# Constraint Propagation of $C^2$-adjusted Equations — Another Recipe for Robust Evolution Systems —

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Abstract

In order to construct a robust evolution system against numerical instability for integrating the Einstein equations, we propose a new set of evolution equations by adjusting BSSN evolution equations with constraints. We apply an adjustment proposed by Fiske (2004) which uses the norm of the constraints, $C^2$. The advantage of this method is that the signature of the effective Lagrange multipliers are determined in advance. We show this feature by eigenvalue-analysis of constraint propagations and perform numerical tests using Gowdy wave propagation which indicates robust evolutions against the violation of the constraints than existing formulations.

## 1 Introduction

In numerical relativity, it is essential to perform stable and accurate simulation. The standard way to integrate the Einstein equations is to split spacetime into three-dimensional space and time. Arnowitt-Deser-Misner (ADM) formulation[1] is the fundamental evolution system of spacetime decompositions. However, it is known that this formulation is not appropriate since the constraints are not satisfied during long-term numerical calculation and in strong gravitational field[2]. Several formulations which modified ADM formulation are suggested, Baumgarte-Shapiro-Shibata-Nakamura(BSSN) formulation[3] is widely used among them.

However, there exists more robust systems than the current standard BSSN system (e.g.[4, 5]) depending on problems. Therefore seeking a robust evolution system against the violation of constraints is still an important issue.

Yoneda and Shinkai[5] systematically investigated adjusted systems, which adds constraints to the evolution equations. With this method, we can predict the stability of numerical simulation by analyzing the eigenvalues of the coefficient matrix which is Fourier-transformed constraint propagation equations under assuming a fixed background metric.

Fiske[6] proposed an adjustment which uses the norm of constraints, $C^2$, and does not require the background metric for specifying effective Lagrange multipliers and applied this method to the Maxwell equations. A good point of his method is what the stability of the numerical simulation can be expected without depending on background metric. We apply his method to the ADM and BSSN formulations, and actually perform the effect of dumping by numerical simulation.

## 2 $C^2$-adjusted Systems

For variables $u^i$ and constraint values $C^i$, evolution equations with constraint equations are generally written as

$$\partial_t u^i = f(u^i, \partial_j u^i, \cdots), \text{ and} \tag{1}$$

$$C^i(u^i, \partial_j u^i. \cdots) \approx 0. \tag{2}$$

Suppose we adjust (1) with $C^2 \equiv C^i C_i$, and evaluate constraint propagation as

$$\partial_t C^2 = \frac{\delta C^2}{\delta u^i}(\partial_t u^i). \tag{3}$$

There exists various combinations of this adjustment. Fiske[6] proposed an adjusted term as

$$\partial_t u^i = [\textbf{Original} \quad \textbf{Terms}] - \kappa^{ij}\frac{\delta C^2}{\delta u^j}, \tag{4}$$

with $\kappa^{ij}$ of positive definite. The constraint propagation, then, becomes

$$\partial_t C^2 = [\textbf{Original} \quad \textbf{Terms}] - \kappa^{ij}\frac{\delta C^2}{\delta u^i}\frac{\delta C^2}{\delta u^j}, \tag{5}$$

which clearly shows the dumping of constraints. If we set $\kappa^{ij}$ so that the second term becomes more dominant of (5) than first term in evolution, then $C^2$ dumps because of $\partial_t C^2 < 0$. Fiske presented an numerical example in the Maxwell system.

**CAFs (constraint amplification facters)** CAFs is a tool for predicting the violation of constraints. The CAFs are the eigenvalues of the coefficient matrix of the constraiant propagation equations. Negative real parts, or non-zero imaginary-parts of CAFs are preferable for stable evolutions.

# 3 Applications to the Einstein equations

## 3.1 For ADM Formulation

Now we apply Fiske's method to the ADM formulation[1], which can be written as

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta(\gamma_{ij}) - \kappa_{\gamma ijmn}\frac{\delta (C^A)^2}{\delta \gamma_{mn}}, \tag{6}$$

$$\partial_t K_{ij} = \alpha({}^{(3)}R_{ij} + KK_{ij} - 2K_{i\ell}K^\ell{}_j) - D_iD_j\alpha + \mathcal{L}_\beta(K_{ij}) - \kappa_{Kijmn}\frac{\delta (C^A)^2}{\delta K_{mn}}, \tag{7}$$

where $(C^A)^2$ is the norm of the constraints,

$$(C^A)^2 \equiv (\mathcal{H}^A)^2 + (\mathcal{M}^A)^i(\mathcal{M}^A)_i, \tag{8}$$

and both of $\kappa_{\gamma ijmn}, \kappa_{Kijmn}$ are positive definite.

For the modified ADM equations, (6)-(7), we confirm this system has better stablility than the standard ADM system by the method proposed by Yoneda and Shinkai[5]. That is, assuming the background metric to Minkowski metric, and setting $\kappa_{\gamma ijmn} = \kappa_{Kijmn} = \delta_{im}\delta_{jn}$, we analyzed the eigenvalues of the constraint propagation matrix . We found that all the real parts of eigenvalues are negative. Therefore the system is expected to dump the violation of constraints.

## 3.2 For BSSN Formulation

The widely used BSSN evolution equations[3, 5] are,

$$\partial_t\varphi = -(1/6)\alpha K + (1/6)(\partial_i\beta^i) + \beta^i(\partial_i\varphi), \tag{9}$$

$$\partial_t K = \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - D_iD^i\alpha + \beta^i(\partial_i K), \tag{10}$$

$$\partial_t\tilde{\gamma}_{ij} = -2\alpha\widetilde{A}_{ij} - (2/3)\widetilde{\gamma}_{ij}(\partial_\ell\beta^\ell) + \widetilde{\gamma}_{j\ell}(\partial_i\beta^\ell) + \widetilde{\gamma}_{i\ell}(\partial_j\beta^\ell) + \beta^\ell(\partial_\ell\widetilde{\gamma}_{ij}), \tag{11}$$

$$\begin{aligned}\partial_t\widetilde{A}_{ij} &= \alpha K\widetilde{A}_{ij} - 2\alpha\widetilde{A}_{i\ell}\widetilde{A}^\ell{}_j + \alpha e^{-4\varphi}R_{ij}{}^{TF} \\ &\quad - e^{-4\varphi}(D_iD_j\alpha)^{TF} - (2/3)\widetilde{A}_{ij}(\partial_\ell\beta^\ell) + (\partial_i\beta^\ell)\widetilde{A}_{j\ell} + (\partial_j\beta^\ell)\widetilde{A}_{i\ell} + \beta^\ell(\partial_\ell\widetilde{A}_{ij}),\end{aligned} \tag{12}$$

$$\begin{aligned}\partial_t\widetilde{\Gamma}^i &= 2\alpha\{6(\partial_j\varphi)\widetilde{A}^{ij} + \widetilde{\Gamma}^i{}_{j\ell}\widetilde{A}^{j\ell} - (2/3)\widetilde{\gamma}^{ij}(\partial_j K)\} \\ &\quad - 2(\partial_j\alpha)\widetilde{A}^{ij} + (2/3)\widetilde{\Gamma}^i(\partial_j\beta^j) + (1/3)\widetilde{\gamma}^{ij}(\partial_\ell\partial_j\beta^\ell) + \beta^\ell(\partial_\ell\widetilde{\Gamma}^i) - \widetilde{\Gamma}^j(\partial_j\beta^i) + \widetilde{\gamma}^{j\ell}(\partial_j\partial_\ell\beta^i),\end{aligned} \tag{13}$$

The BSSN system has 5 constraint equations; both "kinetic" and "algebraic" constraint equations:

$$\mathcal{H}^B \equiv \mathrm{e}^{-4\varphi}\widetilde{R} - 8\mathrm{e}^{-4\varphi}(\widetilde{D}_i\widetilde{D}^i\varphi + (\widetilde{D}^m\varphi)(\widetilde{D}_m\varphi)) + (2/3)K^2 - \widetilde{A}_{ij}\widetilde{A}^{ij} - (2/3)\mathcal{A}K \approx 0, \quad (14)$$

$$(\mathcal{M}^B)_i \equiv -(2/3)\widetilde{D}_iK + 6(\widetilde{D}_j\varphi)\widetilde{A}^j{}_i + \widetilde{D}_j\widetilde{A}^j{}_i - 2(\widetilde{D}_i\varphi)\mathcal{A} \approx 0, \quad (15)$$

$$\mathcal{G}^i \equiv \widetilde{\Gamma}^i - \widetilde{\gamma}^{j\ell}\widetilde{\Gamma}^i{}_{j\ell} \approx 0, \quad (16)$$

$$\mathcal{A} \equiv \widetilde{A}^{ij}\widetilde{\gamma}_{ij} \approx 0, \quad (17)$$

$$\mathcal{S} \equiv \det(\widetilde{\gamma}_{ij}) - 1 \approx 0. \quad (18)$$

The $C^2$-adjusted BSSN evolution equations are written as

$$\partial_t\varphi = (9) - \lambda_\varphi\left(\frac{\delta(C^B)^2}{\delta\varphi}\right), \quad (19)$$

$$\partial_tK = (10) - \lambda_K\left(\frac{\delta(C^B)^2}{\delta K}\right), \quad (20)$$

$$\partial_t\widetilde{\gamma}_{ij} = (11) - \lambda_{\widetilde{\gamma}ijmn}\left(\frac{\delta(C^B)^2}{\delta\widetilde{\gamma}_{mn}}\right), \quad (21)$$

$$\partial_t\widetilde{A}_{ij} = (12) - \lambda_{\widetilde{A}ijmn}\left(\frac{\delta(C^B)^2}{\delta\widetilde{A}_{mn}}\right), \quad (22)$$

$$\partial_t\widetilde{\Gamma}^i = (13) - \lambda_{\widetilde{\Gamma}}^{ij}\left(\frac{\delta(C^B)^2}{\delta\widetilde{\Gamma}^j}\right), \quad (23)$$

where $(C^B)^2$ is the norm of the constraints,

$$(C^B)^2 \equiv (\mathcal{H}^B)^2 + (\mathcal{M}^B)^i(\mathcal{M}^B)_i + \mathcal{G}^i\mathcal{G}_i + \mathcal{A}^2 + \mathcal{S}^2,$$

and all of the coefficients, $\lambda_\varphi, \lambda_K, \lambda_{\widetilde{\gamma}ijmn}, \lambda_{\widetilde{A}ijmn}$ and $\lambda_{\widetilde{\Gamma}}^{ij}$ are supposed to be positive definite.

**CAFs of the $C^2$-adjusted BSSN system** CAFs of the system

$$\partial_t\begin{pmatrix}\hat{\mathcal{H}}^B\\ \hat{\mathcal{M}}^B\\ \hat{\mathcal{G}}_i\\ \hat{\mathcal{A}}\\ \hat{\mathcal{S}}\end{pmatrix} = \begin{pmatrix} & & \\ \cdots & \cdots & \cdots\\ & & \end{pmatrix}\begin{pmatrix}\hat{\mathcal{H}}^B\\ \hat{\mathcal{M}}^B\\ \hat{\mathcal{G}}_i\\ \hat{\mathcal{A}}\\ \hat{\mathcal{S}}\end{pmatrix} \quad (24)$$

are confirmed to be

- three negative real numbers, and
- six complex numbers with negative real part,

if we fix the background metric is Minkowskii metric and set $\lambda_\varphi = \lambda_K = \lambda$, $\lambda_{\widetilde{\gamma}ijmn} = \lambda_{\widetilde{A}ijmn} = \lambda\delta_{im}\delta_{jn}$ and $\lambda_{\widetilde{\Gamma}}^{ij} = \lambda\delta^{ij}$ for simplicity, where $\lambda > 0$.

# 4 Numerical Examples

We show damping of constraint in numerical evolutions using polarized Gowdy wave evolution, which is one of the standard tests for comparisons of formulations in numerical relativity as is known to the Apples-with-Apples testbeds [7].

The metric of polarized Gowdy wave is

$$ds^2 = t^{-1/2}\mathrm{e}^{\lambda/2}(-dt^2 + dx^2) + t(\mathrm{e}^P dy^2 + \mathrm{e}^{-P}dz^2), \quad (25)$$

where $P$ and $\lambda$ are functions of $x$ and $t$. The time coordinate $t$ is chosen such that time increases as the universe expands, this metric is singular at $t = 0$ which corresponds to the cosmological singularity.

### 4.1 Adjusted ADM formulation

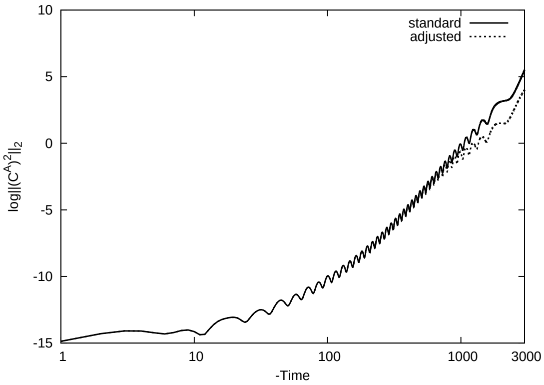


Figure 1: Polarized Gowdy-wave test with the adjusted ADM system. The vertical axis is $\log(||(C^A)^2||_2)$ and the horizontal axis is backward time. The dotted line is the one with (6)-(7) by setting $\kappa_{\gamma ijmn} = 1.0 \times 10^{-4.8}\alpha\gamma_{im}\gamma_{jn}$ and $\kappa_{Kijmn} = 1.0 \times 10^{-5.4}\alpha\gamma_{im}\gamma_{jn}$. The solid line is calculated with the standard ADM.

We see from Figure 1 that the adjusted ADM system, (6)-(7), has better stability than the standard ADM system. The norm $||(C^A)^2||_2$ of the adjusted ADM is $7.24 \times 10^{-1}$ times of that of the standard ADM at time $t = -3000$.

### 4.2 Adjusted BSSN formulation

The comparison of the evolutions between the standard BSSN system and the $C^2$-adjusted BSSN system is shown in Figure 2.

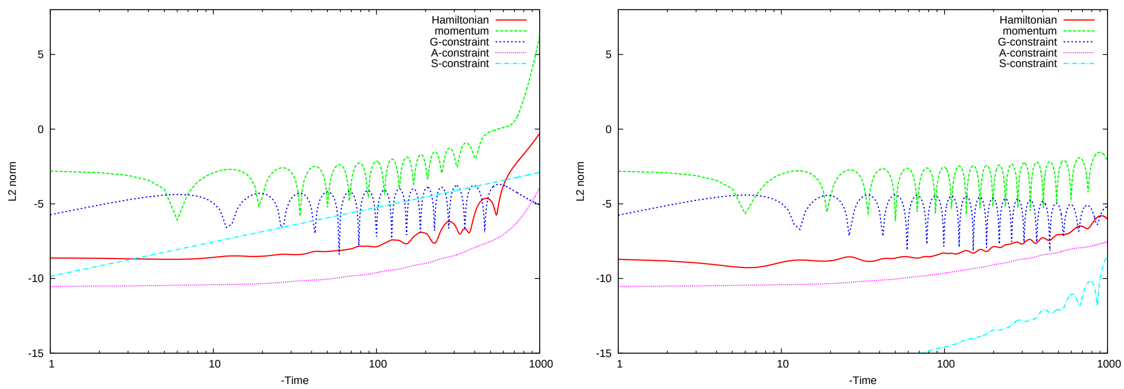


Figure 2: Constraint violations of the standard BSSN system, and of the $C^2$-adjusted BSSN system. We see the adjusted version shows better performance than the standard system. ($\lambda_\varphi = 0, \lambda_K = 10^{-2.7}, \lambda_{\tilde{\gamma}} = 10^{-5.0}\delta_{im}\delta_{jn}, \lambda_{\tilde{A}} = 0, \lambda_{\tilde{\Gamma}} = 10^{-1.4}\delta_{ij}$.)

We think the stability of the adjusted BSSN formulation is explained by the dumping of $\mathcal{M}_i^B$ at the early time (about $t \leq -20$). As was argued by Kiuchi and Shinkai[4], the key of the stability of the evolution with BSSN system is to dump $\mathcal{M}_i^B$ earlier. We see the adjusted version shows better performance than the standard system by improving the violation of $\mathcal{M}_i^B$ in the initial stage.

We also compared the L2 norm of the $(C^B)^2$ of three systems, including another type of adjustment,

$$\partial_t \widetilde{A}_{ij} = (12) + \kappa_A \alpha \widetilde{D}_{(i}\mathcal{M}_{j)}$$

with $\kappa_A = 10^{-2.4}$. $C^2$-adjusted BSSN system keeps the violation of constraint lowest between these three types of formulations.

## 5 Summary

In this report, we applied the adjusting method suggested by Fiske to the ADM and BSSN formulations, and obtained the equations (6)-(7) and (19)-(23). We performed numerical tests with polarized Gowdy wave and showed that the adjusted ADM and BSSN systems have actually better stablility than the standard ADM and BSSN systems.

The advantage of the present systems to the previous adjusted systems [5, 8] is the way of specifying the Lagrange multipliers $\kappa$. In the present systems, $\kappa$s are restricted as "positive definite" from the formulation independent on the background metric, while in the previous systems one needs to specify the signature of $\kappa$s with eigenvalue analysis which depends on the background metric.

The detail numerical analysis on the range of effective parameters and the comparisons with other systems are underway.

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