona et.al.)
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harmonic
Z4-lambda
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asymptotically constrained /
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adjusted BSSN (Yoneda-Shinkai)
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Detweiler
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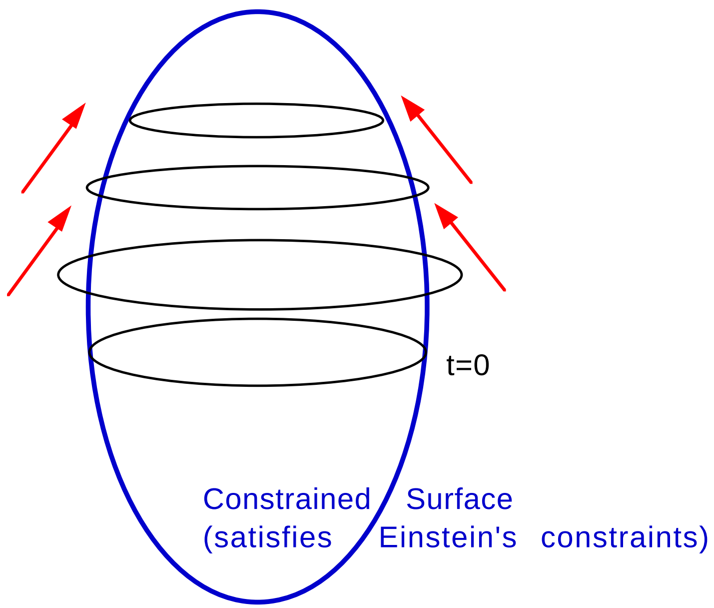
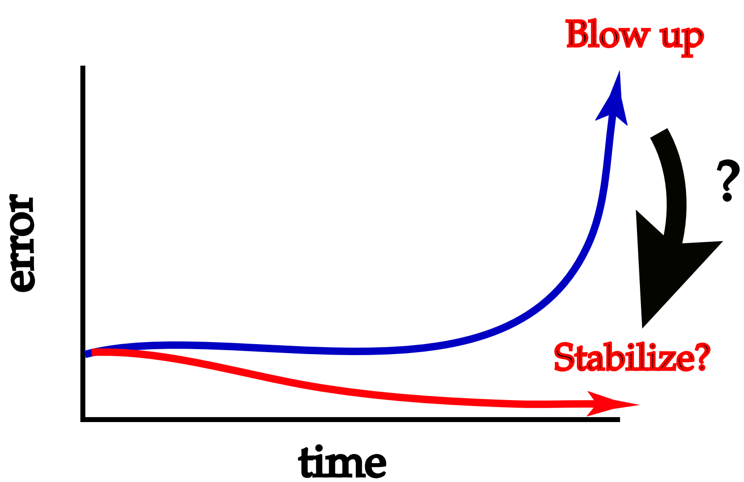
## strategy 3 "Asymptotically Constrained" system / "Constraint Damping" system

Formulate a system which is "asymptotically constrained" against a violation of constraints

Constraint Surface as an Attractor



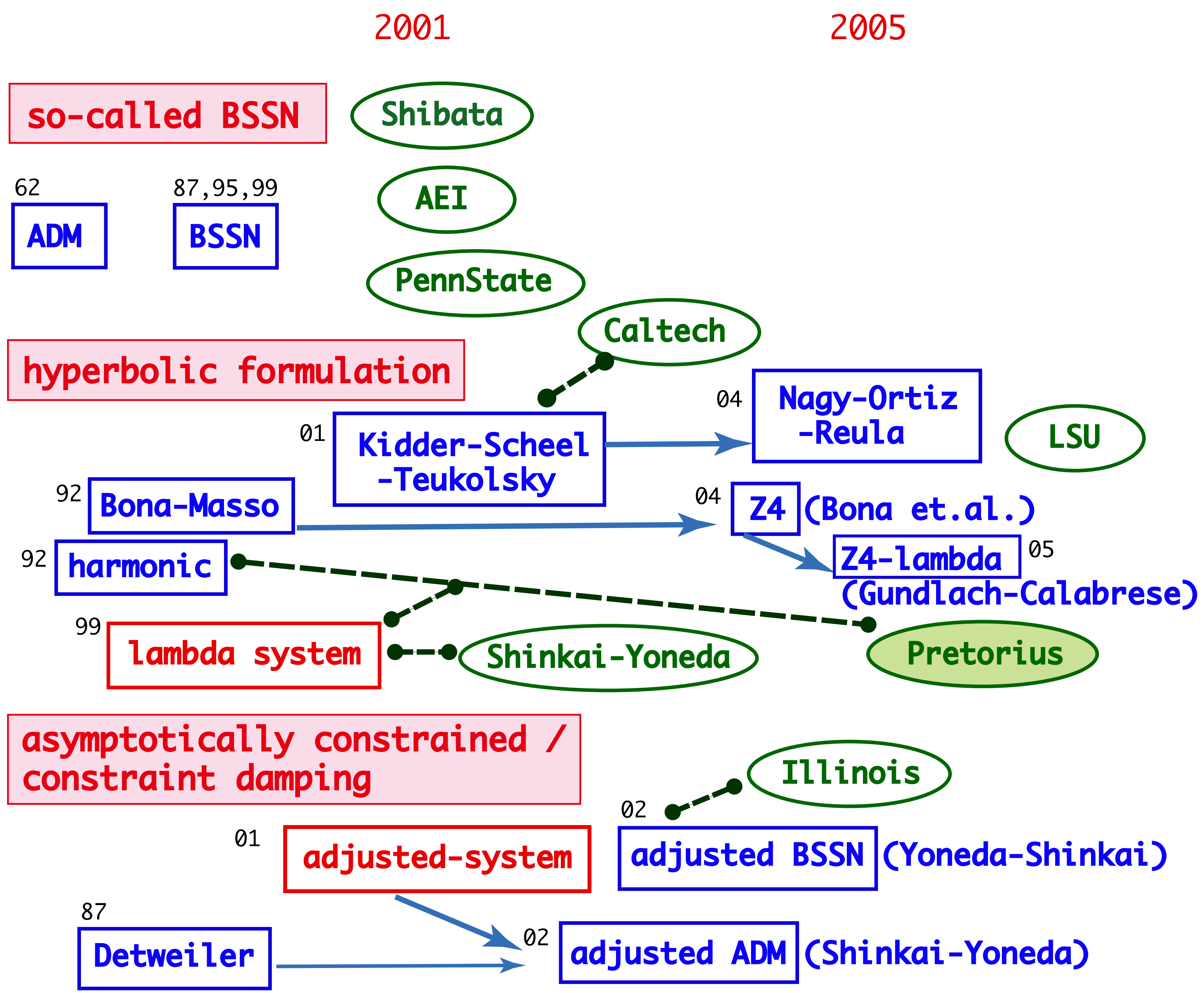

method 1: $\lambda$-system (Brodbeck et al, 2000)

- Add aritificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may prodict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. $\Rightarrow$

  for the ADM/BSSN formulation, too!!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## Idea of $\lambda$-system

Brodbeck, Frittelli, Hübner and Reula, JMP40(99)909

We expect a system that is robust for controlling the violation of constraints

**Recipe**

1. Prepare a symmetric hyperbolic evolution system $\partial_t u = J\partial_i u + K$
2. Introduce $\lambda$ as an indicator of violation of constraint which obeys dissipative eqs. of motion $\partial_t \lambda = \alpha C - \beta \lambda$ $(\alpha \neq 0, \beta > 0)$
3. Take a set of $(u, \lambda)$ as dynamical variables $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$
4. Modify evolution eqs so as to form a symmetric hyperbolic system $\partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$

Remarks

- BFHR used a sym. hyp. formulation by Frittelli-Reula [PRL76(96)4667]
- The version for the Ashtekar formulation by HS-Yoneda [PRD60(99)101502] for controlling the constraints or reality conditions or both.
- Succeeded in evolution of GW in planar spacetime using Ashtekar vars. [CQG18(2001)441]
- Do the recovered solutions represent true evolution? by Siebel-Hübner [PRD64(2001)024021]
- The version for Z4 hyperbolic system by Gundlach-Calabrese-Hinder-MartinGarcia [CQG22(05)3767]
  $\Rightarrow$ Pretorius noticed the idea of "constraint damping" [PRL95(05)121101]

## Idea of "Adjusted system" and Our Conjecture

CQG18 (2001) 441, PRD 63 (2001) 120419, CQG 19 (2002) 1027

### General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) \underbrace{+F(C^a, \partial_b C^a, \cdots)}$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$ to make the system stable evolution

### How to specify $F(C^a, \partial_b C^a, \cdots)$ ?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) \underbrace{+G(C^a, \partial_b C^a, \cdots)}$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = \underbrace{A(\hat{C}^a)} \hat{C}^k$

**Conjecture:** Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs.
If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

# Example: the Maxwell equations

Yoneda HS, CQG 18 (2001) 441

Maxwell evolution equations.

$$\begin{aligned} \partial_t E_i &= c\epsilon_i{}^{jk}\partial_j B_k + P_i C_E + Q_i C_B, \\ \partial_t B_i &= -c\epsilon_i{}^{jk}\partial_j E_k + R_i C_E + S_i C_B, \\ & C_E = \partial_i E^i \approx 0, \quad C_B = \partial_i B^i \approx 0, \end{aligned}$$

$$\begin{cases} \text{sym. hyp} & \Leftrightarrow \quad P_i = Q_i = R_i = S_i = 0, \\ \text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\ \text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0 \end{cases}$$

Constraint propagation equations

$$\begin{aligned} \partial_t C_E &= (\partial_i P^i)C_E + P^i(\partial_i C_E) + (\partial_i Q^i)C_B + Q^i(\partial_i C_B), \\ \partial_t C_B &= (\partial_i R^i)C_E + R^i(\partial_i C_E) + (\partial_i S^i)C_B + S^i(\partial_i C_B), \end{aligned}$$

$$\begin{cases} \text{sym. hyp} & \Leftrightarrow \quad Q_i = R_i, \\ \text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\ \text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0 \end{cases}$$

CAFs?

$$\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} = \begin{pmatrix} \partial_i P^i + P^i k_i & \partial_i Q^i + Q^i k_i \\ \partial_i R^i + R^i k_i & \partial_i S^i + S^i k_i \end{pmatrix} \partial_l \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} \approx \begin{pmatrix} P^i k_i & Q^i k_i \\ R^i k_i & S^i k_i \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} =: T \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix}$$

$$\Rightarrow \text{CAFs} = (P^i k_i + S^i k_i \pm \sqrt{(P^i k_i + S^i k_i)^2 + 4(Q^i k_i R^j k_j - P^i k_i S^j k_j)})/2$$

Therefore CAFs become negative-real when

$$P^i k_i + S^i k_i < 0, \qquad \text{and} \qquad Q^i k_i R^j k_j - P^i k_i S^j k_j < 0$$

## The Adjusted system (essentials):

| | |
|---|---|
| Purpose: | Control the violation of constraints by reformulating the system so as to have a constrained surface an attractor. |
| Procedure: | Add a particular combination of constraints to the evolution equations, and adjust its multipliers. |
| Theoretical support: | Eigenvalue analysis of the constraint propagation equations. |
| Advantages: | Available even if the base system is not a symmetric hyperbolic. |
| Advantages: | Keep the number of the variable same with the original system. |

## Conjecture on Constraint Amplification Factors (CAFs):

(A) If CAF has a negative real-part (the constraints are forced to be diminished), then we see more stable evolution than a system which has positive CAF.

(B) If CAF has a non-zero imaginary-part (the constraints are propagating away), then we see more stable evolution than a system which has zero CAF.

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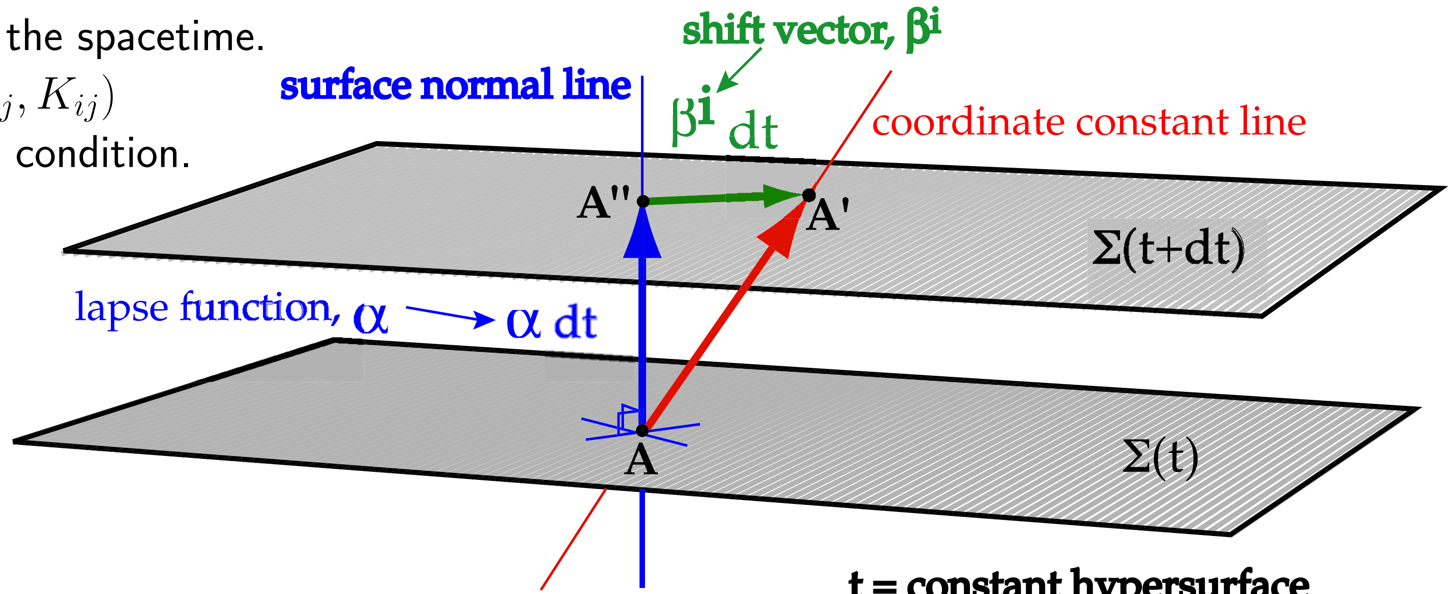
3. Robust system for Constraint Violation
    - Adjusted systems
    - Adjusted ADM system -- why the standard ADM brows up?
    - Adjusted BSSN system
4. Outlook

strategy 0 **The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)**

3+1 decomposition of the spacetime.
Evolve 12 variables $(\gamma_{ij}, K_{ij})$
with a choice of gauge condition.




| | Maxwell eqs. | ADM Einstein eq. |
|---|---|---|
| constraints | div $\mathbf{E} = 4\pi\rho$<br>div $\mathbf{B} = 0$ | ${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$<br>$D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$ |
| evolution eqs. | $\frac{1}{c}\partial_t \mathbf{E} = \text{rot } \mathbf{B} - \frac{4\pi}{c}\mathbf{j}$<br>$\frac{1}{c}\partial_t \mathbf{B} = -\text{rot } \mathbf{E}$ | $\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$<br>$\partial_t K_{ij} = N( {}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$<br>$+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$<br>$- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$ |

## The Standard ADM formulation (aka York 1978):

The fundamental dynamical variables are $(\gamma_{ij}, K_{ij})$, the three-metric and extrinsic curvature. The three-hypersurface $\Sigma$ is foliated with gauge functions, $(\alpha, \beta^i)$, the lapse and shift vector.

- The evolution equations:

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i, \\
\partial_t K_{ij} &= \alpha\ {}^{(3)}R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - D_i D_j \alpha \\
&\quad + (D_i \beta^k) K_{kj} + (D_j \beta^k) K_{ki} + \beta^k D_k K_{ij} \\
&\quad - 8\pi G \alpha \{ S_{ij} + (1/2) \gamma_{ij} (\rho_H - \mathrm{tr} S) \},
\end{aligned}
$$

where $K = K^i{}_i$, and ${}^{(3)}R_{ij}$ and $D_i$ denote three-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$
\begin{aligned}
\text{Hamiltonian constr.} \qquad \mathcal{H}^{ADM} &:= {}^{(3)}R + K^2 - K_{ij} K^{ij} \approx 0, \\
\text{momentum constr.} \qquad \mathcal{M}_i^{ADM} &:= D_j K^j{}_i - D_i K \approx 0,
\end{aligned}
$$

where ${}^{(3)}R = {}^{(3)}R^i{}_i$.

S. Frittelli, Phys. Rev. D55, 5992 (1997)
HS and G. Yoneda, Class. Quant. Grav. 19, 1027 (2002)

**The Constraint Propagations of the Standard ADM:**

$$\begin{aligned} \partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) \\ &\quad +\alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \\ \partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\ &\quad +\alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j. \end{aligned}$$

From these equations, we know that

if the constraints are satisfied on the initial slice $\Sigma$,
then the constraints are satisfied throughout evolution (in principle).

But this is NOT TRUE in NUMERICS....

**Original ADM** The original construction by ADM uses the pair of $(h_{ij}, \pi^{ij})$.

$$\mathcal{L} = \sqrt{-g}R = \sqrt{h}N[{}^{(3)}R - K^2 + K_{ij}K^{ij}], \quad \text{where } K_{ij} = \frac{1}{2}\pounds_n h_{ij}$$

$$\text{then} \quad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}),$$

The Hamiltonian density gives us constraints and evolution eqs.

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} = \sqrt{h}\left\{N\mathcal{H}(h,\pi) - 2N_j\mathcal{M}^j(h,\pi) + 2D_i(h^{-1/2}N_j\pi^{ij})\right\},$$

$$\begin{cases} \partial_t h_{ij} = \dfrac{\delta\mathcal{H}}{\delta\pi^{ij}} = 2\dfrac{N}{\sqrt{h}}(\pi_{ij} - \dfrac{1}{2}h_{ij}\pi) + 2D_{(i}N_{j)}, \\ \partial_t \pi^{ij} = -\dfrac{\delta\mathcal{H}}{\delta h_{ij}} = -\sqrt{h}N({}^{(3)}R^{ij} - \dfrac{1}{2}{}^{(3)}Rh^{ij}) + \dfrac{1}{2}\dfrac{N}{\sqrt{h}}h^{ij}(\pi_{mn}\pi^{mn} - \dfrac{1}{2}\pi^2) - 2\dfrac{N}{\sqrt{h}}(\pi^{in}\pi_n{}^j - \dfrac{1}{2}\pi\pi^{ij}) \\ \qquad + \sqrt{h}(D^iD^jN - h^{ij}D^mD_mN) + \sqrt{h}D_m(h^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)} \end{cases}$$

**Standard ADM (by York)** NRists refer ADM as the one by York with a pair of $(h_{ij}, K_{ij})$.

$$\begin{cases} \partial_t h_{ij} = -2NK_{ij} + D_jN_i + D_iN_j, \\ \partial_t K_{ij} = N({}^{(3)}R_{ij} + KK_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN + (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} \end{cases}$$

In the process of converting, $\mathcal{H}$ was used, i.e. the standard ADM has already adjusted.

# Adjusted ADM systems

PRD 63 (2001) 120419, CQG 19 (2002) 1027

We adjust the standard ADM system using constraints as:

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, && (1)\\
&\quad + P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (2)\\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, && (3)\\
&\quad + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (4)
\end{aligned}
$$

with constraint equations

$$
\begin{aligned}
\mathcal{H} &:= R^{(3)} + K^2 - K_{ij} K^{ij}, && (5)\\
\mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K. && (6)
\end{aligned}
$$

We can write the adjusted constraint propagation equations as

$$
\begin{aligned}
\partial_t \mathcal{H} &= \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], && (7)\\
\partial_t \mathcal{M}_i &= \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. && (8)
\end{aligned}
$$

## 3 Constraint propagation of ADM systems

### 3.1 Original ADM vs Standard ADM

Try the adjustment $R_{ij} = \kappa_1 \alpha \gamma_{ij}$ and other multiplier zero, where $\kappa_1 = \begin{cases} 0 & \text{the standard ADM} \\ -1/4 & \text{the original ADM} \end{cases}$

- The constraint propagation eqs keep the first-order form (cf Frittelli, PRD55(97)5992):

$$\partial_t \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} \simeq \begin{pmatrix} \beta^l & -2\alpha\gamma^{jl} \\ -(1/2)\alpha\delta^l_i + R^l{}_i - \delta^l_i R & \beta^l \delta^j_i \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_j \end{pmatrix}. \qquad (5)$$

  The eigenvalues of the characteristic matrix:

$$\lambda^l = (\beta^l, \beta^l, \beta^l \pm \sqrt{\alpha^2 \gamma^{ll}(1 + 4\kappa_1)})$$

  The hyperbolicity of (5): $\begin{cases} \text{symmetric hyperbolic} & \text{when } \kappa_1 = 3/2 \\ \text{strongly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) > 0 \\ \text{weakly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) \geq 0 \end{cases}$

- On the Minkowskii background metric, the linear order terms of the Fourier-transformed constraint propagation equations gives the eigenvalues

$$\Lambda^l = (0, 0, \pm\sqrt{-k^2(1 + 4\kappa_1)}).$$

  That is, $\begin{cases} \text{(two 0s, two pure imaginary)} & \text{for the standard ADM} \\ \text{(four 0s)} & \text{for the original ADM} \end{cases}$ BETTER STABILITY

## Comparisons of Adjusted ADM systems (Teukolsky wave)

## 3-dim, harmonic slice, periodic BC **HS original Cactus/GR code**

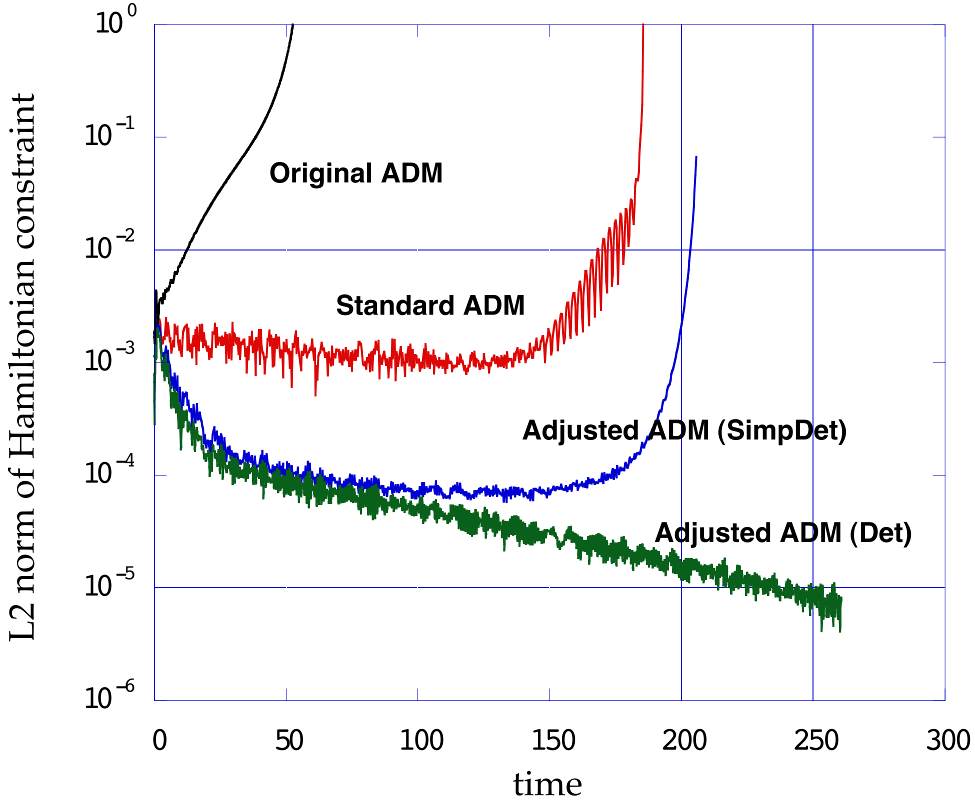



Figure 1: Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR/evolveADMeq code was used. Grid $= 24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method.

## 4 Constraint propagations in spherically symmetric spacetime

### 4.1 The procedure

The discussion becomes clear if we expand the constraint $C_\mu := (\mathcal{H}, \mathcal{M}_i)^T$ using vector harmonics.

$$C_\mu = \sum_{l,m} \left( A^{lm}(t,r) a_{lm}(\theta,\varphi) + B^{lm} b_{lm} + C^{lm} c_{lm} + D^{lm} d_{lm} \right), \tag{1}$$

where we choose the basis of the vector harmonics as

$$a_{lm} = \begin{pmatrix} Y_{lm} \\ 0 \\ 0 \\ 0 \end{pmatrix}, b_{lm} = \begin{pmatrix} 0 \\ Y_{lm} \\ 0 \\ 0 \end{pmatrix}, c_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ \partial_\theta Y_{lm} \\ \partial_\varphi Y_{lm} \end{pmatrix}, d_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\sin\theta} \partial_\varphi Y_{lm} \\ \sin\theta \, \partial_\theta Y_{lm} \end{pmatrix}.$$

The basis are normalized so that they satisfy

$$\langle C_\mu, C_\nu \rangle = \int_0^{2\pi} d\varphi \int_0^\pi C_\mu^* C_\rho \, \eta^{\nu\rho} \sin\theta d\theta,$$

where $\eta^{\nu\rho}$ is Minkowskii metric and the asterisk denotes the complex conjugate. Therefore

$$A^{lm} = \langle a^{lm}_{(\nu)}, C_\nu \rangle, \quad \partial_t A^{lm} = \langle a^{lm}_{(\nu)}, \partial_t C_\nu \rangle, \quad \text{etc.}$$

We also express these evolution equations using the Fourier expansion on the radial coordinate,

$$A^{lm} = \sum_k \hat{A}^{lm}_{(k)}(t) \, e^{ikr} \quad \text{etc.} \tag{2}$$

So that we will be able to obtain the RHS of the evolution equations for $(\hat{A}^{lm}_{(k)}(t), \cdots, \hat{D}^{lm}_{(k)}(t))^T$ in a homogeneous form.

## 4.2 Constraint propagations in Schwarzschild spacetime

1. the standard Schwarzschild coordinate

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1-2M/r} + r^2 d\Omega^2, \qquad \text{(the standard expression)}$$

2. the isotropic coordinate, which is given by, $r = (1 + M/2r_{iso})^2 r_{iso}$:

$$ds^2 = -(\frac{1 - M/2r_{iso}}{1 + M/2r_{iso}})^2 dt^2 + (1 + \frac{M}{2r_{iso}})^4 [dr_{iso}^2 + r_{iso}^2 d\Omega^2], \qquad \text{(the isotropic expression)}$$

3. the ingoing Eddington-Finkelstein (iEF) coordinate, by $t_{iEF} = t + 2M \log(r - 2M)$ :

$$ds^2 = -(1 - \frac{2M}{r})dt_{iEF}^2 + \frac{4M}{r} dt_{iEF} dr + (1 + \frac{2M}{r})dr^2 + r^2 d\Omega^2 \qquad \text{(the iEF expression)}$$

4. the Painlevé-Gullstrand (PG) coordinates,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt_{PG}^2 + 2\sqrt{\frac{2M}{r}} dt_{PG}\, dr + dr^2 + r^2 d\Omega^2 , \qquad \text{(the PG expression)}$$

which is given by $t_{PG} = t + \sqrt{8Mr} - 2M \log\{(\sqrt{r/2M} + 1)/(\sqrt{r/2M} - 1)\}$

## Example 1: standard ADM vs original ADM (in Schwarzschild coordinate)

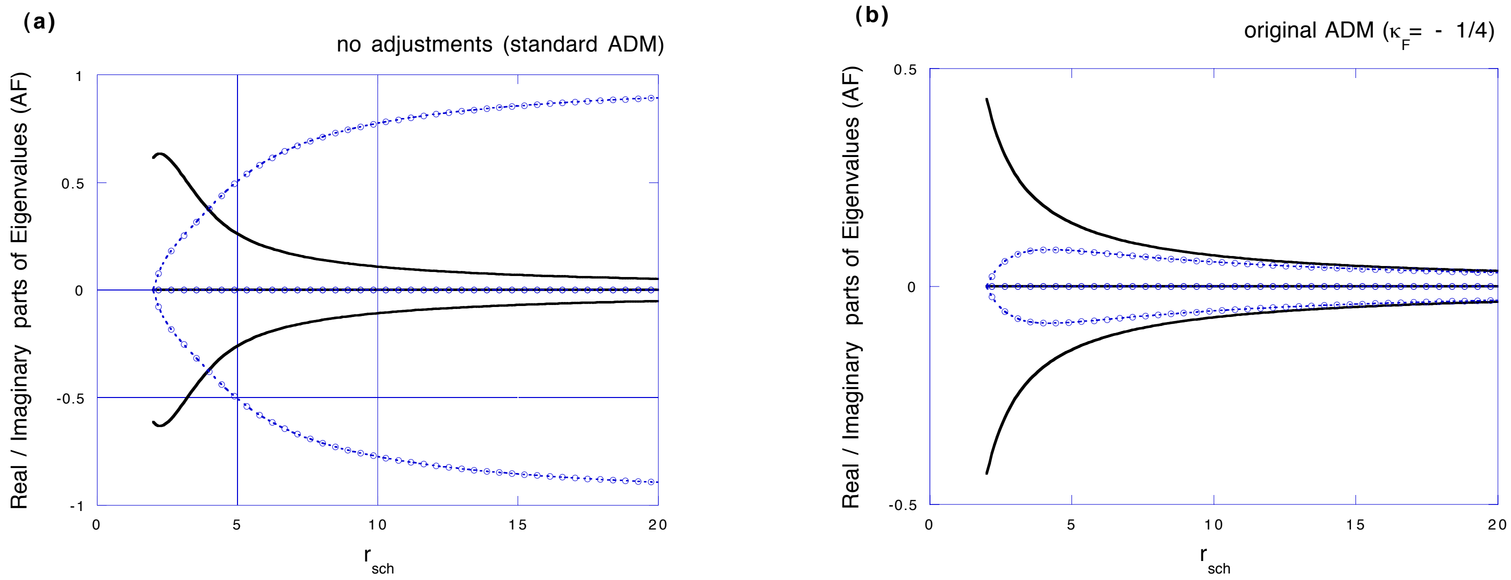



Figure 1: Amplification factors (AFs, eigenvalues of homogenized constraint propagation equations) are shown for the standard Schwarzschild coordinate, with (a) no adjustments, i.e., standard ADM, (b) original ADM ($\kappa_F = -1/4$). The solid lines and the dotted lines with circles are real parts and imaginary parts, respectively. They are four lines each, but actually the two eigenvalues are zero for all cases. Plotting range is $2 < r \leq 20$ using Schwarzschild radial coordinate. We set $k = 1, l = 2$, and $m = 2$ throughout the article.

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + \kappa_F \alpha \gamma_{ij} \mathcal{H},
\end{aligned}
$$

# Example 2: Detweiler-type adjusted (in Schwarzschild coord.)

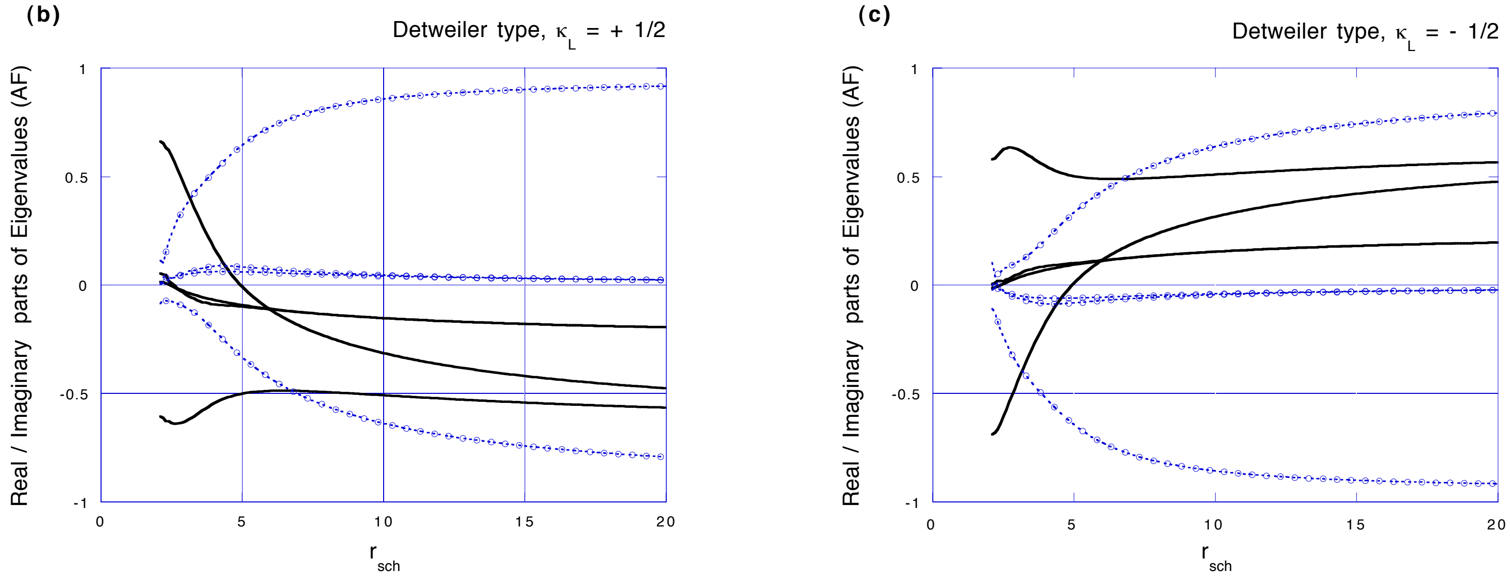



Figure 2: Amplification factors of the standard Schwarzschild coordinate, with Detweiler type adjustments. Multipliers used in the plot are (b) $\kappa_L = +1/2$, and (c) $\kappa_L = -1/2$.

$$\begin{aligned}
\partial_t \gamma_{ij} &= \text{(original terms)} + P_{ij}\mathcal{H}, \\
\partial_t K_{ij} &= \text{(original terms)} + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \\
& \quad \text{where} \quad P_{ij} = -\kappa_L \alpha^3 \gamma_{ij}, \quad R_{ij} = \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}), \\
& \quad S^k{}_{ij} = \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], \quad s^{kl}{}_{ij} = \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}],
\end{aligned}$$

## Detweiler's criteria vs Our criteria

- Detweiler calculated the L2 norm of the constraints, $C_\alpha$, over the 3-hypersurface and imposed its negative definiteness of its evolution,

$$\text{Detweiler's criteria} \Leftrightarrow \partial_t \int \sum_\alpha C_\alpha^2 \, dV < 0,$$

This is rewritten by supposing the constraint propagation to be $\partial_t \hat{C}_\alpha = A_\alpha{}^\beta \hat{C}_\beta$ in the Fourier components,

$$\Leftrightarrow \quad \partial_t \int \sum_\alpha \hat{C}_\alpha \bar{\hat{C}}_\alpha \, dV = \int \sum_\alpha A_\alpha{}^\beta \hat{C}_\beta \bar{\hat{C}}_\alpha + \hat{C}_\alpha \bar{A}_\alpha{}^\beta \bar{\hat{C}}_\beta \, dV < 0, \ \forall \text{ non zero } \hat{C}_\alpha$$

$$\Leftrightarrow \quad \underline{\text{eigenvalues of } (A + A^\dagger) \text{ are all negative for } \forall k}.$$

- Our criteria is that <u>the eigenvalues of $A$ are all negative</u>. Therefore,

$$\boxed{\text{Our criteria} \ni \text{Detweiler's criteria}}$$

- We remark that Detweiler's truncations on higher order terms in $C$-norm corresponds our perturbative analysis, both based on the idea that the deviations from constraint surface (the errors expressed non-zero constraint value) are initially small.

# Constraint propagation of ADM systems

## (2) Detweiler's system

Detweiler's modification to ADM [PRD35(87)1095] can be realized in our notation as:

$$\begin{aligned} P_{ij} &= -L\alpha^3\gamma_{ij}, \\ R_{ij} &= L\alpha^3(K_{ij} - (1/3)K\gamma_{ij}), \\ S^k_{ij} &= L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}], \\ s^{kl}_{ij} &= L\alpha^3[2\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}], \end{aligned}$$

and else zero, where $L$ is a constant.

- This adjustment does not make constraint propagation equation in the first order form, so that we can not discuss the hyperbolicity nor the characteristic speed of the constraints.
- For the Minkowskii background spacetime, the adjusted constraint propagation equations with above choice of multiplier become

$$\begin{aligned} \partial_t\mathcal{H} &= -2(\partial_j\mathcal{M}_j) + 4L(\partial_j\partial_j\mathcal{H}), \\ \partial_t\mathcal{M}_i &= -(1/2)(\partial_i\mathcal{H}) + (L/2)(\partial_k\partial_k\mathcal{M}_i) + (L/6)(\partial_i\partial_k\mathcal{M}_k). \end{aligned}$$

  Constraint Amp. Factors (the eigenvalues of their Fourier expression) are

$$\Lambda^l = (-(L/2)k^2(\text{multiplicity 2}), -(7L/3)k^2 \pm (1/3)\sqrt{k^2(-9+25L^2k^2)}.)$$

  This indicates negative real eigenvalues if we chose small positive $L$.

# Example 3: standard ADM (in isotropic/iEF coord.)

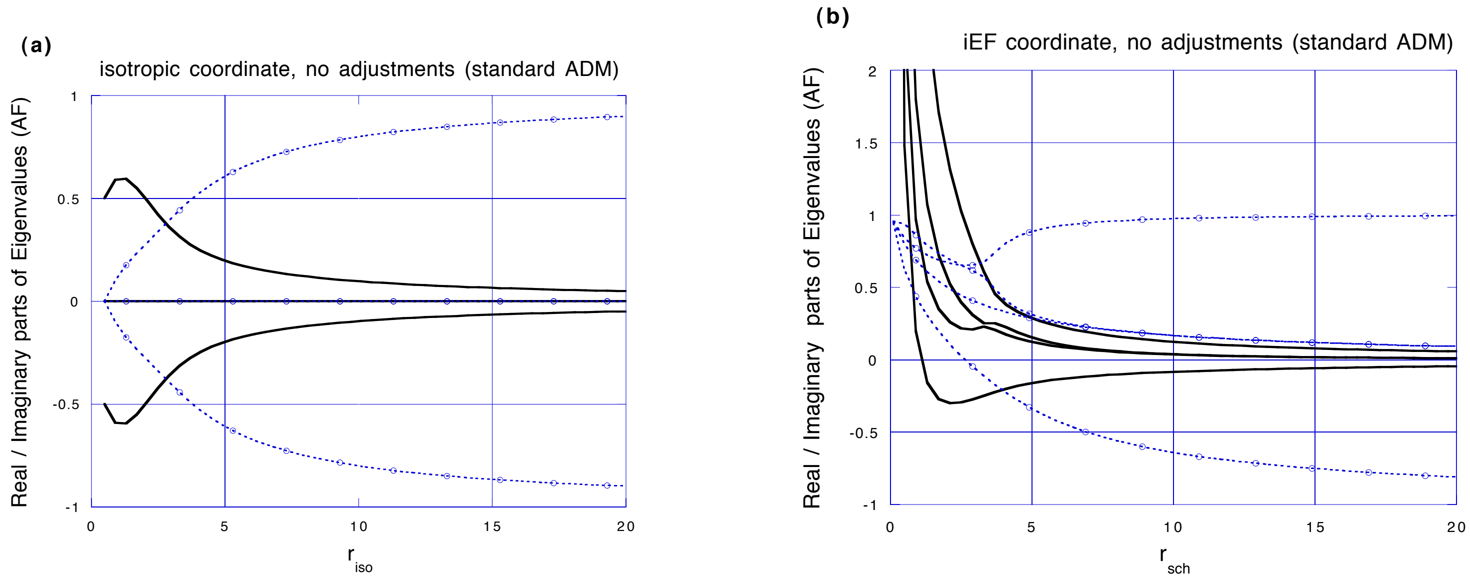



Figure 3: Comparison of amplification factors between different coordinate expressions for the standard ADM formulation (i.e. no adjustments). Fig. (a) is for the isotropic coordinate (1), and the plotting range is $1/2 \leq r_{iso}$. Fig. (b) is for the iEF coordinate (1) and we plot lines on the $t = 0$ slice for each expression. The solid four lines and the dotted four lines with circles are real parts and imaginary parts, respectively.

# Example 4: Detweiler-type adjusted (in iEF/PG coord.)

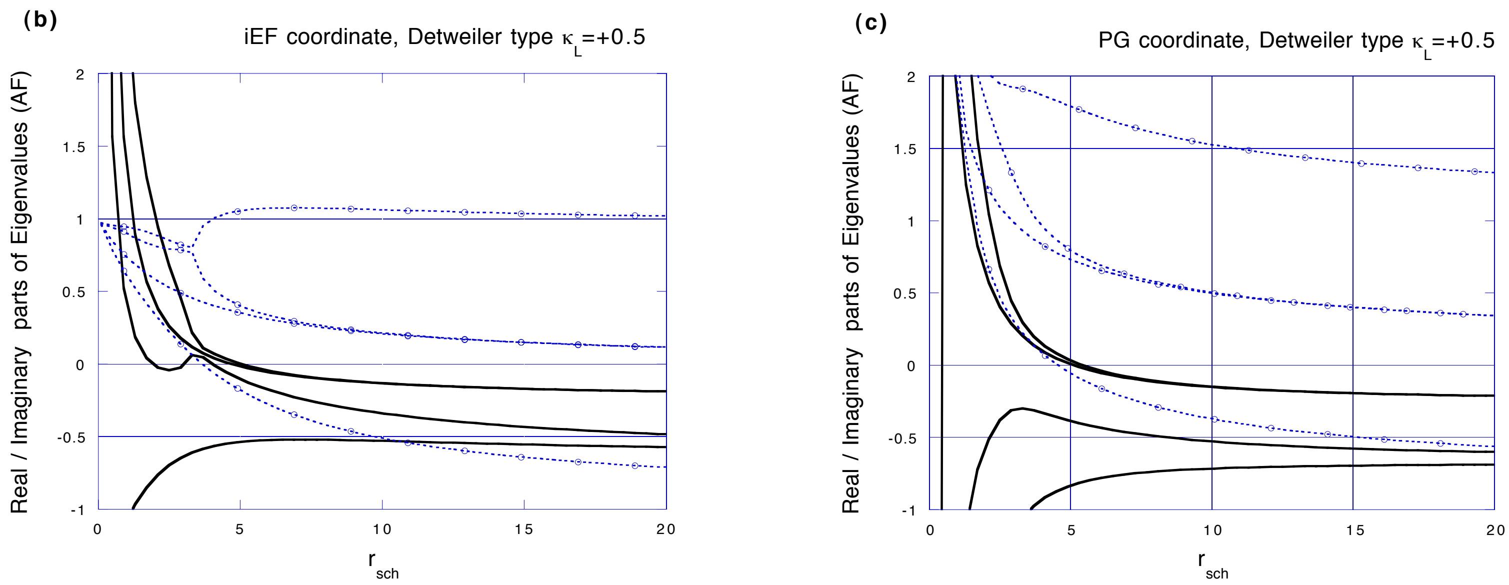



Figure 4: Similar comparison for Detweiler adjustments. $\kappa_L = +1/2$ for all plots.

"Einstein equations" are time-reversal invariant. So ...

Why all negative amplification factors (AFs) are available?

## Explanation by the time-reversal invariance (TRI)

- the adjustment of the system I,

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{K_{ij}}_{(-)} = \kappa_1 \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

preserves TRI. ... so the AFs remain zero (unchange).

- the adjustment by (a part of) Detweiler

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{\gamma_{ij}}_{(+)} = -L \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

violates TRI. ... so the AFs can become negative.

Therefore

| |
|---|
| We can break the time-reversal invariant feature of the "ADM equations". |

# Adjusted ADM systems

PRD 63 (2001) 120419, CQG 19 (2002) 1027

We adjust the standard ADM system using constraints as:

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, && (1)\\
&\quad + P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (2)\\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, && (3)\\
&\quad + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (4)
\end{aligned}$$

with constraint equations

$$\begin{aligned}
\mathcal{H} &:= R^{(3)} + K^2 - K_{ij}K^{ij}, && (5)\\
\mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K. && (6)
\end{aligned}$$

We can write the adjusted constraint propagation equations as

$$\begin{aligned}
\partial_t \mathcal{H} &= \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], && (7)\\
\partial_t \mathcal{M}_i &= \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. && (8)
\end{aligned}$$

**Table 3.** List of adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers in terms of (13) and (14). The column '1st?' and 'TRS' are the same as in table 1. The effects to amplification factors (when $\kappa > 0$) are commented for each coordinate system and for real/imaginary parts of AFs, respectively. The 'N/A' means that there is no effect due to the coordinate properties; 'not apparent' means the adjustment does not change the AFs effectively according to our conjecture; 'enl./red./min.' means enlarge/reduce/minimize, and 'Pos./Neg.' means positive/negative, respectively. These judgements are made at the $r \sim O(10M)$ region on their $t = 0$ slice.

| | | | | | Schwarzschild/isotropic coordinates | | | iEF/PG coordinates | |
|---|---|---|---|---|---|---|---|---|---|
| No | No in table 1 | | Adjustment | 1st? | TRS | Real | Imaginary | Real | Imaginary |
| 0 | 0 | – | no adjustments | yes | – | – | – | – | – |
| P-1 | 2-P | $P_{ij}$ | $-\kappa_L \alpha^3 \gamma_{ij}$ | no | no | makes 2 Neg. | not apparent | makes 2 Neg. | not apparent |
| P-2 | 3 | $P_{ij}$ | $-\kappa_L \alpha \gamma_{ij}$ | no | no | makes 2 Neg. | not apparent | makes 2 Neg. | not apparent |
| P-3 | – | $P_{ij}$ | $P_{rr} = -\kappa$ or $P_{rr} = -\kappa\alpha$ | no | no | slightly enl.Neg. | not apparent | slightly enl.Neg. | not apparent |
| P-4 | – | $P_{ij}$ | $-\kappa \gamma_{ij}$ | no | no | makes 2 Neg. | not apparent | makes 2 Neg. | not apparent |
| P-5 | – | $P_{ij}$ | $-\kappa \gamma_{rr}$ | no | no | red. Pos./enl.Neg. | not apparent | red.Pos./enl.Neg. | not apparent |
| Q-1 | – | $Q^k{}_{ij}$ | $\kappa \alpha \beta^k \gamma_{ij}$ | no | no | N/A | N/A | $\kappa \sim 1.35$ min. vals. | not apparent |
| Q-2 | – | $Q^k{}_{ij}$ | $Q^r{}_{rr} = \kappa$ | no | yes | red. abs vals. | not apparent | red. abs vals. | not apparent |
| Q-3 | – | $Q^k{}_{ij}$ | $Q^r{}_{ij} = \kappa\gamma_{ij}$ or $Q^r{}_{ij} = \kappa\alpha\gamma_{ij}$ | no | yes | red. abs vals. | not apparent | enl.Neg. | enl. vals. |
| Q-4 | – | $Q^k{}_{ij}$ | $Q^r{}_{rr} = \kappa\gamma_{rr}$ | no | yes | red. abs vals. | not apparent | red. abs vals. | not apparent |
| R-1 | 1 | $R_{ij}$ | $\kappa_F \alpha \gamma_{ij}$ | yes | yes | $\kappa_F = -1/4$ min. abs vals. | | $\kappa_F = -1/4$ min. vals. | |
| R-2 | 4 | $R_{ij}$ | $R_{rr} = -\kappa_\mu \alpha$ or $R_{rr} = -\kappa_\mu$ | yes | no | not apparent | not apparent | red.Pos./enl.Neg. | enl. vals. |
| R-3 | – | $R_{ij}$ | $R_{rr} = -\kappa\gamma_{rr}$ | yes | no | enl. vals. | not apparent | red.Pos./enl.Neg. | enl. vals. |
| S-1 | 2-S | $S^k{}_{ij}$ | $\kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}]$ | yes | no | not apparent | not apparent | not apparent | not apparent |
| S-2 | – | $S^k{}_{ij}$ | $\kappa \alpha \gamma^{lk} (\partial_l \gamma_{ij})$ | yes | no | makes 2 Neg. | not apparent | makes 2 Neg. | not apparent |
| p-1 | – | $p^k{}_{ij}$ | $p^r{}_{ij} = -\kappa\alpha\gamma_{ij}$ | no | no | red. Pos. | red. vals. | red. Pos. | enl. vals. |
| p-2 | – | $p^k{}_{ij}$ | $p^r{}_{rr} = \kappa\alpha$ | no | no | red. Pos. | red. vals. | red.Pos/enl.Neg. | enl. vals. |
| p-3 | – | $p^k{}_{ij}$ | $p^r{}_{rr} = \kappa\alpha\gamma_{rr}$ | no | no | makes 2 Neg. | enl. vals. | red. Pos. vals. | red. vals. |
| q-1 | – | $q^{kl}{}_{ij}$ | $q^{rr}{}_{ij} = \kappa\alpha\gamma_{ij}$ | no | no | $\kappa = 1/2$ min. vals. | red. vals. | not apparent | enl. vals. |
| q-2 | – | $q^{kl}{}_{ij}$ | $q^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$ | no | yes | red. abs vals. | not apparent | not apparent | not apparent |
| r-1 | – | $r^k{}_{ij}$ | $r^r{}_{ij} = \kappa\alpha\gamma_{ij}$ | no | yes | not apparent | not apparent | not apparent | enl. vals. |
| r-2 | – | $r^k{}_{ij}$ | $r^r{}_{rr} = -\kappa\alpha$ | no | yes | red. abs vals. | enl. vals. | red. abs vals. | enl. vals. |
| r-3 | – | $r^k{}_{ij}$ | $r^r{}_{rr} = -\kappa\alpha\gamma_{rr}$ | no | yes | red. abs vals. | enl. vals. | red. abs vals. | enl. vals. |
| s-1 | 2-s | $s^{kl}{}_{ij}$ | $\kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}]$ | no | no | makes 4 Neg. | not apparent | makes 4 Neg. | not apparent |
| s-2 | – | $s^{kl}{}_{ij}$ | $s^{rr}{}_{ij} = -\kappa\alpha\gamma_{ij}$ | no | no | makes 2 Neg. | red. vals. | makes 2 Neg. | red. vals. |
| s-3 | – | $s^{kl}{}_{ij}$ | $s^{rr}{}_{rr} = -\kappa\alpha\gamma_{rr}$ | no | no | makes 2 Neg. | red. vals. | makes 2 Neg. | red. vals. |

# Numerical Tests (method)

- Cactus-based original “GR” code
  http://www.cactuscode.org/
  [CactusBase+CactusPUGH+GR]
- 3+1dim,  linear wave evolution
  (Teukolsky wave)
- harmonic slice
- periodic boundary, [-3,+3]
- iterative Crank-Nicholson method
- 12^3, 24^3, 48^3, 96^3

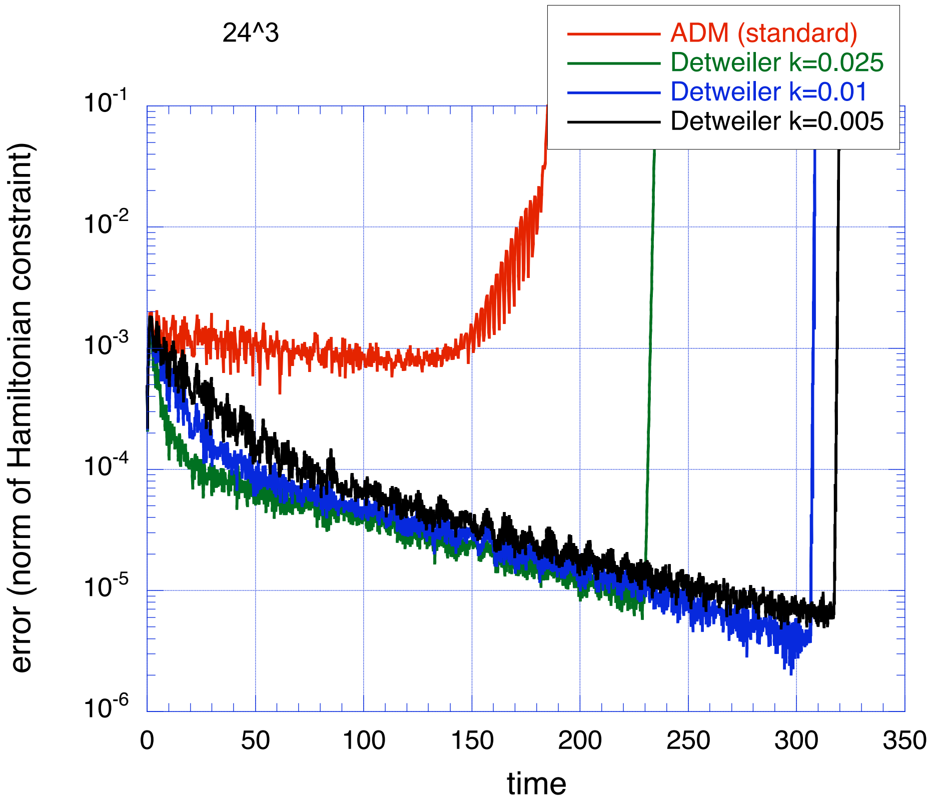
Towards standard testbeds for numerical relativity
Mexico Numerical Relativity Workshop 2002 Participants
CQG 21 (2004) 589-613

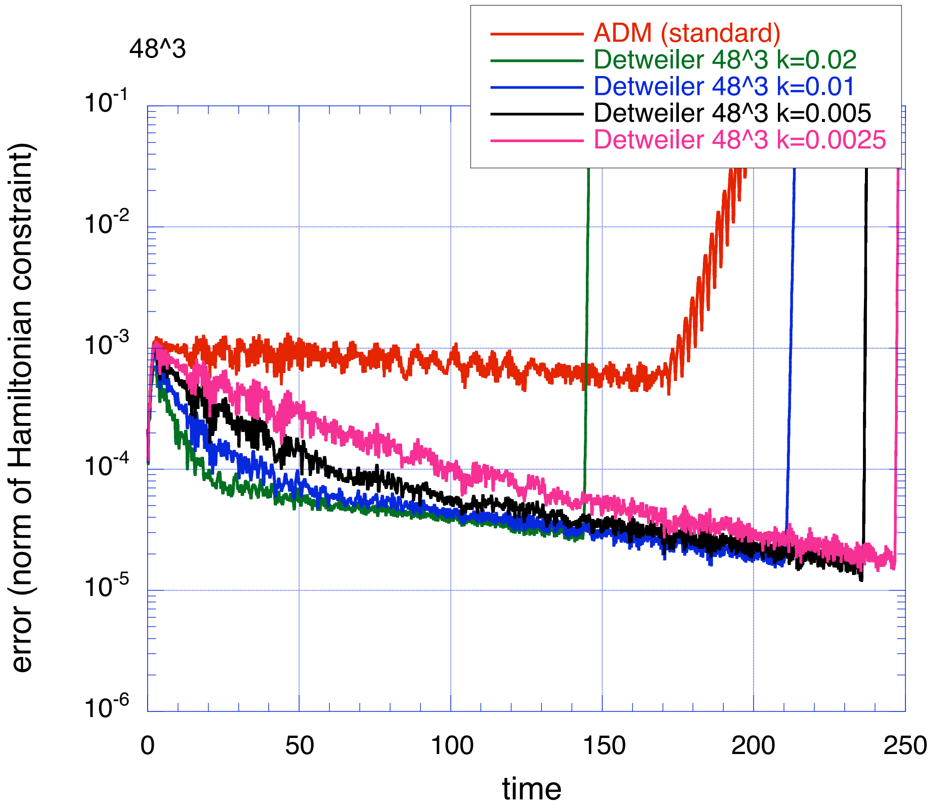
# Numerical Tests (Detweiler-type)

PRD35(1987)1095

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H}$$

$$\begin{aligned}\partial_t K_{ij} = & \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\ & + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}] \mathcal{M}_k \\ & + \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}] (\nabla_k \mathcal{M}_l)\end{aligned}$$

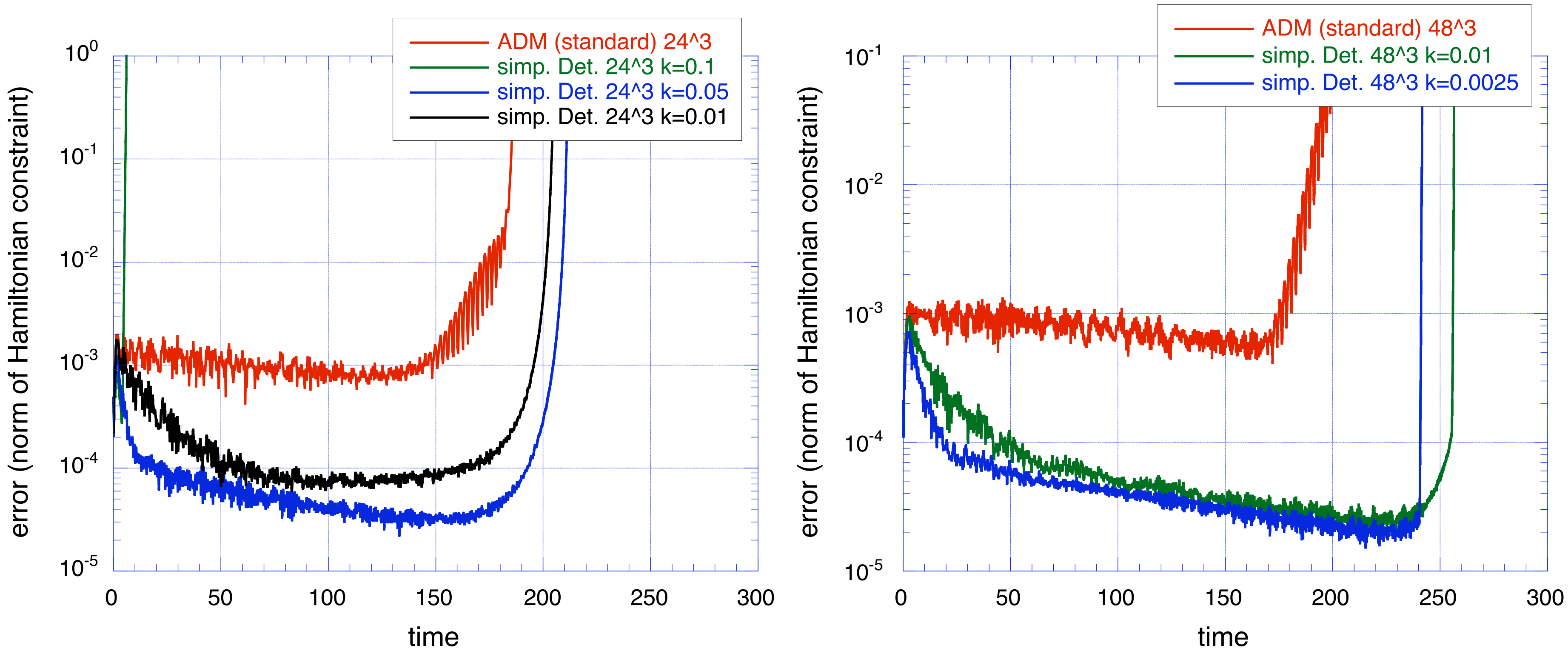


# Numerical Tests (Simplified Detweiler)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha \gamma_{ij} \mathcal{H}$$

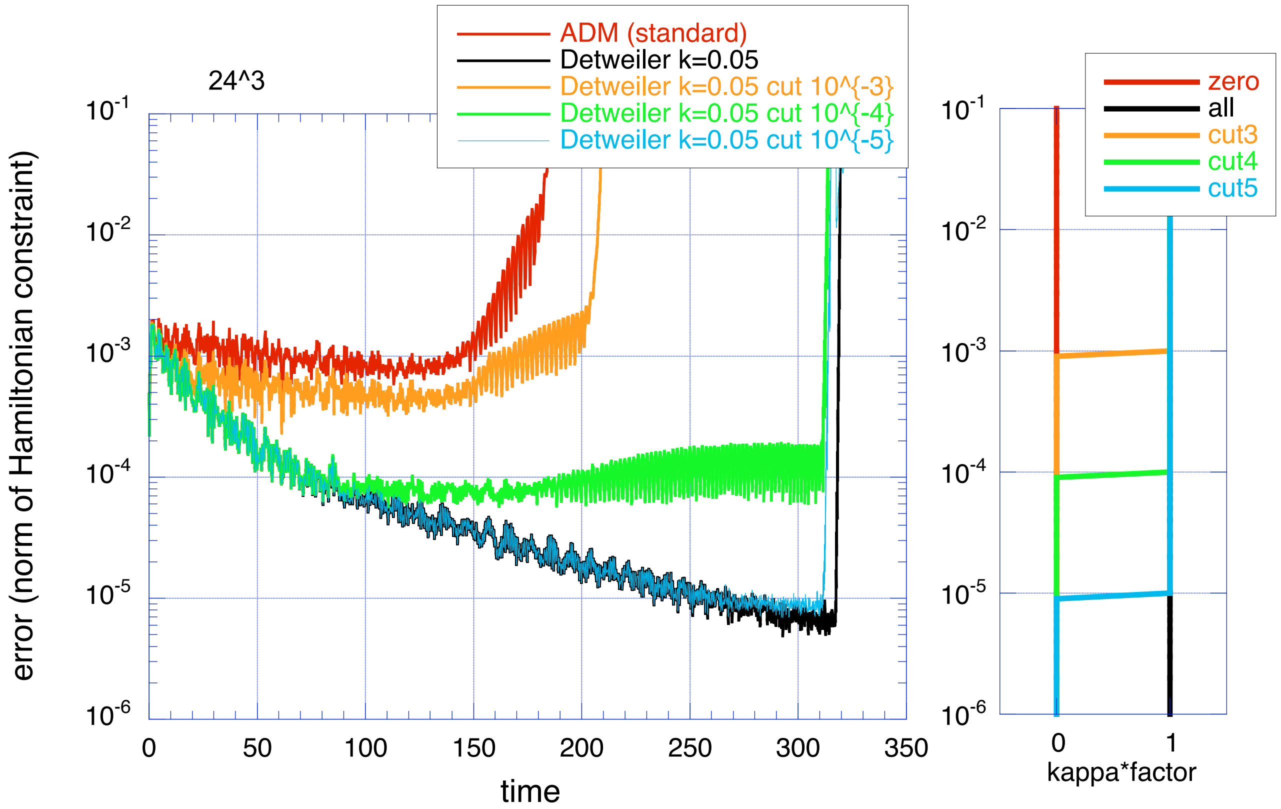
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$



# Numerical Tests (Detweiler, k-adjust)

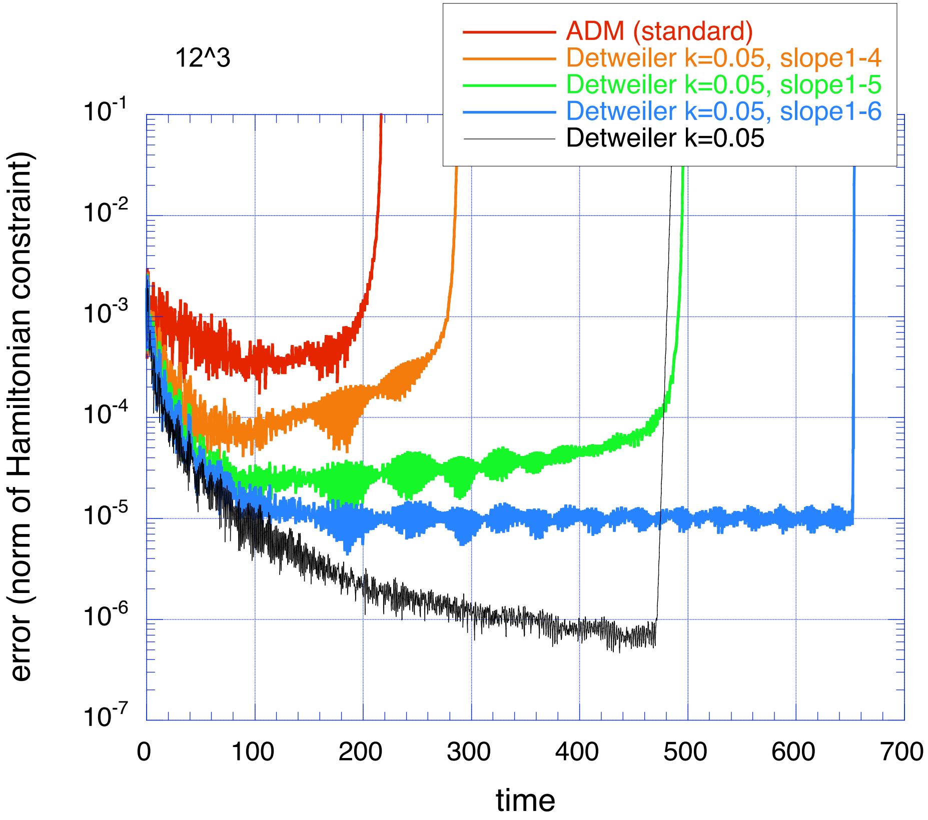
$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H}$$

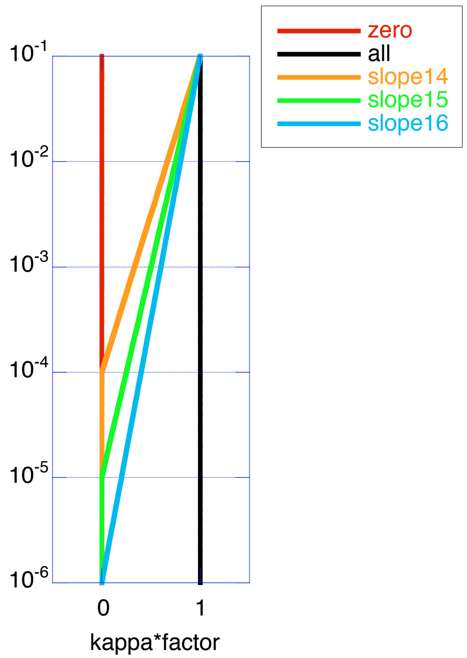
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$
$$+\kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}] \mathcal{M}_k$$
$$+\kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}] (\nabla_k \mathcal{M}_l)$$



# Numerical Tests (Detweiler, k-adjust)

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H} \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i} \alpha) \delta^k_{j)} - (\partial_l \alpha) \gamma_{ij} \gamma^{kl}] \mathcal{M}_k \\
&\quad + \kappa_L \alpha^3 [\delta^k_{(i} \delta^l_{j)} - (1/3) \gamma_{ij} \gamma^{kl}] (\nabla_k \mathcal{M}_l)
\end{aligned}
$$

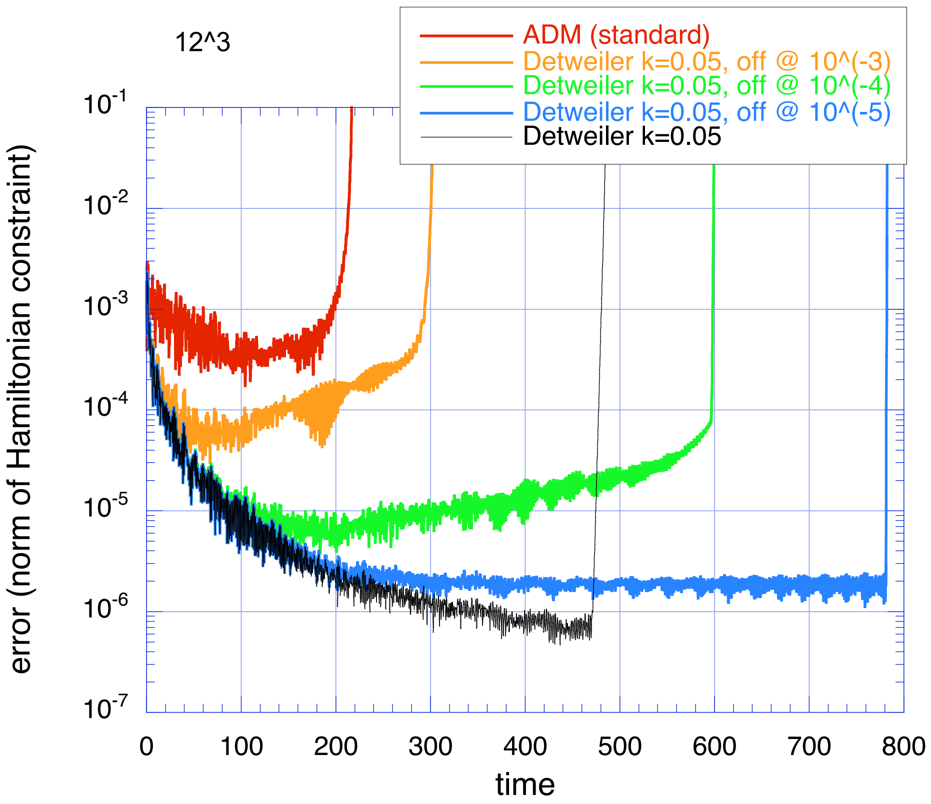


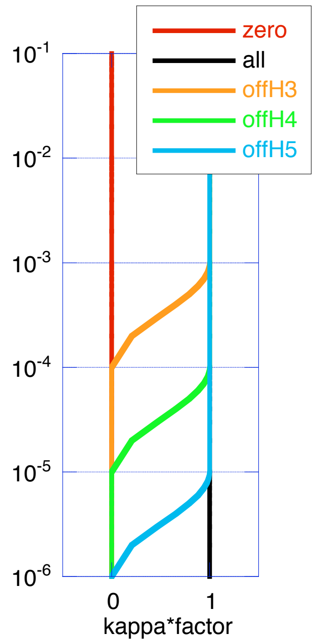



# Numerical Tests (Detweiler, k-adjust)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H}$$

$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$
$$+\kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}] \mathcal{M}_k$$
$$+\kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}] (\nabla_k \mathcal{M}_l)$$



APCTP Winter School, January 25-26, 2008

# Formulation Problem in Numerical Relativity

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신카이 히사아키

1. Introduction
2. The Standard Approach to Numerical Relativity
    - ADM/BSSN/hyperbolic formulations
3. Robust system for Constraint Violation
    - Adjusted systems
    - Adjusted ADM system -- why the standard ADM brows up?
    - Adjusted BSSN system -- should be better than BSSN
4. Outlook

## strategy 1 Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation

T. Nakamura, K. Oohara and Y. Kojima, Prog. Theor. Phys. Suppl. **90**, 1 (1987)
M. Shibata and T. Nakamura, Phys. Rev. D **52**, 5428 (1995)
T.W. Baumgarte and S.L. Shapiro, Phys. Rev. D **59**, 024007 (1999)

The popular approach. Nakamura's idea in 1980s.
BSSN is a tricky nickname. BS (1999) introduced a paper of SN (1995).

- define new set of variables $(\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$, instead of the ADM's $(\gamma_{ij}, K_{ij})$ where

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},$$

  and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

- The set of evolution equations become

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

Momentum constraint was used in $\Gamma^i$-eq.

- Calculate Riemann tensor as

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^{\phi}_{ij}$$

$$R^{\phi}_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi)$$

$$\tilde{R}_{ij} = -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}^{lm}\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}$$

- Constraints are $\mathcal{H}, \mathcal{M}_i$.
  But thre are additional ones, $\mathcal{G}^i, \mathcal{A}, \mathcal{S}$.

Hamiltonian and the momentum constraint equations

$$\mathcal{H}^{BSSN} = R^{BSSN} + K^2 - K_{ij}K^{ij}, \quad (1)$$

$$\mathcal{M}^{BSSN}_i = \mathcal{M}^{ADM}_i, \quad (2)$$

Additionally, we regard the following three as the constraints:

$$\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \quad (3)$$

$$\mathcal{A} = \tilde{A}_{ij}\tilde{\gamma}^{ij}, \quad (4)$$

$$\mathcal{S} = \tilde{\gamma} - 1, \quad (5)$$

Why BSSN better than ADM?
Is the BSSN best? Are there any alternatives?

## Constraints in BSSN system

The normal Hamiltonian and momentum constraints

$$\mathcal{H}^{BSSN} = R^{BSSN} + K^2 - K_{ij}K^{ij}, \tag{1}$$

$$\mathcal{M}_i^{BSSN} = \mathcal{M}_i^{ADM}, \tag{2}$$

Additionally, we regard the following three as the constraints:

$$\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \tag{3}$$

$$\mathcal{A} = \tilde{A}_{ij}\tilde{\gamma}^{ij}, \tag{4}$$

$$\mathcal{S} = \tilde{\gamma} - 1, \tag{5}$$

## Adjustments in evolution equations

$$\partial_t^B \varphi = \partial_t^A \varphi + (1/6)\alpha\mathcal{A} - (1/12)\tilde{\gamma}^{-1}(\partial_j \mathcal{S})\beta^j, \tag{6}$$

$$\partial_t^B \tilde{\gamma}_{ij} = \partial_t^A \tilde{\gamma}_{ij} - (2/3)\alpha\tilde{\gamma}_{ij}\mathcal{A} + (1/3)\tilde{\gamma}^{-1}(\partial_k \mathcal{S})\beta^k\tilde{\gamma}_{ij}, \tag{7}$$

$$\partial_t^B K = \partial_t^A K - (2/3)\alpha K\mathcal{A} - \alpha\mathcal{H}^{BSSN} + \alpha e^{-4\varphi}(\tilde{D}_j\mathcal{G}^j), \tag{8}$$

$$\begin{aligned}\partial_t^B \tilde{A}_{ij} =\ & \partial_t^A \tilde{A}_{ij} + ((1/3)\alpha\tilde{\gamma}_{ij}K - (2/3)\alpha\tilde{A}_{ij})\mathcal{A} + \alpha e^{-4\varphi}((1/2)(\partial_k\tilde{\gamma}_{ij}) - (1/6)\tilde{\gamma}_{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}))\mathcal{G}^k \\ & + \alpha e^{-4\varphi}\tilde{\gamma}_{k(i}(\partial_{j)}\mathcal{G}^k) - (1/3)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}(\partial_k\mathcal{G}^k)\end{aligned} \tag{9}$$

$$\begin{aligned}\partial_t^B \tilde{\Gamma}^i =\ & \partial_t^A \tilde{\Gamma}^i - ((2/3)(\partial_j\alpha)\tilde{\gamma}^{ji} + (2/3)\alpha(\partial_j\tilde{\gamma}^{ji}) + (1/3)\alpha\tilde{\gamma}^{ji}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) - 4\alpha\tilde{\gamma}^{ij}(\partial_j\varphi))\mathcal{A} \\ & - (2/3)\alpha\tilde{\gamma}^{ji}(\partial_j\mathcal{A}) + 2\alpha\tilde{\gamma}^{ij}\mathcal{M}_j - (1/2)(\partial_k\beta^i)\tilde{\gamma}^{kj}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (1/6)(\partial_j\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}) \\ & + (1/3)(\partial_k\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (5/6)\beta^k\tilde{\gamma}^{-2}\tilde{\gamma}^{ij}(\partial_k\mathcal{S})(\partial_j\mathcal{S}) + (1/2)\beta^k\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\mathcal{S}) \\ & + (1/3)\beta^k\tilde{\gamma}^{-1}(\partial_j\tilde{\gamma}^{ji})(\partial_k\mathcal{S}).\end{aligned} \tag{10}$$

## A Full set of BSSN constraint propagation eqs.

$$
\partial_t^{BS} \begin{pmatrix} \mathcal{H}^{BS} \\ \mathcal{M}_i \\ \mathcal{G}^i \\ \mathcal{S} \\ \mathcal{A} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ -(1/3)(\partial_i \alpha) + (1/6)\partial_i & \alpha K & A_{23} & 0 & A_{25} \\ 0 & \alpha \tilde{\gamma}^{ij} & 0 & A_{34} & A_{35} \\ 0 & 0 & 0 & \beta^k(\partial_k \mathcal{S}) & -2\alpha\tilde{\gamma} \\ 0 & 0 & 0 & 0 & \alpha K + \beta^k \partial_k \end{pmatrix} \begin{pmatrix} \mathcal{H}^{BS} \\ \mathcal{M}_j \\ \mathcal{G}^j \\ \mathcal{S} \\ \mathcal{A} \end{pmatrix}
$$

$$
\begin{aligned}
A_{11} &= +(2/3)\alpha K + (2/3)\alpha \mathcal{A} + \beta^k \partial_k \\
A_{12} &= -4e^{-4\varphi}\alpha(\partial_k \varphi)\tilde{\gamma}^{kj} - 2e^{-4\varphi}(\partial_k \alpha)\tilde{\gamma}^{jk} \\
A_{13} &= -2\alpha e^{-4\varphi}\tilde{A}^k{}_j \partial_k - \alpha e^{-4\varphi}(\partial_j \tilde{A}_{kl})\tilde{\gamma}^{kl} - e^{-4\varphi}(\partial_j \alpha)\mathcal{A} - e^{-4\varphi}\beta^k \partial_k \partial_j - (1/2)e^{-4\varphi}\beta^k \tilde{\gamma}^{-1}(\partial_j \mathcal{S})\partial_k \\
&\quad +(1/6)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_j \beta^k)(\partial_k \mathcal{S}) - (2/3)e^{-4\varphi}(\partial_k \beta^k)\partial_j \\
A_{14} &= 2\alpha e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{lk}(\partial_l \varphi)\mathcal{A}\partial_k + (1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l \mathcal{A})\tilde{\gamma}^{lk}\partial_k + (1/2)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l \alpha)\tilde{\gamma}^{lk}\mathcal{A}\partial_k + (1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m \tilde{\gamma}^{lk}\partial_m \partial_l \partial_k \\
&\quad -(5/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^m \tilde{\gamma}^{lk}(\partial_m \mathcal{S})\partial_l \partial_k + e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m(\partial_m \tilde{\gamma}^{lk})\partial_l \partial_k + (1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^i(\partial_j \partial_i \tilde{\gamma}^{jk})\partial_k \\
&\quad +(3/4)e^{-4\varphi}\tilde{\gamma}^{-3}\beta^i \tilde{\gamma}^{jk}(\partial_i \mathcal{S})(\partial_j \mathcal{S})\partial_k - (3/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^i(\partial_i \tilde{\gamma}^{jk})(\partial_j \mathcal{S})\partial_k + (1/3)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{pj}(\partial_j \beta^k)\partial_p \partial_k \\
&\quad -(5/12)e^{-4\varphi}\tilde{\gamma}^{-2}\tilde{\gamma}^{jk}(\partial_k \beta^i)(\partial_i \mathcal{S})\partial_j + (1/3)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_k \tilde{\gamma}^{ij})(\partial_j \beta^k)\partial_i - (1/6)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{mk}(\partial_k \partial_l \beta^l)\partial_m \\
A_{15} &= (4/9)\alpha K \mathcal{A} - (8/9)\alpha K^2 + (4/3)\alpha e^{-4\varphi}(\partial_i \partial_j \varphi)\tilde{\gamma}^{ij} + (8/3)\alpha e^{-4\varphi}(\partial_k \varphi)(\partial_l \tilde{\gamma}^{lk}) + \alpha e^{-4\varphi}(\partial_j \tilde{\gamma}^{jk})\partial_k \\
&\quad +8\alpha e^{-4\varphi}\tilde{\gamma}^{jk}(\partial_j \varphi)\partial_k + \alpha e^{-4\varphi}\tilde{\gamma}^{jk}\partial_j \partial_k + 8e^{-4\varphi}(\partial_l \alpha)(\partial_k \varphi)\tilde{\gamma}^{lk} + e^{-4\varphi}(\partial_l \alpha)(\partial_k \tilde{\gamma}^{lk}) + 2e^{-4\varphi}(\partial_l \alpha)\tilde{\gamma}^{lk}\partial_k \\
&\quad +e^{-4\varphi}\tilde{\gamma}^{lk}(\partial_l \partial_k \alpha) \\
A_{23} &= \alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k \varphi)(\partial_j \tilde{\gamma}_{mi}) - (1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}(\partial_j \tilde{\gamma}_{mi}) \\
&\quad +(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_k \partial_j \tilde{\gamma}_{mi}) + (1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-2}(\partial_i \mathcal{S})(\partial_j \mathcal{S}) - (1/4)\alpha e^{-4\varphi}(\partial_i \tilde{\gamma}_{kl})(\partial_j \tilde{\gamma}^{kl}) + \alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k \varphi)\tilde{\gamma}_{ji}\partial_m \\
&\quad +\alpha e^{-4\varphi}(\partial_j \varphi)\partial_i - (1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}\tilde{\gamma}_{ji}\partial_m + \alpha e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\Gamma}_{ijk}\partial_m + (1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{lk}\tilde{\gamma}_{ji}\partial_k \partial_l \\
&\quad +(1/2)e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_j \tilde{\gamma}_{im})(\partial_k \alpha) + (1/2)e^{-4\varphi}(\partial_j \alpha)\partial_i + (1/2)e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\gamma}_{ji}(\partial_k \alpha)\partial_m \\
A_{25} &= -\tilde{A}^k{}_i(\partial_k \alpha) + (1/9)(\partial_i \alpha)K + (4/9)\alpha(\partial_i K) + (1/9)\alpha K \partial_i - \alpha \tilde{A}^k{}_i \partial_k \\
A_{34} &= -(1/2)\beta^k \tilde{\gamma}^{il}\tilde{\gamma}^{-2}(\partial_l \mathcal{S})\partial_k - (1/2)(\partial_l \beta^i)\tilde{\gamma}^{lk}\tilde{\gamma}^{-1}\partial_k + (1/3)(\partial_l \beta^l)\tilde{\gamma}^{ik}\tilde{\gamma}^{-1}\partial_k - (1/2)\beta^l \tilde{\gamma}^{in}(\partial_l \tilde{\gamma}_{mn})\tilde{\gamma}^{mk}\tilde{\gamma}^{-1}\partial_k \\
&\quad +(1/2)\beta^k \tilde{\gamma}^{il}\tilde{\gamma}^{-1}\partial_l \partial_k \\
A_{35} &= -(\partial_k \alpha)\tilde{\gamma}^{ik} + 4\alpha \tilde{\gamma}^{ik}(\partial_k \varphi) - \alpha \tilde{\gamma}^{ik}\partial_k
\end{aligned}
$$

## BSSN Constraint propagation analysis in flat spacetime

- The set of the constraint propagation equations, $\partial_t(\mathcal{H}^{BSSN}, \mathcal{M}_i, \mathcal{G}^i, \mathcal{A}, \mathcal{S})^T$ ?
- For the flat background metric $g_{\mu\nu} = \eta_{\mu\nu}$, the first order perturbation equations of (6)-(10):

$$\partial_t {}^{(1)}\varphi = -(1/6)^{(1)}K + (1/6)\kappa_\varphi {}^{(1)}\mathcal{A} \tag{11}$$
$$\partial_t {}^{(1)}\tilde{\gamma}_{ij} = -2^{(1)}\tilde{A}_{ij} - (2/3)\kappa_{\tilde{\gamma}}\delta_{ij}{}^{(1)}\mathcal{A} \tag{12}$$
$$\partial_t {}^{(1)}K = -(\partial_j\partial_j {}^{(1)}\alpha) + \kappa_{K1}\partial_j {}^{(1)}\mathcal{G}^j - \kappa_{K2}{}^{(1)}\mathcal{H}^{BSSN} \tag{13}$$
$$\partial_t {}^{(1)}\tilde{A}_{ij} = {}^{(1)}(R_{ij}^{BSSN})^{TF} - {}^{(1)}(\tilde{D}_i\tilde{D}_j\alpha)^{TF} + \kappa_{A1}\delta_{k(i}(\partial_{j)}{}^{(1)}\mathcal{G}^k) - (1/3)\kappa_{A2}\delta_{ij}(\partial_k {}^{(1)}\mathcal{G}^k) \tag{14}$$
$$\partial_t {}^{(1)}\tilde{\Gamma}^i = -(4/3)(\partial_i {}^{(1)}K) - (2/3)\kappa_{\tilde{\Gamma}1}(\partial_i {}^{(1)}\mathcal{A}) + 2\kappa_{\tilde{\Gamma}2}{}^{(1)}\mathcal{M}_i \tag{15}$$

We express the adjustements as

$$\kappa_{adj} := (\kappa_\varphi, \kappa_{\tilde{\gamma}}, \kappa_{K1}, \kappa_{K2}, \kappa_{A1}, \kappa_{A2}, \kappa_{\tilde{\Gamma}1}, \kappa_{\tilde{\Gamma}2}). \tag{16}$$

- Constraint propagation equations at the first order in the flat spacetime:

$$\partial_t {}^{(1)}\mathcal{H}^{BSSN} = (\kappa_{\tilde{\gamma}} - (2/3)\kappa_{\tilde{\Gamma}1} - (4/3)\kappa_\varphi + 2)\,\partial_j\partial_j {}^{(1)}\mathcal{A} + 2(\kappa_{\tilde{\Gamma}2} - 1)(\partial_j {}^{(1)}\mathcal{M}_j), \tag{17}$$
$$\begin{aligned}\partial_t {}^{(1)}\mathcal{M}_i = {}& (-(2/3)\kappa_{K1} + (1/2)\kappa_{A1} - (1/3)\kappa_{A2} + (1/2))\,\partial_i\partial_j {}^{(1)}\mathcal{G}^j \\ & +(1/2)\kappa_{A1}\partial_j\partial_j {}^{(1)}\mathcal{G}^i + ((2/3)\kappa_{K2} - (1/2))\,\partial_i {}^{(1)}\mathcal{H}^{BSSN},\end{aligned} \tag{18}$$
$$\partial_t {}^{(1)}\mathcal{G}^i = 2\kappa_{\tilde{\Gamma}2}{}^{(1)}\mathcal{M}_i + (-(2/3)\kappa_{\tilde{\Gamma}1} - (1/3)\kappa_{\tilde{\gamma}})(\partial_i {}^{(1)}\mathcal{A}), \tag{19}$$
$$\partial_t {}^{(1)}\mathcal{S} = -2\kappa_{\tilde{\gamma}}{}^{(1)}\mathcal{A}, \tag{20}$$
$$\partial_t {}^{(1)}\mathcal{A} = (\kappa_{A1} - \kappa_{A2})(\partial_j {}^{(1)}\mathcal{G}^j). \tag{21}$$

# Effect of adjustments

| No. | | Constraints (number of components) | | | | | Amplification Factors (AFs) in Minkowskii background |
|---|---|---|---|---|---|---|---|
| | | $\mathcal{H}$ (1) | $\mathcal{M}_i$ (3) | $\mathcal{G}^i$ (3) | $\mathcal{A}$ (1) | $\mathcal{S}$ (1) | |
| 0. | standard ADM | use | use | - | - | - | $(0, 0, \Im, \Im)$ |
| 1. | BSSN no adjustment | use | use | use | use | use | $(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$ |
| 2. | the BSSN | use+adj | use+adj | use+adj | use+adj | use+adj | $(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$ |
| 3. | no $\mathcal{S}$ adjustment | use+adj | use+adj | use+adj | use+adj | use | no difference in flat background |
| 4. | no $\mathcal{A}$ adjustment | use+adj | use+adj | use+adj | use | use+adj | $(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$ |
| 5. | no $\mathcal{G}^i$ adjustment | use+adj | use+adj | use | use+adj | use+adj | $(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$ |
| 6. | no $\mathcal{M}_i$ adjustment | use+adj | use | use+adj | use+adj | use+adj | $(0, 0, 0, 0, 0, 0, 0, \Re, \Re)$ Growing modes! |
| 7. | no $\mathcal{H}$ adjustment | use | use+adj | use+adj | use+adj | use+adj | $(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$ |
| 8. | ignore $\mathcal{G}^i$, $\mathcal{A}$, $\mathcal{S}$ | use+adj | use+adj | - | - | - | $(0, 0, 0, 0)$ |
| 9. | ignore $\mathcal{G}^i$, $\mathcal{A}$ | use+adj | use+adj | use+adj | - | - | $(0, \Im, \Im, \Im, \Im, \Im, \Im)$ |
| 10. | ignore $\mathcal{G}^i$ | use+adj | use+adj | - | use+adj | use+adj | $(0, 0, 0, 0, 0, 0)$ |
| 11. | ignore $\mathcal{A}$ | use+adj | use+adj | use+adj | - | use+adj | $(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$ |
| 12. | ignore $\mathcal{S}$ | use+adj | use+adj | use+adj | use+adj | - | $(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$ |

## New Proposals :: Improved (adjusted) BSSN systems

### TRS breaking adjustments

In order to break time reversal symmetry (TRS) of the evolution eqs, to adjust $\partial_t\phi$,$\partial_t\tilde{\gamma}_{ij}$,$\partial_t\tilde{\Gamma}^i$ using $\mathcal{S}, \mathcal{G}^i$, or to adjust $\partial_t K, \partial_t \tilde{A}_{ij}$ using $\tilde{\mathcal{A}}$.

$$
\begin{aligned}
\partial_t\phi &= \partial_t^{BS}\phi + \kappa_{\phi\mathcal{H}}\alpha\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\alpha\tilde{D}_k\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}\alpha\mathcal{S} + \kappa_{\phi\mathcal{S}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{S} \\
\partial_t\tilde{\gamma}_{ij} &= \partial_t^{BS}\tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}\mathcal{H}}\alpha\tilde{\gamma}_{ij}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{G}2}\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{S}1}\alpha\tilde{\gamma}_{ij}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{S} \\
\partial_t K &= \partial_t^{BS}K + \kappa_{K\mathcal{M}}\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k) + \kappa_{K\tilde{\mathcal{A}}1}\alpha\tilde{\mathcal{A}} + \kappa_{K\tilde{\mathcal{A}}2}\alpha\tilde{D}^j\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{A}_{ij} &= \partial_t^{BS}\tilde{A}_{ij} + \kappa_{A\mathcal{M}1}\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k) + \kappa_{A\mathcal{M}2}\alpha(\tilde{D}_{(i}\mathcal{M}_{j)}) + \kappa_{A\tilde{\mathcal{A}}1}\alpha\tilde{\gamma}_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{\mathcal{A}}2}\alpha\tilde{D}_i\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{\Gamma}^i &= \partial_t^{BS}\tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}\mathcal{H}}\alpha\tilde{D}^i\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}\alpha\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\alpha\tilde{D}^i\mathcal{H}^{BS}
\end{aligned}
$$

or in the flat background

$$
\begin{aligned}
\partial_t^{ADJ(1)}\phi &= +\kappa_{\phi\mathcal{H}}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\partial_k{}^{(1)}\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}{}^{(1)}\mathcal{S} + \kappa_{\phi\mathcal{S}2}\partial_j\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}\tilde{\gamma}_{ij} &= +\kappa_{\tilde{\gamma}\mathcal{H}}\delta_{ij}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{G}^k + (1/2)\kappa_{\tilde{\gamma}\mathcal{G}2}(\partial_j{}^{(1)}\mathcal{G}^i + \partial_i{}^{(1)}\mathcal{G}^j) + \kappa_{\tilde{\gamma}\mathcal{S}1}\delta_{ij}{}^{(1)}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\partial_i\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}K &= +\kappa_{K\mathcal{M}}\partial_j{}^{(1)}\mathcal{M}_j + \kappa_{K\tilde{\mathcal{A}}1}{}^{(1)}\tilde{\mathcal{A}} + \kappa_{K\tilde{\mathcal{A}}2}\partial_j\partial_j{}^{(1)}\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{A}_{ij} &= +\kappa_{A\mathcal{M}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{M}_k + (1/2)\kappa_{A\mathcal{M}2}(\partial_i\mathcal{M}_j + \partial_j\mathcal{M}_i) + \kappa_{A\tilde{\mathcal{A}}1}\delta_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{\mathcal{A}}2}\partial_i\partial_j\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{\Gamma}^i &= +\kappa_{\tilde{\Gamma}\mathcal{H}}\partial_i{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\partial_j\partial_j{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\partial_i\partial_j{}^{(1)}\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\partial_i{}^{(1)}\mathcal{S}
\end{aligned}
$$

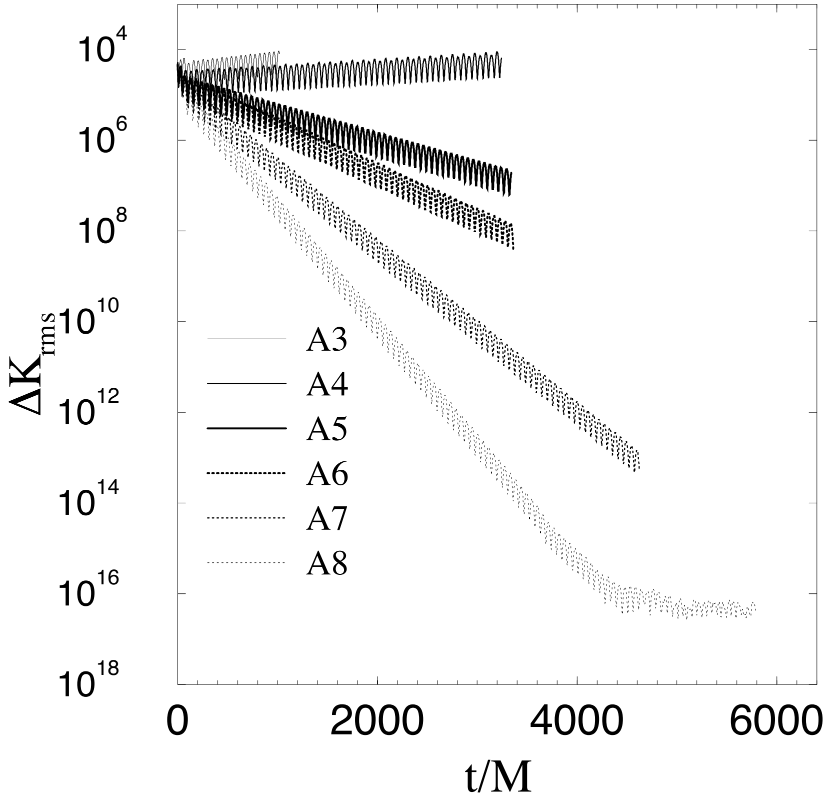
# Constraint Amplification Factors with each adjustment

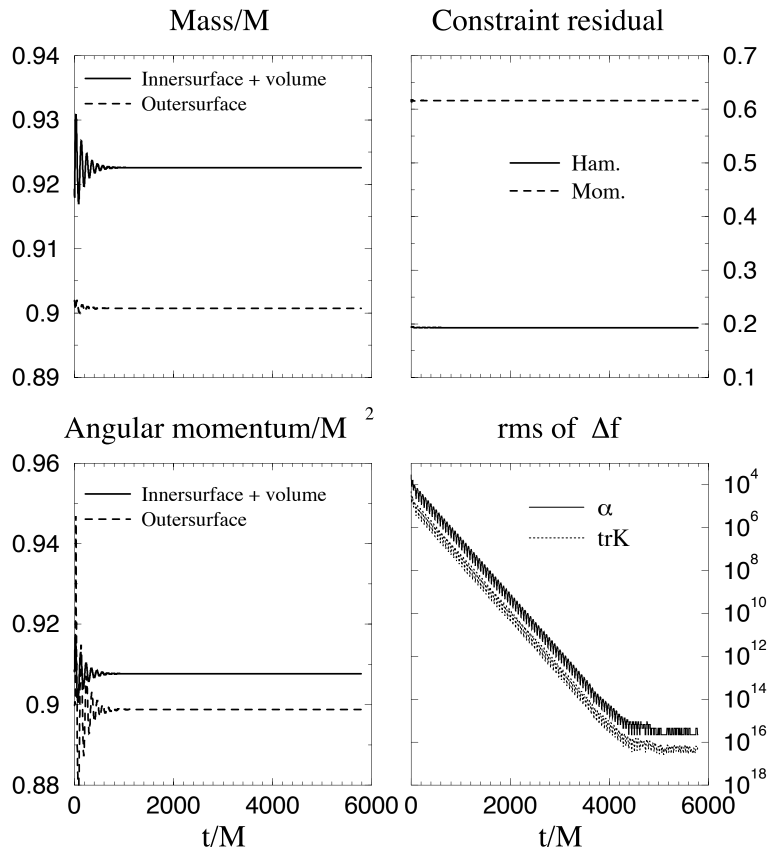
| adjustment | | CAFs | diag? | effect of the adjustment | |
|---|---|---|---|---|---|
| $\partial_t\phi$ | $\kappa_{\phi\mathcal{H}}\,\alpha\mathcal{H}$ | $(0,0,\pm\sqrt{-k^2}(*3), 8\kappa_{\phi\mathcal{H}}k^2)$ | no | $\kappa_{\phi\mathcal{H}}<0$ makes 1 Neg. | |
| $\partial_t\phi$ | $\kappa_{\phi\mathcal{G}}\,\alpha\tilde{D}_k\mathcal{G}^k$ | $(0,0,\pm\sqrt{-k^2}(*2)$, long expressions) | yes | $\kappa_{\phi\mathcal{G}}<0$ makes 2 Neg. 1 Pos. | |
| $\partial_t\tilde{\gamma}_{ij}$ | $\kappa_{SD}\,\alpha\tilde{\gamma}_{ij}\mathcal{H}$ | $(0,0,\pm\sqrt{-k^2}(*3),(3/2)\kappa_{SD}k^2)$ | yes | $\kappa_{SD}<0$ makes 1 Neg. | Case (B) |
| $\partial_t\tilde{\gamma}_{ij}$ | $\kappa_{\tilde{\gamma}\mathcal{G}1}\,\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k$ | $(0,0,\pm\sqrt{-k^2}(*2)$, long expressions) | yes | $\kappa_{\tilde{\gamma}\mathcal{G}1}>0$ makes 1 Neg. | |
| $\partial_t\tilde{\gamma}_{ij}$ | $\kappa_{\tilde{\gamma}\mathcal{G}2}\,\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k$ | (0,0, $(1/4)k^2\kappa_{\tilde{\gamma}\mathcal{G}2}\pm\sqrt{k^2(-1+k^2\kappa_{\tilde{\gamma}\mathcal{G}2}/16)}(*2)$,<br>long expressions) | yes | $\kappa_{\tilde{\gamma}\mathcal{G}2}<0$ makes 6 Neg. 1 Pos. | Case (E1) |
| $\partial_t\tilde{\gamma}_{ij}$ | $\kappa_{\tilde{\gamma}\mathcal{S}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{S}$ | $(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{\tilde{\gamma}\mathcal{S}1})$ | no | $\kappa_{\tilde{\gamma}\mathcal{S}1}<0$ makes 1 Neg. | |
| $\partial_t\tilde{\gamma}_{ij}$ | $\kappa_{\tilde{\gamma}\mathcal{S}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{S}$ | $(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{\tilde{\gamma}\mathcal{S}2}k^2)$ | no | $\kappa_{\tilde{\gamma}\mathcal{S}2}>0$ makes 1 Neg. | |
| $\partial_t K$ | $\kappa_{K\mathcal{M}}\,\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k)$ | $(0,0,0,\pm\sqrt{-k^2}(*2)$,<br>$(1/3)\kappa_{K\mathcal{M}}k^2\pm(1/3)\sqrt{k^2(-9+k^2\kappa_{K\mathcal{M}}^2)})$ | no | $\kappa_{K\mathcal{M}}<0$ makes 2 Neg. | |
| $\partial_t\tilde{A}_{ij}$ | $\kappa_{A\mathcal{M}1}\,\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k)$ | $(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{M}1}k^2)$ | yes | $\kappa_{A\mathcal{M}1}>0$ makes 1 Neg. | |
| $\partial_t\tilde{A}_{ij}$ | $\kappa_{A\mathcal{M}2}\,\alpha(\tilde{D}_{(i}\mathcal{M}_{j)})$ | (0,0, $-k^2\kappa_{A\mathcal{M}2}/4\pm\sqrt{k^2(-1+k^2\kappa_{A\mathcal{M}2}/16)}(*2)$ ,<br>long expressions) | yes | $\kappa_{A\mathcal{M}2}>0$ makes 7 Neg | Case (D) |
| $\partial_t\tilde{A}_{ij}$ | $\kappa_{A\mathcal{A}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{A}$ | $(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{A\mathcal{A}1})$ | yes | $\kappa_{A\mathcal{A}1}<0$ makes 1 Neg. | |
| $\partial_t\tilde{A}_{ij}$ | $\kappa_{A\mathcal{A}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{A}$ | $(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2}k^2)$ | yes | $\kappa_{A\mathcal{A}2}>0$ makes 1 Neg. | |
| $\partial_t\tilde{\Gamma}^i$ | $\kappa_{\tilde{\Gamma}\mathcal{H}}\,\alpha\tilde{D}^i\mathcal{H}$ | $(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2}k^2)$ | no | $\kappa_{\tilde{\Gamma}\mathcal{H}}>0$ makes 1 Neg. | |
| $\partial_t\tilde{\Gamma}^i$ | $\kappa_{\tilde{\Gamma}\mathcal{G}1}\,\alpha\mathcal{G}^i$ | $(0,0,(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}1}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}1}^2}(*2)$ , long.) | yes | $\kappa_{\tilde{\Gamma}\mathcal{G}1}<0$ makes 6 Neg. 1 Pos. | Case (E2) |
| $\partial_t\tilde{\Gamma}^i$ | $\kappa_{\tilde{\Gamma}\mathcal{G}2}\,\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i$ | $(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}2}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}2}^2}(*2)$ , long.) | yes | $\kappa_{\tilde{\Gamma}\mathcal{G}2}>0$ makes 2 Neg. 1 Pos. | |
| $\partial_t\tilde{\Gamma}^i$ | $\kappa_{\tilde{\Gamma}\mathcal{G}3}\,\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j$ | $(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}3}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}3}^2}(*2)$ , long.) | yes | $\kappa_{\tilde{\Gamma}\mathcal{G}3}>0$ makes 2 Neg. 1 Pos. | |

Yoneda-HS, PRD66 (2002) 124003

# An Evolution of Adjusted BSSN Formulation

**by Yo-Baumgarte-Shapiro, PRD 66 (2002) 084026**

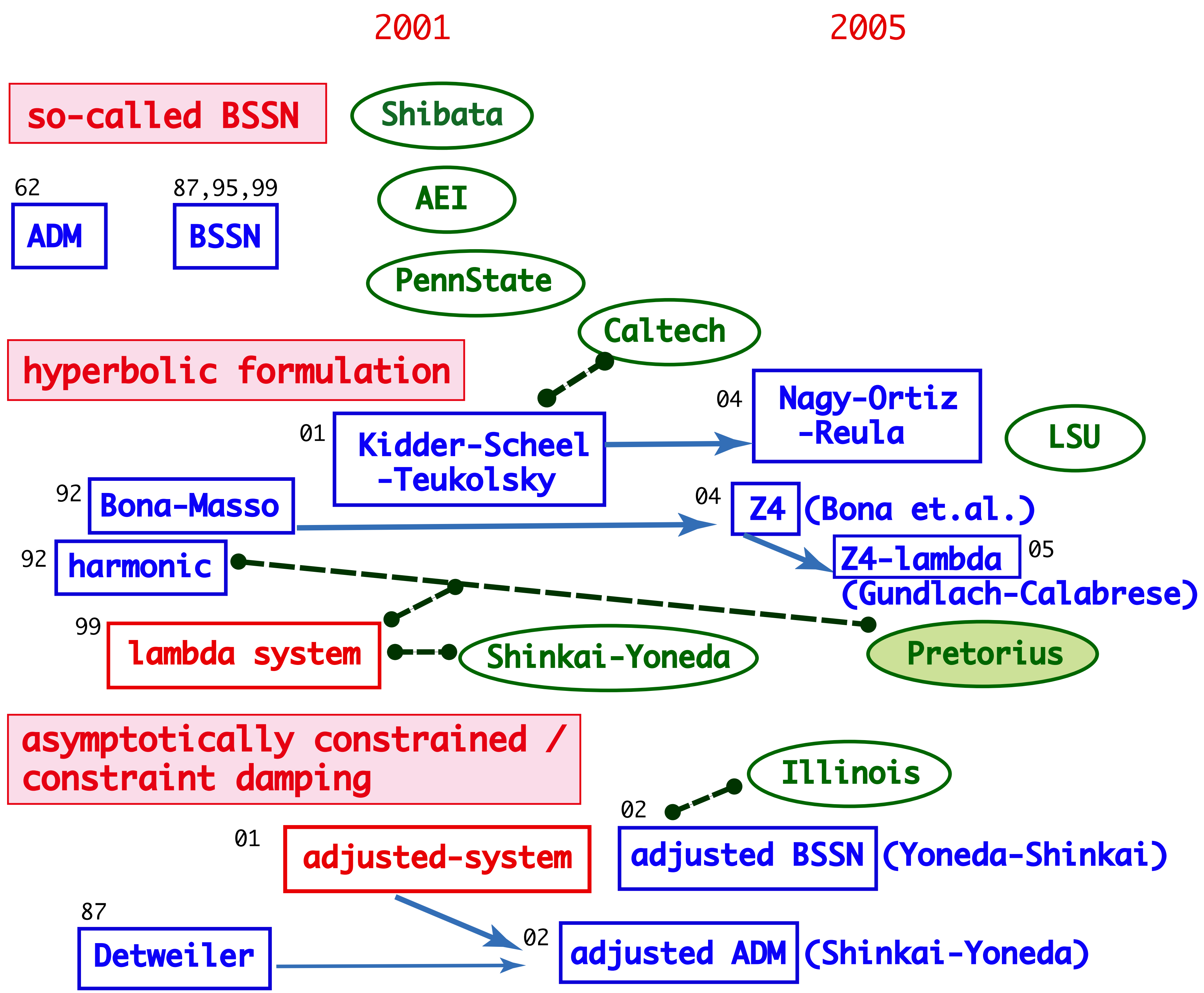


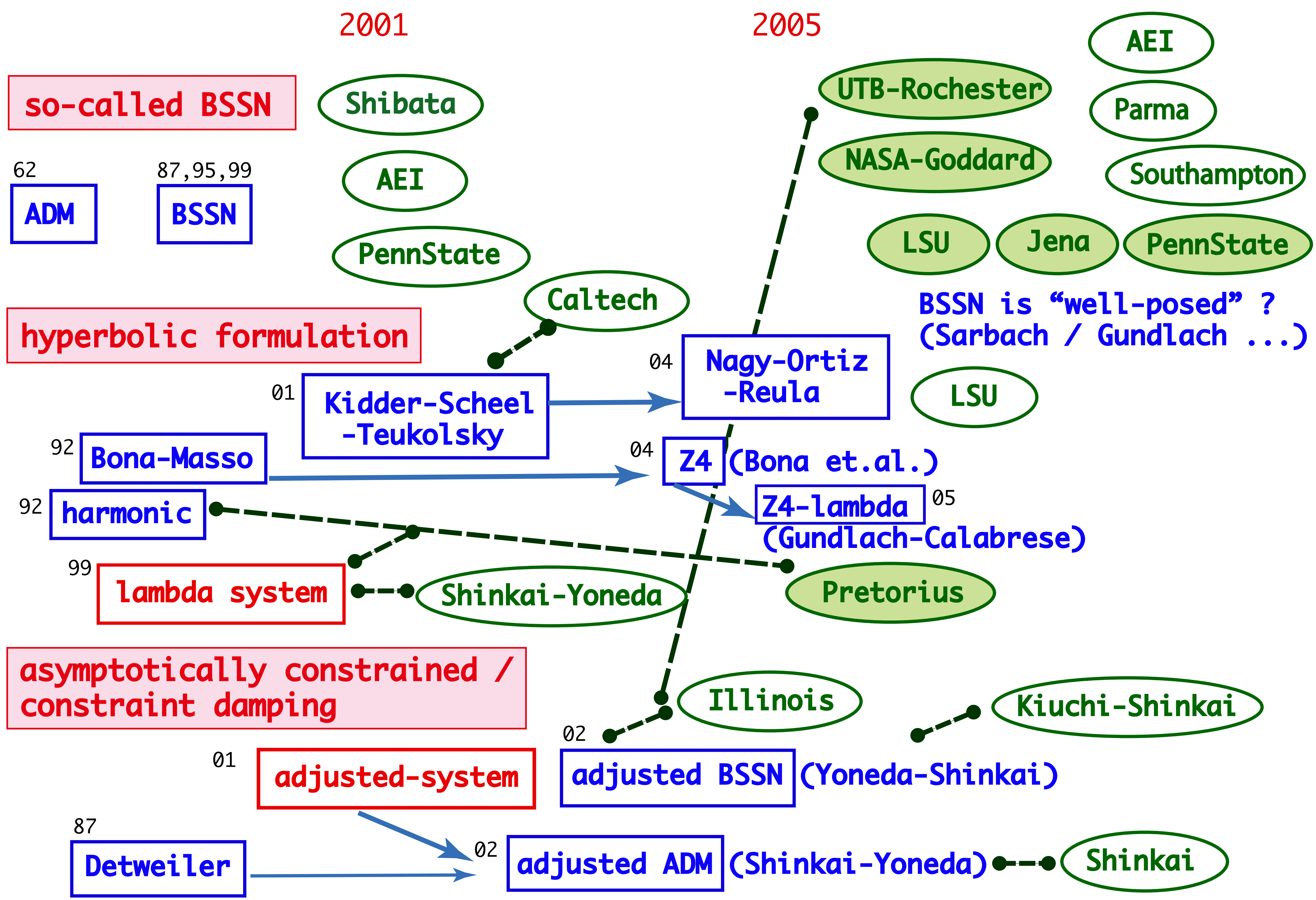




**Kerr-Schild BH (0.9 J/M), excision with cube, $1+\log$-lapse, $\Gamma$-driver shift.**

$$\partial_t \tilde{\Gamma}^i = (\cdots) + \frac{2}{3}\tilde{\Gamma}^i \beta^i{}_{,j} - (\chi + \frac{2}{3})\mathcal{G}^i \beta^j{}_{,j} \qquad \chi = 2/3 \ \ \textbf{for (A4)-(A8)}$$

$$\partial_t \tilde{\gamma}_{ij} = (\cdots) - \kappa\alpha\tilde{\gamma}_{ij}\mathcal{H} \qquad \kappa = 0.1 \sim 0.2 \ \ \textbf{for (A5), (A6) and (A8)}$$


2001
2005
so-called BSSN
Shibata
62
ADM
87,95,99
BSSN
AEI
PennState
Caltech
hyperbolic formulation
04
Nagy-Ortiz -Reula
LSU
01
Kidder-Scheel -Teukolsky
92
Bona-Masso
04
Z4 (Bona et.al.)
92
harmonic
Z4-lambda
05
(Gundlach-Calabrese)
99
lambda system
Shinkai-Yoneda
Pretorius
asymptotically constrained / constraint damping
Illinois
02
01
adjusted-system
adjusted BSSN (Yoneda-Shinkai)
87
Detweiler
02
adjusted ADM (Shinkai-Yoneda)



2001
2005
so-called BSSN
62
ADM
87,95,99
BSSN
Shibata
AEI
PennState
UTB-Rochester
NASA-Goddard
AEI
Parma
Southampton
LSU
Jena
PennState
BSSN is "well-posed" ?
(Sarbach / Gundlach ...)
hyperbolic formulation
Caltech
01
Kidder-Scheel
-Teukolsky
04
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constraint damping
Illinois
Kiuchi-Shinkai
02
adjusted BSSN (Yoneda-Shinkai)
01
adjusted-system
87
Detweiler
02
adjusted ADM (Shinkai-Yoneda)
Shinkai

## Some known fact (technical):

- Trace-out $A_{ij}$ at every time step helps the stability.

  Alcubierre, et al, [PRD 62 (2000) 044034]

- "The essential improvement is in the process of replacing terms by the momentum constraints",

  Alcubierre, et al, [PRD 62 (2000) 124011]

- $\tilde{\Gamma}^i$ is replaced by $-\partial_j \tilde{\gamma}^{ij}$ where it is not differentiated,

  Campanelli, et al, [PRL96 (2006) 111101; PRD 73 (2006) 061501R]

- $\tilde{\Gamma}^i$-equation has been modified as suggested in Yo-Baumgarte-Shapiro [PRD 66 (2002) 084026]

  Baker et al, [PRL96 (2006) 111102; PRD73 (2006) 104002]

Some known fact (technical):

- Trace-out $A_{ij}$ at every time step helps the stability.

  Alcubierre, et al, [PRD 62 (2000) 044034]

  This is because $\mathcal{A}$-violation affects to all other constraint violations.

- "The essential improvement is in the process of replacing terms by the momentum constraints",

  Alcubierre, et al, [PRD 62 (2000) 124011]

  This is because $\mathcal{M}$-replacement in $\Gamma^i$ equation kills the positive real eigenvalues of CAFs. eigenvalues

- $\tilde{\Gamma}^i$ is replaced by $-\partial_j \tilde{\gamma}^{ij}$ where it is not differentiated,

  Campanelli, et al, [PRL96 (2006) 111101; PRD 73 (2006) 061501R]

  This is because $\mathcal{G}$-violation affects to $\mathcal{H}, \mathcal{M}_i$-violation constraint violations.

- $\tilde{\Gamma}^i$-equation has been modified as suggested in Yo-Baumgarte-Shapiro [PRD 66 (2002) 084026]

  Baker et al, [PRL96 (2006) 111102; PRD73 (2006) 104002]

  No doubt about this.

# Numerical Experiments of Adjusted BSSN Systems


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- BSSN vs adjusted BSSN Numerical tests
- gauge-wave, linear wave, and Gowdy-wave tests, proposed by the Mexico workshop 2002
- 3 adjusted BSSN systems.
- Work as Expected
  - When the original BSSN system already shows satisfactory good evolutions (e.g., linear wave test), the adjusted versions also coincide with those evolutions.
  - For some cases (e.g., gauge-wave or Gowdy-wave tests) the simulations using the adjusted systems last 10 times longer than the standard BSSN.

arXiv:0711.3575, to be published in Phys. Rev. D. (2008)

## Adjusted BSSN systems; we tested

from the proposals in Yoneda & HS, Phys. Rev. D66 (2002) 124003

1. $\tilde{A}$-equation with the momentum constraint:

$$\partial_t \tilde{A}_{ij} = \partial_t^B \tilde{A}_{ij} + \kappa_A \alpha \tilde{D}_{(i} \mathcal{M}_{j)}, \tag{1}$$

   with $\kappa_{\mathcal{A}} > 0$ (predicted from the eigenvalue analysis).

2. $\tilde{\gamma}$-equation with $\mathcal{G}$ constraint:

$$\partial_t \tilde{\gamma}_{ij} = \partial_t^B \tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}} \alpha \tilde{\gamma}_{k(i} \tilde{D}_{j)} \mathcal{G}^k, \tag{2}$$

   with $\kappa_{\tilde{\gamma}} < 0$.

3. $\tilde{\Gamma}$-equation with $\mathcal{G}$ constraint:

$$\partial_t \tilde{\Gamma}^i = \partial_t^B \tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}} \alpha \mathcal{G}^i. \tag{3}$$

   with $\kappa_{\tilde{\Gamma}} < 0$.

## Numerical Testbed Models — A: Gauge-wave testbed

from the proposals in Mexico Workshop 2002, Class. Quant. Gravity 21 (2004) 589

The trivial Minkowski space-time, but time-dependent tilded slice.

$$ds^2 = -Hdt^2 + Hdx^2 + dy^2 + dz^2,$$
$$H = H(x-t) = 1 - A\sin\left(\frac{2\pi(x-t)}{d}\right),$$

Parameters:

- Gauge-wave parameters: $d = 1$ and $A = 10^{-2}$
- Simulation domain: $x \in [-0.5, 0.5]$, $y = z = 0$
- Grid: $x^i = -0.5 + (n - \frac{1}{2})dx$ with $n = 1, \cdots 50\rho$, where $dx = 1/(50\rho)$ with $\rho = 2, 4, 8$
- Time step: $dt = 0.25dx$
- Periodic boundary condition in $x$ direction
- Gauge conditions: $\partial_t \alpha = -\alpha^2 K, \quad \beta^i = 0.$

The 1D simulation is carried out for a $T = 1000$ crossing-time or until the code crashes, where one crossing-time is defined by the length of the simulation domain.

## Error evaluation methods

It should be emphasized that the adjustment effect has two meanings, improvement of stability and of accuracy. Even if a simulation is stable, it does not imply that the result is accurate.

- We judge the **stability of the evolution** by monitoring the L2 norm of each constraint,

$$||\delta\mathcal{C}||_2(t) \equiv \sqrt{\frac{1}{N}\sum_{x,y,z}\left(\mathcal{C}(t;x,y,z)\right)^2},$$

  where $N$ is the total number of grid points,

- We judge the **accuracy** by the difference of the metric components $g_{ij}(t;x,y,z)$ from the exact solution $g_{ij}^{(\text{exact})}(t;x,y,z)$,

$$||\delta g_{ij}||_2(t) \equiv \sqrt{\frac{1}{N}\sum_{x,y,z}\left(g_{ij} - g_{ij}^{(\text{exact})}\right)^2}.$$

# Numerical Results    A: Gauge-wave test (1)

## A.1 The plain BSSN system

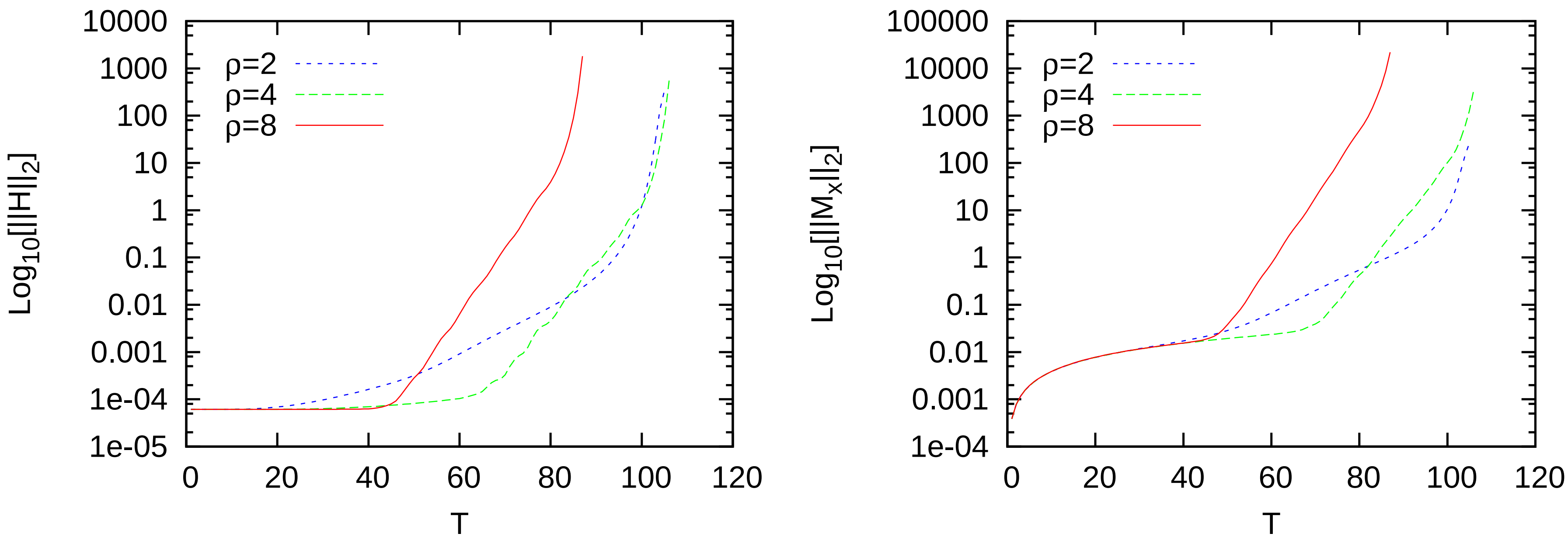



FIG. 1: The one-dimensional gauge-wave test with the plain BSSN system. The L2 norm of $\mathcal{H}$ and $\mathcal{M}_x$, rescaled by $\rho^2/4$, are plotted with a function of the crossing-time. The amplitude of the wave is $A = 0.01$. The loss of convergence at the early time, near the 20 crossing-time, can be seen, and it will produce the blow-ups of the calculation in the end.

- The poor performance of the plain BSSN system has been reported.
  Jansen, Bruegmann, & Tichy, PRD 74 (2006) 084022.
- The 4th-order finite differencing scheme improves the results.
  Zlochower, Baker, Campanelli, & Lousto, PRD 72 (2005) 024021.

Numerical Results    A: Gauge-wave test (2)

## A.2 Adjusted BSSN with $\tilde{A}$-equation

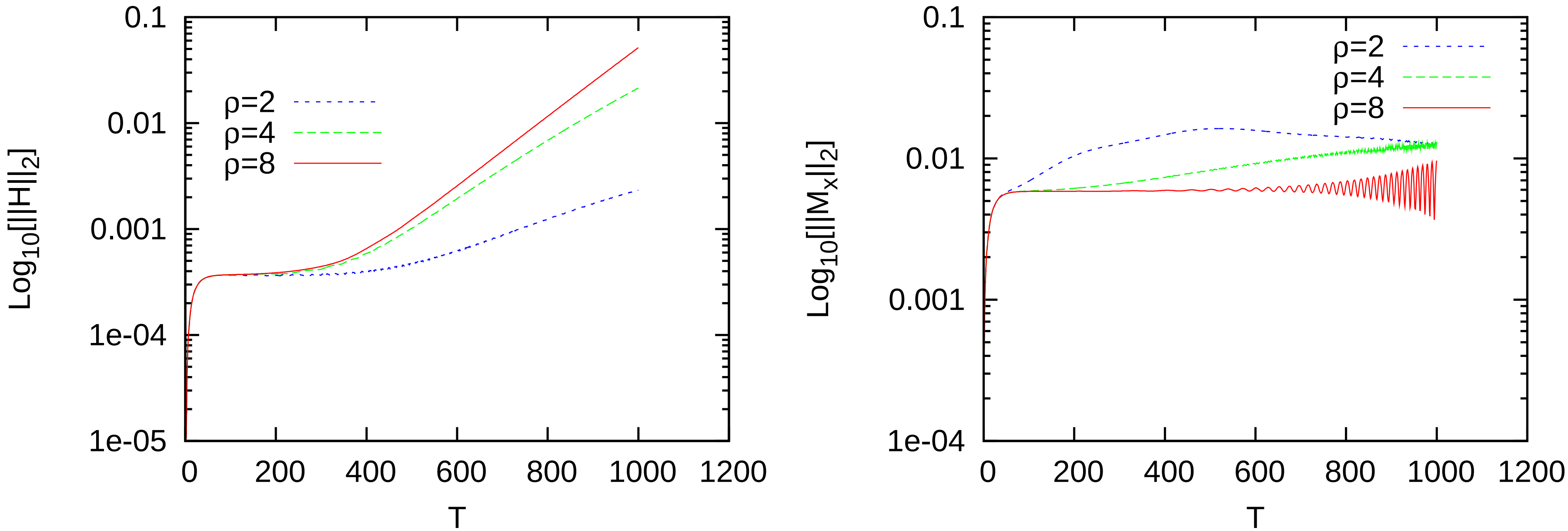



FIG. 2: The one-dimensional gauge-wave test with the adjusted BSSN system in the $\tilde{A}$-equation (1). The L2 norm of $\mathcal{H}$ and $\mathcal{M}_x$, rescaled by $\rho^2/4$, are plotted with a function of the crossing-time. The wave parameter is the same as with Fig. 1, and the adjustment parameter $\kappa_A$ is set to $\kappa_A = 0.005$. We see the higher resolution runs show convergence longer, i.e., the 300 crossing-time in $\mathcal{H}$ and the 200 crossing-time in $\mathcal{M}_x$ with $\rho = 4$ and 8 runs. All runs can stably evolve up to the 1000 crossing-time.

- We found that the simulation continues 10 times longer.
- Convergence behaviors are apparently improved than those of the plain BSSN.
- However, growth of the error in later time at higher resolution.

$$\partial_t \tilde{A}_{ij} = -e^{-4\phi}\left[D_i D_j \alpha + \alpha R_{ij}\right]^{\mathrm{TF}} + \alpha K \tilde{A}_{ij} - 2\alpha \tilde{A}_{ik}\tilde{A}^k{}_j + \partial_i \beta^k \tilde{A}_{kj} + \partial_j \beta^k \tilde{A}_{ki} - \frac{2}{3}\partial_k \beta^k \tilde{A}_{ij} + \beta^k \partial_k \tilde{A}_{ij} + \kappa_A \alpha \tilde{D}_{(i}\mathcal{M}_{j)}$$

## A.4 Evaluation of Accuracy

- L2 norm of the error in $\gamma_{xx}$, (4), with the function of time.
- The error is induced by distortion of the wave; the both phase and amplitude errors.

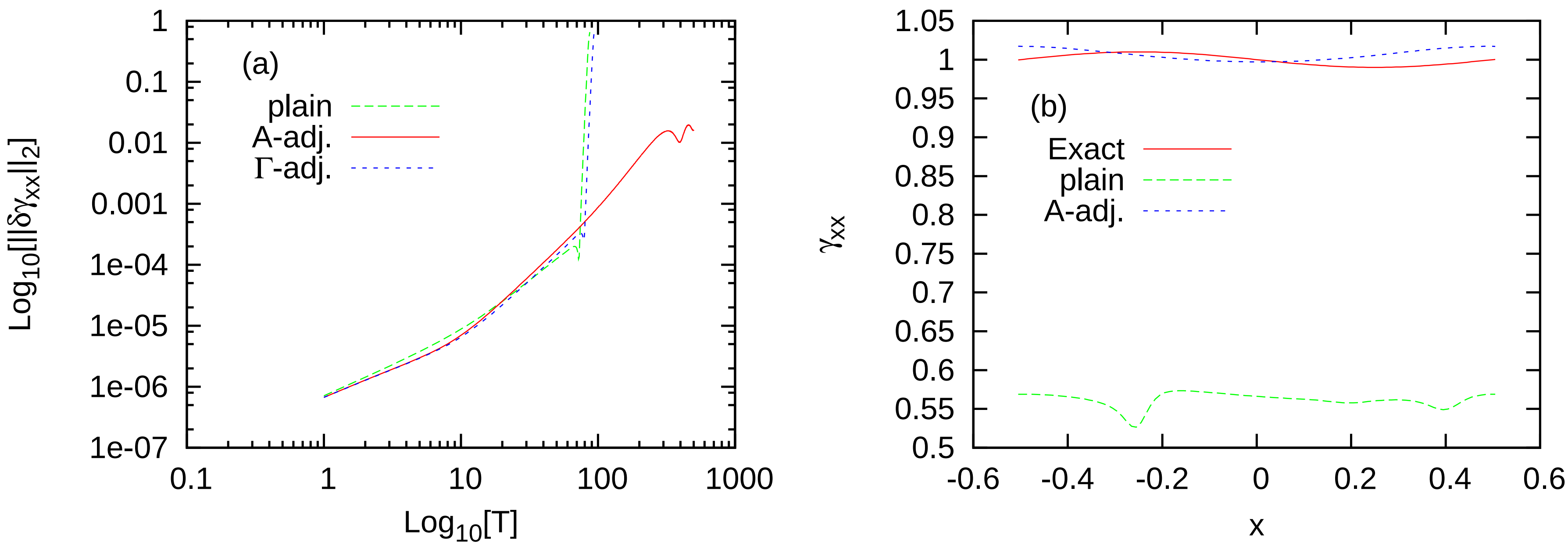



FIG. 4: Evaluation of the accuracy of the one-dimensional gauge-wave testbed. Lines show the plain BSSN, the adjusted BSSN with $\mathcal{A}$-equation, and with $\tilde{\Gamma}$-equation. (a) The L2 norm of the error in $\gamma_{xx}$, using (4). (b) A snapshot of the exact and numerical solution at $T = 100$.

## Numerical Testbed Models — B: Linear wave testbed

from the proposals in Mexico Workshop 2002, Class. Quant. Gravity 21 (2004) 589

Check the ability of handling a travelling gravitational wave.

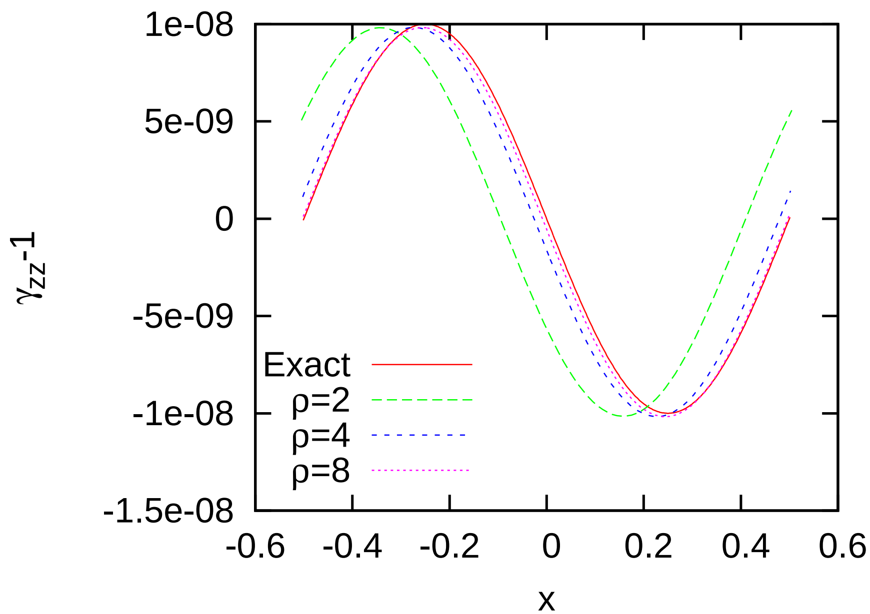
$$ds^2 = -dt^2 + dx^2 + (1+b)dy^2 + (1-b)dz^2,$$
$$b = A \sin\left(\frac{2\pi(x-t)}{d}\right)$$

Parameters:

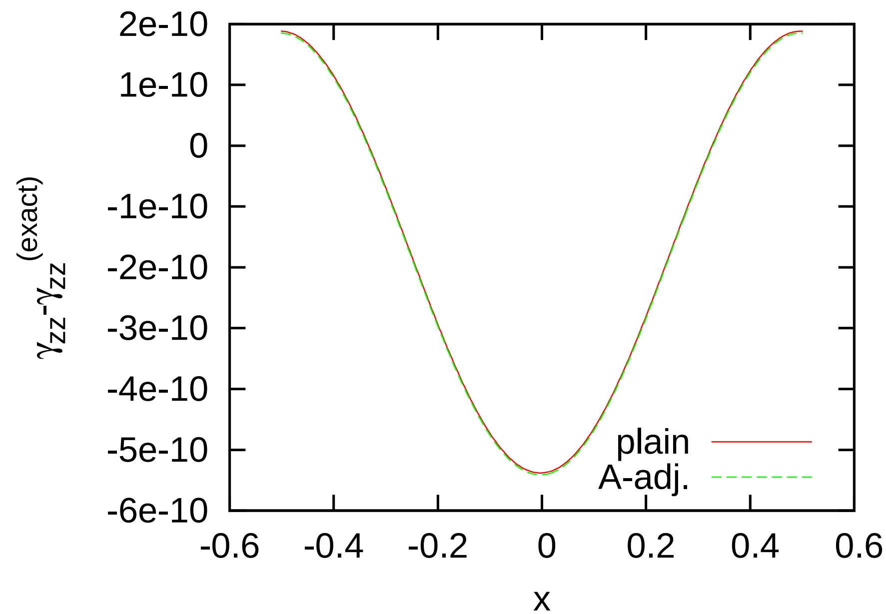
- Linear wave parameters: $d = 1$ and $A = 10^{-8}$
- Simulation domain: $x \in [-0.5, 0.5]$, $y = 0$, $z = 0$
- Grid: $x^i = -0.5 + (n - \frac{1}{2})dx$ with $n = 1, \cdots 50\rho$, where $dx = 1/(50\rho)$ with $\rho = 2, 4, 8$
- Time step: $dt = 0.25dx$
- Periodic boundary condition in $x$ direction
- Gauge conditions: $\alpha = 1$ and $\beta^i = 0$

The 1D simulation is carried out for a $T = 1000$ crossing-time or until the code crashes.

# Numerical Results — B: Linear Wave Test




Snapshots of the one-dimensional linear wave at different resolutions with the plain BSSN system at the simulation time 500 crossing-time. We see there exists phase error, but they are convergent away at higher resolution runs.




Snapshot of errors with the exact solution for the Linear Wave testbed with the plain BSSN system and the adjusted BSSN system with the $\tilde{A}$ equation at $T = 500$. The highest resolution $\rho = 8$ is used in both runs. The difference between the plain and the adjusted BSSN system with the $\tilde{A}$ equation is indistinguishable. Note that the maximum amplitude is set to be $10^{-8}$ in this problem.

- The linear wave testbed does not produce a significant constraint violation.
- The plain BSSN and adjusted BSSN results are indistinguishable.
  This is because the adjusted terms of the equations are small due to the small violations of constraints.

## Numerical Testbed Models — C: Collapsing polarized Gowdy-wave testbed

from the proposals in Mexico Workshop 2002, Class. Quant. Gravity 21 (2004) 589

Check the formulation in a strong field context using the polarized Gowdy metric.

$$\begin{aligned} ds^2 &= t^{-1/2}e^{\lambda/2}(-dt^2+dz^2)+t(e^P dx^2+e^{-P}dy^2). \\ P &= J_0(2\pi t)\cos(2\pi z), \\ \lambda &= -2\pi t J_0(2\pi t)J_1(2\pi t)\cos^2(2\pi z)+2\pi^2 t^2[J_0^2(2\pi t)+J_1^2(2\pi t)] \\ &\quad -\frac{1}{2}[(2\pi)^2[J_0^2(2\pi)+J_1^2(2\pi)]-2\pi J_0(2\pi)J_1(2\pi)], \end{aligned}$$

where $J_n$ is the Bessel function.

Parameters:

- Perform the evolution in the collapsing (i.e. backward in time) direction.
- Simulation domain: $z \in [-0.5, 0.5]$, $x = y = 0$
- Grid: $z = -0.5 + (n - \frac{1}{2})dz$ with $n = 1, \cdots 50\rho$, where $dz = 1/(50\rho)$ with $\rho = 2, 4, 8$
- Time step: $dt = 0.25dz$
- Periodic boundary condition in $z$-direction
- Gauge conditions: the harmonic slicing $\partial_t \alpha = -\alpha^2 K, \quad \beta^i = 0.$ and $\beta^i = 0$
- Set the initial lapse function is 1, using coordinate transformation.

The 1D simulation is carried out for a $T = 1000$ crossing-time or until the code crashes.

## Numerical Results — C: Collapsing polarized Gowdy-wave testbed (1)

### C.1 The plain BSSN

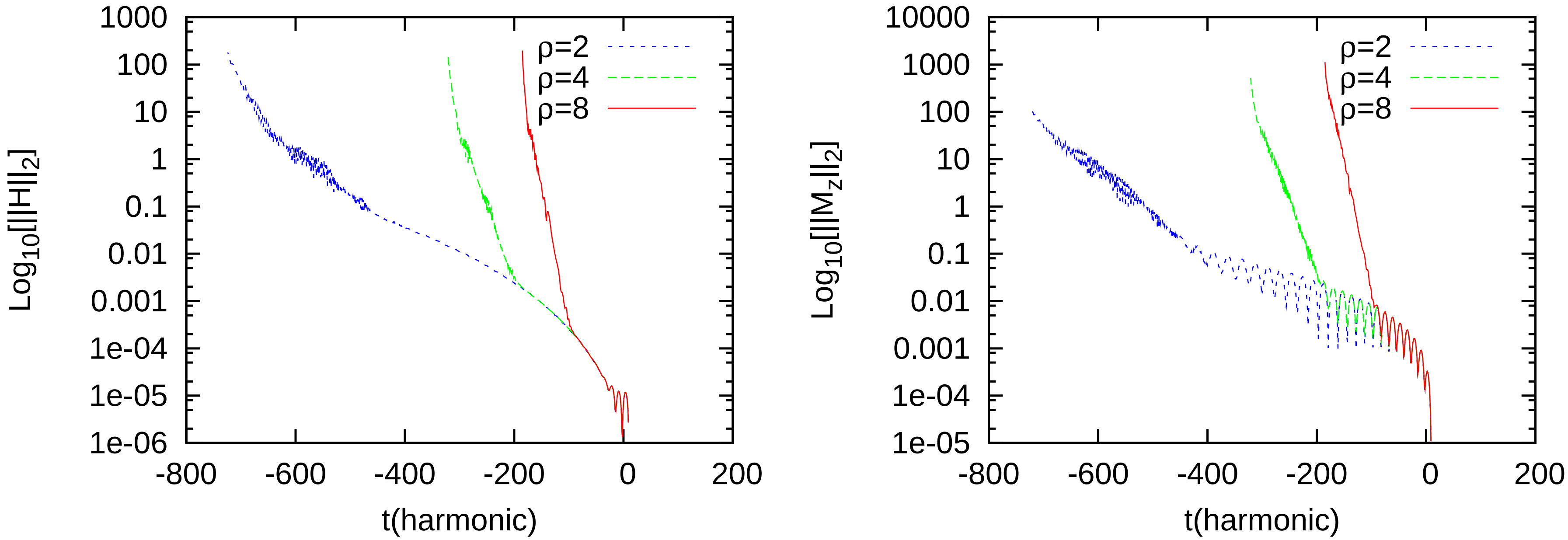



FIG. 5: Collapsing polarized Gowdy-wave test with the plain BSSN system. The L2 norm of $\mathcal{H}$ and $\mathcal{M}_z$, rescaled by $\rho^2/4$, are plotted with a function of the crossing-time. (Simulation proceeds backwards from $t = 0$.) We see almost perfect overlap for the initial 100 crossing-time, and the higher resolution runs crash earlier. This result is quite similar to those achieved with the Cactus BSSN code, reported by [**?** ].

- Our result shows similar crashing time with that of *Cactus* BSSN code.
  Alcubierre et al. CQG **21**, 589 (2004)
- Higher order differencing scheme with Kreiss-Oliger dissipation term improves the results.
  Zlochower, Baker, Campanelli & Lousto, PRD **72**, 024021 (2005)

## Numerical Results    C: Collapsing polarized Gowdy-wave testbed (2)

### C.2 Adjusted BSSN with $\tilde{A}$-equation

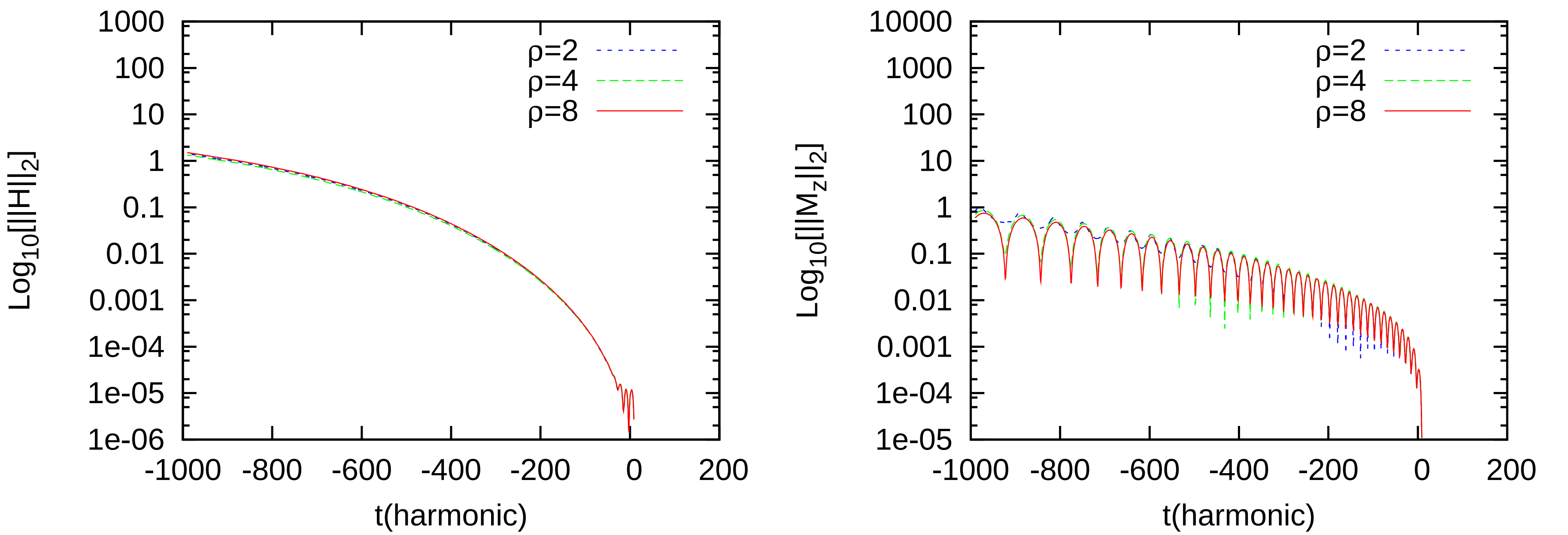



FIG. 6: Collapsing polarized Gowdy-wave test with the adjusted BSSN system in the $\tilde{A}$-equation (1), with $\kappa_{\mathcal{A}} = -0.001$. The style is the same as in Fig. 5 and note that both constraints are normalized by $\rho^2/4$. We see almost perfect overlap for the initial 1000 crossing-time in both constraint equations, $\mathcal{H}$ and $\mathcal{M}_z$, even for the highest resolution run.

- Adjustment extends the life-time of the simulation 10 times longer.
- Almost perfect convergence upto $t = 1000 t_{cross}$ for both $\mathcal{H}$ and $\mathcal{M}_z$, while we find oscillations in $\mathcal{M}_z$ later time.

$$\partial_t \tilde{A}_{ij} = -e^{-4\phi}\left[D_i D_j \alpha + \alpha R_{ij}\right]^{\mathrm{TF}} + \alpha K \tilde{A}_{ij} - 2\alpha \tilde{A}_{ik}\tilde{A}^k{}_j + \partial_i \beta^k \tilde{A}_{kj} + \partial_j \beta^k \tilde{A}_{ki} - \frac{2}{3}\partial_k \beta^k \tilde{A}_{ij} + \beta^k \partial_k \tilde{A}_{ij} + \kappa_A \alpha \tilde{D}_{(i}\mathcal{M}_{j)}$$

## C.3 Adjusted BSSN with $\tilde{\gamma}$-equation

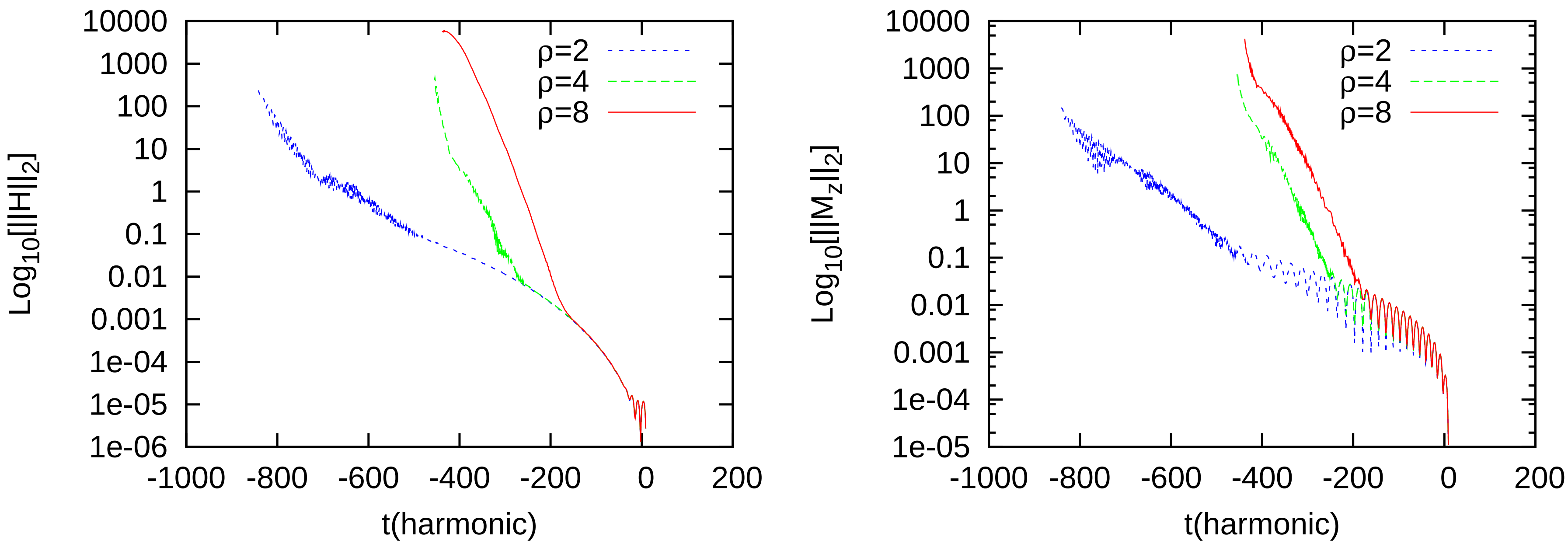



FIG. 7: Collapsing polarized Gowdy-wave test with the adjusted BSSN system in the $\tilde{\gamma}$-equation (2), with $\kappa_{\tilde{\gamma}} = 0.000025$. The figure style is the same as Figure 5. Note the almost perfect overlap for 200 crossing-time in the both the Hamiltonian and Momentum constraint and the $\rho = 2$ run can evolve stably for 1000 crossing-time.

- Almost perfect convergence up to $t = 200 t_{cross}$ in both $\mathcal{H}$ and $\mathcal{M}_z$.

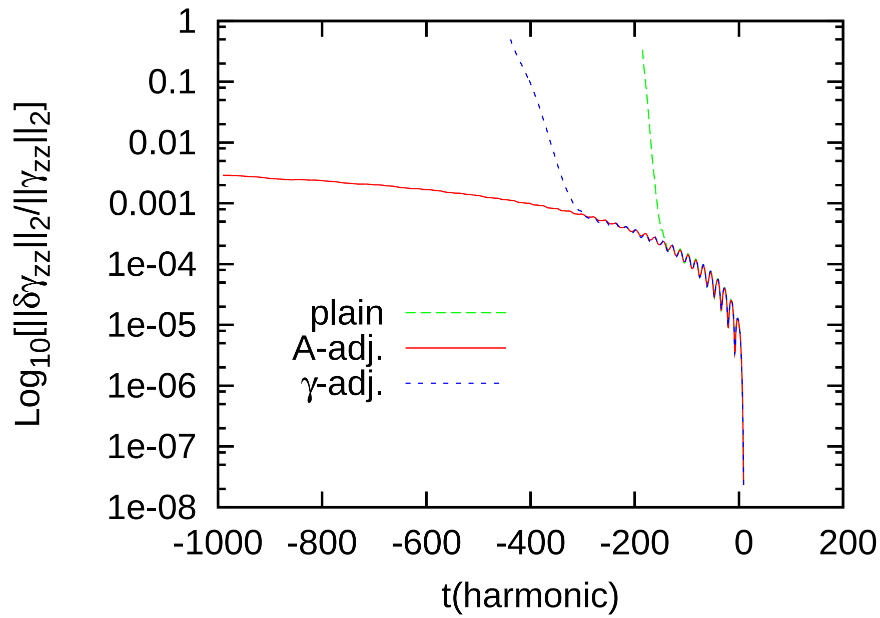
$$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta^k + \tilde{\gamma}_{jk}\partial_i \beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k + \beta^k \partial_k \tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}} \alpha \tilde{\gamma}_{k(i} \tilde{D}_{j)} \mathcal{G}^k$$

## C.4 Adjustment works for Accuracy

Error of $\gamma_{zz}$ to the exact solution normalized by $\gamma_{zz}$.

- *Accurate Evolution* $\Leftrightarrow$ Error $< 1$ %.
  (Zlochower, et al., PRD72 (2005) 024021 )

the Plain BSSN $\approx t = 200 t_{cross}$
adjusted BSSN $\tilde{\mathcal{A}}$-eq $\approx t = 1000 t_{cross}$
adjusted BSSN $\tilde{\gamma}$-eq $\approx t = 400 t_{cross}$




Comparisons of systems in the collapsing polarized Gowdy-wave test. The L2 norm of the error in $\gamma_{zz}$, rescaled by the L2 norm of $\gamma_{zz}$, for the plain BSSN, adjusted BSSN with $\tilde{\mathcal{A}}$-equation, and with $\tilde{\gamma}$-equation are shown. The highest resolution run, $\rho = 8$, is depicted for the plots. We can conclude that the adjustments make longer accurate runs available. Note that the evolution is backwards in time.

# A Full set of BSSN constraint propagation eqs.

$$\partial_t^{BS}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_i\\ \mathcal{G}^i\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12} & A_{13} & A_{14} & A_{15}\\ -(1/3)(\partial_i\alpha)+(1/6)\partial_i & \alpha K & A_{23} & 0 & A_{25}\\ 0 & \alpha\tilde{\gamma}^{ij} & 0 & A_{34} & A_{35}\\ 0 & 0 & 0 & \beta^k(\partial_k\mathcal{S}) & -2\alpha\tilde{\gamma}\\ 0 & 0 & 0 & 0 & \alpha K+\beta^k\partial_k\end{pmatrix}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_j\\ \mathcal{G}^j\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix}$$

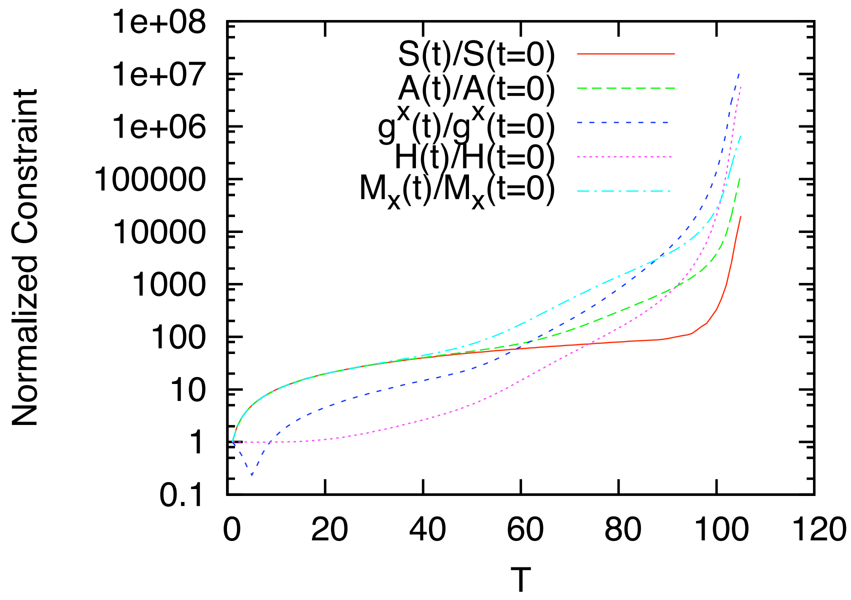
$$\begin{aligned}
A_{11} &= +(2/3)\alpha K+(2/3)\alpha\mathcal{A}+\beta^k\partial_k\\
A_{12} &= -4e^{-4\varphi}\alpha(\partial_k\varphi)\tilde{\gamma}^{kj}-2e^{-4\varphi}(\partial_k\alpha)\tilde{\gamma}^{jk}\\
A_{13} &= -2\alpha e^{-4\varphi}\tilde{A}^k{}_j\partial_k-\alpha e^{-4\varphi}(\partial_j\tilde{A}_{kl})\tilde{\gamma}^{kl}-e^{-4\varphi}(\partial_j\alpha)\mathcal{A}-e^{-4\varphi}\beta^k\partial_k\partial_j-(1/2)e^{-4\varphi}\beta^k\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\partial_k\\
&\quad +(1/6)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_j\beta^k)(\partial_k\mathcal{S})-(2/3)e^{-4\varphi}(\partial_k\beta^k)\partial_j\\
A_{14} &= 2\alpha e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{lk}(\partial_l\varphi)\mathcal{A}\partial_k+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l\mathcal{A})\tilde{\gamma}^{lk}\partial_k+(1/2)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l\alpha)\tilde{\gamma}^{lk}\mathcal{A}\partial_k+(1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m\tilde{\gamma}^{lk}\partial_m\partial_l\partial_k\\
&\quad -(5/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^m\tilde{\gamma}^{lk}(\partial_m\mathcal{S})\partial_l\partial_k+e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m(\partial_m\tilde{\gamma}^{lk})\partial_l\partial_k+(1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^i(\partial_j\partial_i\tilde{\gamma}^{jk})\partial_k\\
&\quad +(3/4)e^{-4\varphi}\tilde{\gamma}^{-3}\beta^i\tilde{\gamma}^{jk}(\partial_i\mathcal{S})(\partial_j\mathcal{S})\partial_k-(3/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^i(\partial_i\tilde{\gamma}^{jk})(\partial_j\mathcal{S})\partial_k+(1/3)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{pj}(\partial_j\beta^k)\partial_p\partial_k\\
&\quad -(5/12)e^{-4\varphi}\tilde{\gamma}^{-2}\tilde{\gamma}^{jk}(\partial_k\beta^i)(\partial_i\mathcal{S})\partial_j+(1/3)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\beta^k)\partial_i-(1/6)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{mk}(\partial_k\partial_l\beta^l)\partial_m\\
A_{15} &= (4/9)\alpha K\mathcal{A}-(8/9)\alpha K^2+(4/3)\alpha e^{-4\varphi}(\partial_i\partial_j\varphi)\tilde{\gamma}^{ij}+(8/3)\alpha e^{-4\varphi}(\partial_k\varphi)(\partial_l\tilde{\gamma}^{lk})+\alpha e^{-4\varphi}(\partial_j\tilde{\gamma}^{jk})\partial_k\\
&\quad +8\alpha e^{-4\varphi}\tilde{\gamma}^{jk}(\partial_j\varphi)\partial_k+\alpha e^{-4\varphi}\tilde{\gamma}^{jk}\partial_j\partial_k+8e^{-4\varphi}(\partial_l\alpha)(\partial_k\varphi)\tilde{\gamma}^{lk}+e^{-4\varphi}(\partial_l\alpha)(\partial_k\tilde{\gamma}^{lk})+2e^{-4\varphi}(\partial_l\alpha)\tilde{\gamma}^{lk}\partial_k\\
&\quad +e^{-4\varphi}\tilde{\gamma}^{lk}(\partial_l\partial_k\alpha)\\
A_{23} &= \alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k\varphi)(\partial_j\tilde{\gamma}_{mi})-(1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}(\partial_j\tilde{\gamma}_{mi})\\
&\quad +(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_k\partial_j\tilde{\gamma}_{mi})+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-2}(\partial_i\mathcal{S})(\partial_j\mathcal{S})-(1/4)\alpha e^{-4\varphi}(\partial_i\tilde{\gamma}_{kl})(\partial_j\tilde{\gamma}^{kl})+\alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k\varphi)\tilde{\gamma}_{ji}\partial_m\\
&\quad +\alpha e^{-4\varphi}(\partial_j\varphi)\partial_i-(1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}\tilde{\gamma}_{ji}\partial_m+\alpha e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\Gamma}_{ijk}\partial_m+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{lk}\tilde{\gamma}_{ji}\partial_k\partial_l\\
&\quad +(1/2)e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_j\tilde{\gamma}_{im})(\partial_k\alpha)+(1/2)e^{-4\varphi}(\partial_j\alpha)\partial_i+(1/2)e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\gamma}_{ji}(\partial_k\alpha)\partial_m\\
A_{25} &= -\tilde{A}^k{}_i(\partial_k\alpha)+(1/9)(\partial_i\alpha)K+(4/9)\alpha(\partial_i K)+(1/9)\alpha K\partial_i-\alpha\tilde{A}^k{}_i\partial_k\\
A_{34} &= -(1/2)\beta^k\tilde{\gamma}^{il}\tilde{\gamma}^{-2}(\partial_l\mathcal{S})\partial_k-(1/2)(\partial_l\beta^i)\tilde{\gamma}^{lk}\tilde{\gamma}^{-1}\partial_k+(1/3)(\partial_l\beta^l)\tilde{\gamma}^{ik}\tilde{\gamma}^{-1}\partial_k-(1/2)\beta^l\tilde{\gamma}^{in}(\partial_l\tilde{\gamma}_{mn})\tilde{\gamma}^{mk}\tilde{\gamma}^{-1}\partial_k\\
&\quad +(1/2)\beta^k\tilde{\gamma}^{il}\tilde{\gamma}^{-1}\partial_l\partial_k\\
A_{35} &= -(\partial_k\alpha)\tilde{\gamma}^{ik}+4\alpha\tilde{\gamma}^{ik}(\partial_k\varphi)-\alpha\tilde{\gamma}^{ik}\partial_k
\end{aligned}$$

## Which constraint should be monitored?

Yoneda & HS, PRD 66 (2002) 124003 Kiuchi & HS, arXiv:0711.3575, PRD (2008)

Order of constraint violation?

- $\mathcal{A}$ and $\mathcal{S}$ constraints propagate independently of the other constraints.
- $\mathcal{G}$-constraint is triggered by the violation of the momentum constraint.
- $\mathcal{H}$ and $\mathcal{M}$ constraints are affected by all the other constraints.




The violation of all constraints normalized with their initial values, $||\delta\mathcal{C}||_2(t)/||\delta\mathcal{C}||_2(0)$, are plotted with a function of time. The evolutions of the gauge-wave testbeds with the plain BSSN system are shown.

By observing which constraint triggers the other constraint's violation from the constraint propagation equations, we may guess the mechanism by which the entire system is violating accuracy and stability.

# Summary up to here (2nd half)

[Keyword 1] Adjusted Systems

Adjusting the EoM with constraints is common to all previous approaches.
Just add constraints to evolution eqs, while lambda-system requires symmetric hyperbolicity.

[Keyword 2] Constraint Propagation Analysis -> Constraint Damping System

By evaluating the propagation eqs of constraints, we can predict the suitable adjustments to the EoM in advance.

(Step 1) Fourier mode expression of all terms of constraint propagation eqs.

(Step 2) Eigenvalues and Diagonalizability of constraint propagation matrix.
Eigenvalues = Constraint Amplification Factors

(Step 3) If CAF=negatives -> Constraint surface becomes the attractor.

[Keyword 3] Adjusted ADM systems

We show the standard ADM has constraint violating mode.
We predict several adjustments, which give better stability.

[Keyword 3] Adjusted BSSN systems

We show the advantage of BSSN is the adjustment using M.
We predict several adjustments, which give better stability.

## Discussion
## Application 1 : Constraint Propagation in $N+1$ dim. space-time

HS-Yoneda, GRG 36 (2004) 1931

### Dynamical equation has $N$-dependency

Only the matter term in $\partial_t K_{ij}$ has $N$-dependency.

$$\begin{aligned}
0 \approx \mathcal{C}_H &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu n^\nu = \frac{1}{2}({}^{(N)}R + K^2 - K^{ij}K_{ij}) - 8\pi\rho_H - \Lambda, \\
0 \approx \mathcal{C}_{Mi} &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu \perp_i^\nu = D_j K_i^j - D_i K - 8\pi J_i, \\
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_j\beta_i + D_i\beta_j, \\
\partial_t K_{ij} &= \alpha^{(N)}R_{ij} + \alpha K K_{ij} - 2\alpha K^\ell{}_j K_{i\ell} - D_i D_j \alpha \\
&\quad + \beta^k (D_k K_{ij}) + (D_j \beta^k) K_{ik} + (D_i \beta^k) K_{kj} - 8\pi\alpha \left( S_{ij} - \frac{1}{N-1}\gamma_{ij} T \right) - \frac{2\alpha}{N-1}\gamma_{ij}\Lambda,
\end{aligned}$$

### Constraint Propagations remain the same

From the Bianchi identity, $\nabla^\nu \mathcal{S}_{\mu\nu} = 0$ with $\mathcal{S}_{\mu\nu} = X n_\mu n_\nu + Y_\mu n_\nu + Y_\nu n_\mu + Z_{\mu\nu}$, we get

$$\begin{aligned}
0 = n^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= -Z_{\mu\nu}(\nabla^\mu n^\nu) - \nabla^\mu Y_\mu + Y_\nu n^\mu \nabla_\mu n^\nu - 2Y_\mu n_\nu (\nabla^\nu n^\mu) - X(\nabla^\mu n_\mu) - n_\mu (\nabla^\mu X), \\
0 = h_i{}^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= \nabla^\mu Z_{i\mu} + Y_i (\nabla^\mu n_\mu) + Y_\mu (\nabla^\mu n_i) + X(\nabla^\mu n_i) n_\mu + n_\mu (\nabla^\mu Y_i).
\end{aligned}$$

- $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (T_{\mu\nu}, \rho_H, J_i, S_{ij})$ with $\nabla^\mu T_{\mu\nu} = 0 \Rightarrow$ matter eq.
- $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (G_{\mu\nu} - 8\pi T_{\mu\nu}, \mathcal{C}_H, \mathcal{C}_{Mi}, \kappa\gamma_{ij}\mathcal{C}_H)$ with $\nabla^\mu (G_{\mu\nu} - 8\pi T_{\mu\nu}) = 0 \Rightarrow$ CP eq.

# Discussion
# Future : Construct a robust adjusted system

HS-Yoneda, in preparation

(1) dynamic & automatic determination of $\kappa$ under a suitable principle.

e.g.) Efforts in Multi-body Constrained Dynamics simulations

$$\frac{\partial}{\partial t} p_i = F_i + \lambda_a \frac{\partial C^a}{\partial x^i}, \qquad \text{with} \quad C^a(x_i, t) \approx 0$$

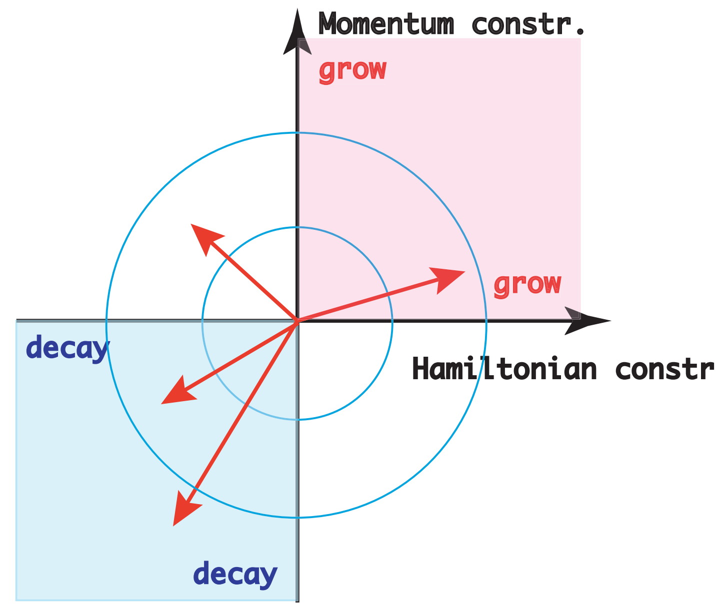
- J. Baumgarte (1972, Comp. Methods in Appl. Mech. Eng.)
  Replace a holonomic constraint $\partial_t^2 C = 0$ as $\partial_t^2 C + \alpha \partial_t C + \beta^2 C = 0$.
- Park-Chiou (1988, J. Guidance), "penalty method"
  Derive "stabilization eq." for Lagrange multiplier $\lambda(t)$.
- Nagata (2002, Multibody Dyn.)
  Introduce a scaled norm, $J = C^T SC$, apply $\partial_t J + w^2 J = 0$, and adjust $\lambda(t)$.

e.g.) Efforts in Molecular Dynamics simulations

- Constant pressure $\cdots\cdots$ potential piston!
- Constant temperature $\cdots\cdots$ potential thermostat!! (Nosé, 1991, PTP)

(2) target to control each constraint violation by adjusting multipliers.

CP-eigenvectors indicate directions of constraint grow/decay, if CP-matrix is diagonalizable.




(3) clarify the reasons of non-linear violation in the last stage of current test evolutions.

(4) Alternative new ideas?

– control theories, optimization methods (convex functional theories), mathematical programming methods, or ....

(5) Numerical comparisons of formulations, links to other systems, ...

– "Comparisons of Formulations" (e.g. Mexico NR workshop, 2002-2003); more formulations to be tested, ...

Find a RECIPE for all. Avoid un-essential techniques.

# Goals of the Lecture

What is the guiding principle for selecting evolution equations for simulations in GR?

Why many groups use the BSSN equations?

Are there an alternative formulation better than the BSSN?

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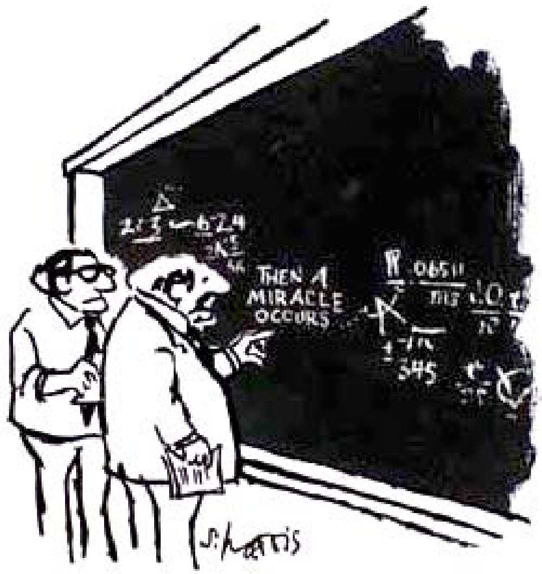
Why many groups use the BSSN equations?

--- Just rush, not to be late.

Are there an alternative formulation better than the BSSN?

--- Yes, there are. But we do not the best one.




"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

## Discussion
## Application 2 : Constraint Propagation of Maxwell field in Curved space

HS-Yoneda, in preparation

Towards a robust GR-MHD system:

- Maxwell eqs in curved space-time

$$\begin{aligned} \partial_t E^i &= \epsilon^{ijk} D_j(\alpha B_k) - 4\pi\alpha J^i + \alpha K E^i + \pounds_\beta E^i \\ \partial_t B^i &= -\epsilon^{ijk} D_j(\alpha E_k) + \alpha K B^i + \pounds_\beta B^i \\ \mathcal{C}_E &:= D_i E^i - 4\pi\rho_e \\ \mathcal{C}_B &:= D_i B^i \end{aligned}$$

- CP of Maxwell system in curved space-time

$$\begin{aligned} \partial_t C_E &= \alpha K C_E + \beta^j D_j C_E \\ \partial_t C_B &= \alpha K C_B + \beta^j D_j C_B \end{aligned}$$

- CP of ADM+Maxwell

$$\partial_t \begin{pmatrix} \mathcal{C}_E \\ \mathcal{C}_B \\ \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \begin{pmatrix} \mathcal{C}_E \\ \mathcal{C}_B \\ \mathcal{H} \\ \mathcal{M}_i \end{pmatrix}$$

- CP of ADM+Maxwell+Hydro
  in progress.