Constraint Propagation Revisited
– Adjusted ADM Formulations for Numerical Relativity –

Hisa-aki Shinkai
真貝寿明　@　大阪工業大学　情報科学部
Dept of Information Science, Osaka Institute of Technology, Japan
shinkai@is.oit.ac.jp

Outline

A search of a formulation for stable numerical evolution

- Adjust ADM formulation with constraints $\Longrightarrow$ Attractor System
- A New Criteria for adjusting rules
- Numerical test with 3D Teukolsky wave evolution $\Longrightarrow$ Longer Stability

Work with **Gen Yoneda** Math. Sci. Dept., Waseda Univ., Japan

@ 16th JGRG, Niigata, November, 2006

Plan of the talk

1. Introduction
   (0) Arnowitt-Deser-Misner
2. Three approaches
   (1) Baumgarte-Shapiro-Shibata-Nakamura formulation
   (2) Hyperbolic formulations
   (3) Attractor systems – "Adjusted Systems"
3. Adjusted ADM systems
   Flat background
   Schwarzschild background
   + Numerical Examples
4. Adjusted BSSN systems
   Flat background
   + Numerical Examples
5. Summary and Future Issues

General Relativity: historical overview

特殊相対性理論 (Special Theory of Relativity) 1905 Einstein

基本原理１　相対性原理「すべての自然法則はあらゆる慣性系において同一である」
All inertial observers are equivalent.

基本原理２　光速不変の原理　$\longleftarrow$　伝播速度が有限　＋　相対性原理
The velocity of light is the same in all inertial systems.

$\Longrightarrow$ 時間も絶対的なものではない．time is relative!
$\Longrightarrow$ 光速近くの慣性系どうしの座標変換は,「Gallilei 変換」に代わる「Lorentz 変換」
$\Longrightarrow$ 相対論的運動学，共変的 Maxwell 方程式

一般相対性理論 (General Theory of Relativity) 1916 Einstein

一般化１　加速運動している系においても成立する力学法則？ in an accelerated frame?
一般化２　重力を含む系の力学？ including gravity?

$\Longrightarrow$ 等価原理と局所慣性系の導入　equivalence principle, local inertial frame
$\Longrightarrow$ 「Euclid 幾何学」から「Riemann 幾何学」へ．重力の正体は，時空の歪みである

Riemannian Geometry

$\boxed{\textbf{metric (計量)}}$

$$
\begin{aligned}
ds^2 &= \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu := g_{\mu\nu} dx^\mu dx^\nu \qquad\qquad \mu,\nu = 0,\cdots,3 \\
g_{\mu\nu} &= \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ & g_{xx} & g_{xy} & g_{xz} \\ & & g_{yy} & g_{yz} \\ sym. & & & g_{zz} \end{pmatrix}
\end{aligned}
$$

for flat spacetime (Minkowskii spacetime):

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\
&= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)
\end{aligned}
$$

$\boxed{\textbf{Christoffel symbol}}$

$$\Gamma^{\alpha}_{\mu\nu} \; = \; \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$$

Covariant derivative

$$\nabla_\mu A^\alpha \; \equiv \; A^\alpha_{;\mu} \equiv A^\alpha{}_{,\mu} + \Gamma^\alpha_{\mu\nu} A^\nu$$
$$\nabla_\mu A_\alpha \; \equiv \; A_{\alpha;\mu} \equiv A_{\alpha,\mu} - \Gamma^\nu_{\alpha\mu} A_\nu$$

Geodesic deviation equation

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

$\boxed{\textbf{Riemann tensor}}$

$$R^\mu{}_{\nu\alpha\beta} \; \equiv \; \Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta} + \Gamma^\mu{}_{\sigma\alpha}\Gamma^\sigma{}_{\nu\beta} - \Gamma^\mu{}_{\sigma\beta}\Gamma^\sigma{}_{\nu\alpha}$$

$\boxed{\textbf{Ricci tensor}}$

$$R_{ab} \equiv R^\mu{}_{a\mu b} \; \equiv \; \Gamma^\mu_{ab,\mu} - \Gamma^\mu_{a\mu,b} + \Gamma^\mu{}_{\nu\mu}\Gamma^\nu{}_{ab} - \Gamma^\mu{}_{\nu b}\Gamma^\nu{}_{a\mu}$$

Ricci scalar

$$R \equiv R^\mu_\mu$$

$\boxed{\textbf{Einstein tensor}}$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G \, T_{\mu\nu}$$

For the perfect fluid,
(no shear, no viscosity, and isotropic pressure)

$$T_{\mu\nu} = \begin{pmatrix} -\rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$ **and** $p = p(\rho)$

0.1 Application 1: Black Holes

Schwarzschild (1917) : spherical symmetric, vacuum spacetime

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

or in the Kruskal coordinate

$$ds^2 = -\frac{32M^3}{r}e^{-r/2M}d\tilde{u}d\tilde{v} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

where $\tilde{u} = -(r/2M - 1)^{1/2}e^{(r-t)/4M}, \quad \tilde{v} = (r/2M - 1)^{1/2}e^{(r-t)/4M}$

The solution has a horizon at $r = 2M$**, a singularity at** $r = 0$ **(**$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 48M^2/r^4$**).**

0.2 Application 2: Cosmology

Friedman-Robertson-Walker spacetime: homogeneous and isotropic with perfect fluid

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right], \qquad T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$$

Friedman equation

$$\begin{aligned} \left(\frac{a_{,t}}{a}\right)^2 &= -\frac{k}{a^2} + \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \\ 2\left(\frac{a_{,tt}}{a}\right)^2 &= -\left(\frac{a_{,t}}{a}\right)^2 - \frac{k}{a^2} - 8\pi G p + \Lambda \end{aligned}$$

The standard Big-Bang model is supported by

- **observation of the expanding Universe**
- **cosmological microwave background (2.7K)**
- **light element amount from nucleosynthesis theory**

The inflationary scenario explains

- **flatness/horizon problem in FRW model**
- **topological defect problem in high energy theory**
- **origin of the structure formation**

0.3 Application 3: Gravitational Wave

LIGO	**USA**	**4Km**	$\sim$ **2002?**	**http://www.ligo.caltech.edu/**
VIRGO	**Italy/France**	**3Km**	$\sim$ **2002?**	**http://www.virgo.infn.it/**
GEO	**Germany/UK**	**600m**	$\sim$ **2002?**	**http://www.geo600.uni-hannover.de/**
TAMA	**Japan**	**300m**	**1999** $\sim$	**http://tamago.mtk.nao.ac.jp/**
LISA	**NASA/ESA**	**5** 10^6 **Km**	**2008?** $\sim$	**http://lisa.jpl.nasa.gov/**

0.4 Application 3: Weak field limit

•

Numerical Relativity := Solve the Einstein eq. numerically

TARGET
To figure out gravitational wave form in the final process of coalescence of binary blackholes and/or neutron stars.

What are the difficulties?

- **Einstein eq. is for 10-component metric, mixed with 4 elliptic eqs and 6 dynamical eqs.**
- **Completely free to choose coordinates, gauge conditions, and even for decomposition of the space-time.**
- **Has singularity in its nature.**
- **How to construct a realistic initial data?**
- **Cannot evolve the system stably in long-term evolution. Why?**

Not only computational power, but still theoretical understanding, we need

The standard approach :: ADM formulation (1962)

- **Use Arnowitt-Deser-Misner formulation, 3+1 decomposition of the spacetime. Evolve 12 variables in (x, y, z) Cartesian grid, with a choice of appropriate gauge condition.**

	Maxwell eqs.	**ADM Einstein eq.**
constraints	$\textbf{div } \mathbf{E} = 4\pi\rho$ $\textbf{div } \mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} = \textbf{rot } \mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -\textbf{rot } \mathbf{E}$	$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_j\beta_i + D_i\beta_j,$ $\partial_t K_{ij} = \alpha({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2\alpha K_{il}K^l{}_j - D_iD_j\alpha$ $+ (D_j\beta^m)K_{mi} + (D_i\beta^m)K_{mj} + \beta^m D_m K_{ij} - \alpha\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

Status of Numerical Relativity

May, 2001

problems	have done	not yet	refs
現実的な初期条件 realistic initial data	NSNS 連星の場合，星の内部回転が，synchronized/irrotational の両極端の場合解けた	中間段階の回転則では? 一般の状態方程式の場合? 背景重力波の影響? BHBH の場合?	Uryu et al. PRD61(2000)124023 Uryu et al. PRD62(2001)104015 Meudon PRD63 (2001) 064029 . . . and etc etc . . .
計量の安定な発展方法 spacetime evolution	Standard ADM では不安定である CT-ADM 変形で一歩前進 双曲型運動方程式の提案と実験 漸近的拘束システムの提案と実験	提案されている他のシステムでは? より一般的な状況では? より統一的な理解は?	Shibata-Nakamura, PRD52(1995)542 Illinois, PRD59(1999)24007 . . . and etc etc . . . **this talk!** . . . . . .
相対論的流体の発展 matter evolution	「ほぼ解決」と言われているが... ??	星の表面の取り扱い?	Shibata, PRD60(1999)104052 WashU, PRD61(2000)044011 . . . and etc etc . . .
ゲージ条件 gauge condition	試行錯誤的 trials and errors	BH 発生後どこまで発展できるか? 新たな工夫が必要か? 解析的に議論できないか?	Smarr-York PRD17(1978)1945 Shibata, PTP101(1999)1199 . . . and etc etc . . .
BH 判定方法 (AH) horizon finding methods	解決	より収束の早い方法? 新たな BH の定義との関連?	Texas, Cornell, Postdam, Japan, ... and etc etc . . .
BH の切り抜き BH excision	球対称時空では成功 一般時空では試行錯誤	未解決の問題多い	Grand Challenge, PRL80 (1998) 251 Alcubierre et al, PRD63 (2001) 1040 PennState, PRL85 (2000) 5496 Postdam, gr-qc/0104020 . . . etc
重力波の抽出 GW extraction	3+1 から 2+2 への変換 3+1 から 2+2 への接続	精度を上げるためには AMR? 新たな近似方法の有効性? 準局所的質量の導入?	Shinkai et al, CQG12 (1995) 133 Grand Challenge, PRL80 (1998) 181 Grand Challenge, PRL80 (1998) 391 Postdam, Texas, PennState, . . . etc