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$N + 1$ formalism in Einstein-Gauss-Bonnet gravity

Takashi Torii[1,*] and Hisa-aki Shinkai[2,+]

[1]*Department of General Education, Osaka Institute of Technology, Omiya, Asahi-ku, Osaka 535-8585, Japan*
[2]*Department of Information Systems, Osaka Institute of Technology, Kitayama, Hirakata, Osaka 573-0196, Japan*
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Towards the investigation of the full dynamics in a higher-dimensional and/or a stringy gravitational model, we present the basic equations of the Einstein-Gauss-Bonnet gravity theory. We show the $(N + 1)$-dimensional version of the Arnowitt-Deser-Misner decomposition including Gauss-Bonnet terms, which shall be the standard approach to treat the space-time as a Cauchy problem. Because of the quasilinear property of the Gauss-Bonnet gravity, we find that the evolution equations can be in a treatable form in numerics. We also show the conformally transformed constraint equations for constructing the initial data. We discuss how the constraints can be simplified by tuning the powers of conformal factors. Our equations can be used both for timelike and spacelike foliations.

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I. INTRODUCTION

General relativity (GR) has been tested with many experiments and observations in both the strong and the weak gravitational field regimes (see e.g. [1]), and none of them are contradictory to GR. However, the theory also predicts the appearance of the space-time singularities under natural conditions [2,3], which also indicates that GR is still incomplete as a physics theory that describes the whole of the gravity and the space-time structure.

We expect that the true fundamental theory will resolve these theoretical problems. Up to now, several quantum theories of gravity have been proposed. Among them, the superstring/M theory, formulated in higher dimensional space-time, is the most promising candidate. In these prospects, a considerable number of studies concerned with gravitational phenomena and cosmology have been made in the string theoretical framework beyond GR.

Since the present knowledge is still far from understanding the full aspects of the string theory, several kinds of approaches, which are fundamentally approximations, are usually taken. Among them, the perturbative approach plays important roles. There are two particular parameters which characterize the system in the superstring theory. One is string coupling parameter $g_s^2 = e^{\phi}$, where ϕ is the dilaton field. The other is the inverse string tension α'. When the tension is strong (i.e., small α') compared to the energy scale of the system, it is difficult to excite strings, and the size of the strings becomes small enough to be regarded as particles in the zeroth order approximation. In this limit GR (with other light fields) is recovered. This is called α' expansion [4].

In the higher order terms of α', curvature corrections appear. The Gauss-Bonnet (GB) term is the next leading order of the α' expansion of type IIB superstring theory [4,5] and has nice properties such that it is ghost-free combinations [6] and does not give higher derivative equations but an ordinary set of equations with up to the second derivative in spite of the higher curvature combinations.

The models with the GB term and/or other higher curvature terms have been intensively studied in high energy physics. One of them is a series of studies in string cosmology. The pre-big-bang scenario [7] is a fantastic scenario which tries to avoid the big-bang singularity by making use of T duality [8] (or scale factor duality). Furthermore, the pre-big-bang scenario gives a natural inflation mechanism, since the solution in the pre-big-bang phase is inflating from the beginning at least in the string frame. Although these analyses show that the singularity problem has not been resolved yet completely, there are some cosmological solutions which do not start from an initial singularity [9–11].

In addition to cosmology, the string effects can be also seen in the study of black hole physics. As the size of a black hole becomes small, and the curvature around the black hole becomes large, it is expected that the curvature corrections cannot be negligible. The singularity inside of the event horizon would be also modified or even disappear by string effects. For these reasons, the static or stationary black hole solutions in effective string theories were investigated in the systems both without higher curvature terms [12,13] and with such terms [14–16]. Besides the static or stationary solutions, there are some dynamical solutions which are motivated by gravitational collapse [17].

These analyses are performed on the assumption of highly symmetric space-time because the system is much more complicated than that in GR. To obtain a deeper understanding of the early stage of the Universe, singularity, and/or black holes, we should consider a less symmetric and/or dynamical space-time; the analyses require the direct numerical integration of the equations. None of the fully dynamical simulations in GB gravity has been performed.

*torii@ge.oit.ac.jp
+shinkai@is.oit.ac.jp

In this article, we present the basic equations of the Einstein-GB gravity theory. We show the $(N+1)$-dimensional version of the Arnowitt-Deser-Misner (ADM) decomposition, which is the standard approach to treat the space-time as a Cauchy problem. The topic was first discussed by Choquet-Bruhat [18], but the full set of equations and the methodology have not yet been presented. In four-dimensional GR, numerical simulations of binary compact objects are available in the past years, and many groups apply the modified ADM equations in order to obtain long-term stable and accurate simulations. However, such modifications depend on the problem to consider, and the "robustest" formulation is not yet known (see e.g. [19]). Therefore, as the first step, we in this paper just present the fundamental space-time decomposition of the GB equations, focusing on the GB term.

The ADM decomposition is supposed to construct the space-time with foliations of the constant-time hypersurfaces. This method can be also applied to study the brane-world model [20], which states the visible space-time is embedded in higher dimensional "bulk" space-time. As was first investigated by Chamblin *et al.* [21], it is possible to study the bulk structure by switching the normal vector of the hypersurface from timelike to spacelike. We therefore present all sets of equations for both cases for future convenience.

The outline of this paper is as follows. In Sec. II, we show that the set of equations is divided into two constraints and evolution equations according to the standard procedure. In Sec. III, we present the conformal approach to solve the constraints which shall be used for preparing the initial data. In Sec. IV, we show the dynamical equations in GR and in GB theory separately. Section V is devoted to the discussions and summary. We think these expressions are useful for future dynamical investigations.

II. $(N+1)$ DECOMPOSITION IN EINSTEIN-GAUSS-BONNET GRAVITY

A. Model and basic equations

We start from the Einstein-Gauss-Bonnet action in $(N+1)$-dimensional space-time $(\mathcal{M}, g_{\mu\nu})$ which is described as [22]

$$S = \int_{\mathcal{M}} d^{N+1}X\sqrt{-g}\Big[\frac{1}{2\kappa^2}(\mathcal{R} - 2\Lambda + \alpha_{\rm GB}\mathcal{L}_{\rm GB}) + \mathcal{L}_{\rm matter}\Big], \tag{1}$$

with

$$\mathcal{L}_{\rm GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}, \tag{2}$$

where κ^2 is the $(N+1)$-dimensional gravitational constant and $\mathcal{R}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}_{\mu\nu\rho\sigma}$, and $\mathcal{L}_{\rm matter}$ are the $(N+1)$-dimensional scalar curvature, Ricci tensor, Riemann curvature, and matter Lagrangian, respectively. This action reproduces the standard $(N+1)$-dimensional Einstein gravity, if we set the coupling constant $\alpha_{\rm GB}(\geq 0)$ equal to zero.

The action (1) gives the gravitational equation as

$$\mathcal{G}_{\mu\nu} + \alpha_{\rm GB}\mathcal{H}_{\mu\nu} = \kappa^2\mathcal{T}_{\mu\nu}, \tag{3}$$

where

$$\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R} + g_{\mu\nu}\Lambda, \tag{4}$$

$$\mathcal{H}_{\mu\nu} = 2[\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}{}^{\alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}] - \tfrac{1}{2}g_{\mu\nu}\mathcal{L}_{\rm GB}, \tag{5}$$

$$\mathcal{T}_{\mu\nu} = -2\frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\rm matter}. \tag{6}$$

B. Projections to hypersurface

In order to investigate the space-time structure as the foliations of the N-dimensional (spacelike or timelike) hypersurface Σ, we introduce the projection operator to Σ as

$$\perp_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu, \tag{7}$$

where n_μ is the unit-normal vector to Σ with $n_\mu n^\mu = \varepsilon$, with which we define n_μ as timelike (if $\varepsilon = -1$) or spacelike (if $\varepsilon = 1$). Therefore, Σ is spacelike (timelike) if n_μ is timelike (spacelike).

The projections of the gravitational equation (3) give the following three equations:

$$(\mathcal{G}_{\mu\nu} + \alpha_{\rm GB}\mathcal{H}_{\mu\nu})n^\mu n^\nu = \kappa^2\mathcal{T}_{\mu\nu}n^\mu n^\nu = \kappa^2\rho, \tag{8}$$

$$(\mathcal{G}_{\mu\nu} + \alpha_{\rm GB}\mathcal{H}_{\mu\nu})n^\mu \perp^\nu{}_\rho = \kappa^2\mathcal{T}_{\mu\nu}n^\mu\perp^\nu_\rho = -\kappa^2 J_\rho, \tag{9}$$

$$(\mathcal{G}_{\mu\nu} + \alpha_{\rm GB}\mathcal{H}_{\mu\nu})\perp^\mu{}_\rho\perp^\nu{}_\sigma = \kappa^2\mathcal{T}_{\mu\nu}\perp^\mu{}_\rho\perp^\nu{}_\sigma = \kappa^2 S_{\rho\sigma}, \tag{10}$$

where we defined the components of the energy-momentum tensor as

$$\mathcal{T}_{\mu\nu} = \rho n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}, \tag{11}$$

and we also define $\mathcal{T} = \varepsilon\rho + S^\alpha{}_\alpha$ for later convenience.

Projection of the $(N+1)$-dimensional Riemann tensor onto the N-dimensional hypersurface can be written as

$$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha{}_\mu\perp^\beta{}_\nu\perp^\gamma{}_\rho\perp^\delta{}_\sigma = R_{\mu\nu\rho\sigma} - \varepsilon(K_{\mu\rho}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\rho}), \tag{12}$$

$$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha{}_\mu\perp^\beta{}_\nu\perp^\gamma{}_\rho n^\delta = -D_{[\mu}K_{\nu]\rho}, \tag{13}$$

$$\mathcal{R}_{\alpha\beta\gamma\delta}\perp^\alpha{}_\mu\perp^\gamma{}_\rho n^\beta n^\delta = \mathcal{L}_n K_{\mu\rho} + K_{\mu\alpha}K^\alpha{}_\rho, \tag{14}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the induced metric $\gamma_{\mu\nu}(=\perp_{\mu\nu})$, D_μ is the covariant differentiation with respect to $\gamma_{\mu\nu}$, $\mathcal{L}_n$ denotes the Lie derivative in the n direction, and $K_{\mu\nu}$ is the extrinsic curvature defined as

$$K_{\mu\nu} = -\tfrac{1}{2}\mathcal{L}_n \gamma_{\mu\nu} = -\perp^\alpha{}_\mu \perp^\beta{}_\nu \nabla_\alpha n_\beta. \tag{15}$$

Equation (12) is called the Gauss equation, and the contraction of (13) of μ and ν gives the Codacci equation.

Using these projections, the $(N+1)$-dimensional Riemann curvature and its contractions (the Ricci tensor and scalar curvature) are described by the N-dimensional variables on the hypersurface Σ as

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} - \varepsilon(K_{\mu\rho}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\rho} - n_\mu D_\rho K_{\sigma\nu} + n_\mu D_\sigma K_{\rho\nu} + n_\nu D_\rho K_{\sigma\mu} - n_\nu D_\sigma K_{\rho\mu} - n_\rho D_\mu K_{\nu\sigma} \\ &\quad + n_\rho D_\nu K_{\mu\sigma} + n_\sigma D_\mu K_{\nu\rho} - n_\sigma D_\nu K_{\mu\rho}) + n_\mu n_\rho K_{\nu\alpha}K^\alpha{}_\sigma - n_\mu n_\sigma K_{\nu\alpha}K^\alpha{}_\rho - n_\nu n_\rho K_{\mu\alpha}K^\alpha{}_\sigma + n_\nu n_\sigma K_{\mu\alpha}K^\alpha{}_\rho \\ &\quad + n_\mu n_\rho \mathcal{L}_n K_{\nu\sigma} - n_\mu n_\sigma \mathcal{L}_n K_{\nu\rho} - n_\nu n_\rho \mathcal{L}_n K_{\mu\sigma} + n_\nu n_\sigma \mathcal{L}_n K_{\mu\rho},\end{aligned} \tag{16}$$

$$\begin{aligned}\mathcal{R}_{\mu\nu} &= R_{\mu\nu} - \varepsilon[KK_{\mu\nu} - 2K_{\mu\alpha}K^\alpha{}_\nu + n_\mu(D_\alpha K^\alpha{}_\nu - D_\nu K) + n_\nu(D_\alpha K^\alpha{}_\mu - D_\mu K)] + n_\mu n_\nu K_{\alpha\beta}K^{\alpha\beta} + \varepsilon \mathcal{L}_n K_{\mu\nu} \\ &\quad + n_\mu n_\nu \gamma^{\alpha\beta}\mathcal{L}_n K_{\alpha\beta},\end{aligned} \tag{17}$$

$$\mathcal{R} = R - \varepsilon(K^2 - 3K_{\alpha\beta}K^{\alpha\beta} - 2\gamma^{\alpha\beta}\mathcal{L}_n K_{\alpha\beta}), \tag{18}$$

where $K = K^\alpha{}_\alpha$.

Substituting these relations into the field equation (3) or (8)–(10), we find the equations are decomposed as (a) the Hamiltonian constraint equation

$$\begin{aligned}&M + \alpha_{\rm GB}(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}) \\ &\quad = -2\varepsilon\kappa^2\rho_H + 2\Lambda,\end{aligned} \tag{19}$$

(b) the momentum constraint equation

$$\begin{aligned}&N_i + 2\alpha_{\rm GB}(MN_i - 2M_i{}^a N_a + 2M^{ab}N_{iab} - M_i{}^{cab}N_{abc}) \\ &\quad = \kappa^2 J_i,\end{aligned} \tag{20}$$

and (c) the evolution equations for γ_{ij}

$$\begin{aligned}&M_{ij} - \tfrac{1}{2}M\gamma_{ij} - \varepsilon(-K_{ia}K^a{}_j + \gamma_{ij}K_{ab}K^{ab} - \mathcal{L}_n K_{ij} \\ &\quad + \gamma_{ij}\gamma^{ab}\mathcal{L}_n K_{ab}) + 2\alpha_{\rm GB}[H_{ij} + \varepsilon(M\mathcal{L}_n K_{ij} \\ &\quad - 2M_i{}^a \mathcal{L}_n K_{aj} - 2M_j{}^a \mathcal{L}_n K_{ai} - W_{ij}{}^{ab}\mathcal{L}_n K_{ab})] \\ &= \kappa^2 S_{ij} - \gamma_{ij}\Lambda,\end{aligned} \tag{21}$$

respectively, where

$$M_{ijkl} = R_{ijkl} - \varepsilon(K_{ik}K_{jl} - K_{il}K_{jk}), \tag{22}$$

$$M_{ij} = \gamma^{ab}M_{iajb} = R_{ij} - \varepsilon(KK_{ij} - K_{ia}K^a{}_j), \tag{23}$$

$$M = \gamma^{ab}M_{ab} = R - \varepsilon(K^2 - K_{ab}K^{ab}), \tag{24}$$

$$N_{ijk} = D_i K_{jk} - D_j K_{ik}, \tag{25}$$

$$N_i = \gamma^{ab}N_{aib} = D_a K_i{}^a - D_i K, \tag{26}$$

$$\begin{aligned}H_{ij} &= MM_{ij} - 2(M_{ia}M^a{}_j + M^{ab}M_{iajb}) + M_{iabc}M_j{}^{abc} - 2\varepsilon[-K_{ab}K^{ab}M_{ij} - \tfrac{1}{2}MK_{ia}K_j{}^a + K_{ia}K^a{}_b M^b{}_j + K_{ja}K^a{}_b M^b{}_i \\ &\quad + K^{ac}K_c{}^b M_{iajb} + N_i N_j - N^a(N_{aij} + N_{aji}) - \tfrac{1}{2}N_{abi}N^{ab}{}_j - N_{iab}N_j{}^{ab}] - \tfrac{1}{4}\gamma_{ij}[M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}] \\ &\quad - \varepsilon\gamma_{ij}[K_{ab}K^{ab}M - 2M_{ab}K^{ac}K_c{}^b - 2N_a N^a + N_{abc}N^{abc}],\end{aligned} \tag{27}$$

$$W_{ij}{}^{kl} = M\gamma_{ij}\gamma^{kl} - 2M_{ij}\gamma^{kl} - 2\gamma_{ij}M^{kl} + 2M_{iajb}\gamma^{ak}\gamma^{bl}. \tag{28}$$

We remark that the terms of $\mathcal{L}_n K_{ij}$ appear only in the linear form in (21). This is due to the quasilinear property of the GB gravity.

C. Cauchy approach

Using the Bianchi identity, Choquet-Bruhat [18] showed that the set of equations forms the first-class system the same as in GR; that is, a spacelike hypersurface which satisfies the constraints (19) and (20) will also satisfy the constraints after the evolution using (21).

The most standard procedures for following the dynamics of space-time consist of three steps: (i) solve the constraint equations (19) and (20) for $(\gamma_{ij}, K_{ij}, \rho_H, J_i)$ on $\Sigma(t=0)$ and prepare them as the initial data, (ii) evolve $(\gamma_{ij}, K_{ij}, \rho_H, J_i)$ using (21) and the matter equations, and

(iii) monitor the accuracy of the evolutions by checking constraint equations on the evolved $\Sigma(t)$.

In the case of seeking the "dynamics" along a spacelike direction χ such as a study of the bulk structure in the brane-world model, the above strategy can be switched to the evolution in χ coordinates instead of t (using the set of equations of $\varepsilon = +1$). The initial data, in this case, is a timelike hypersurface which should satisfy the constraints. Such initial data can be obtained either by solving the dynamics of the "brane" part or by taking double Wick rotation after the above step (i), depending on the models and motivations.

In the following sections, we describe a way of solving constraints (Sec. III) and a way of solving evolution equations (Sec. IV).

III. CONFORMAL APPROACH TO SOLVE THE CONSTRAINTS

A. "Conformal approach"

In order to prepare the initial data for dynamical evolution, we have to solve two constraints: (19) and (20). The standard approach is to apply a conformal transformation on the initial hypersurface [24]. The idea is to introduce a conformal factor ψ between the initial trial metric $\hat{\gamma}_{ij}$ and the solution γ_{ij}, as

$$\gamma_{ij} = \psi^{2m}\hat{\gamma}_{ij}, \qquad \gamma^{ij} = \psi^{-2m}\hat{\gamma}^{ij}, \tag{29}$$

where m is a constant, and solve for ψ so the solution satisfies the constraints.

For N-dimensional space-time, the Ricci scalar is transformed as

$$R = \psi^{-2m}\{\hat{R} - 2(N-1)m\psi^{-1}(\hat{D}^a\hat{D}_a\psi) + (N-1)[2-(N-2)m]m\psi^{-2}(\hat{D}\psi)^2\}, \tag{30}$$

$$R_{ij} = \hat{R}_{ij} - m\hat{\gamma}_{ij}\psi^{-1}\hat{D}_a\hat{D}^a\psi - (N-2)m\psi^{-1}\hat{D}_i\hat{D}_j\psi + (N-2)m(m+1)\psi^{-2}\hat{D}_i\psi\hat{D}_j\psi - m[(N-2)m-1]\psi^{-2}(\hat{D}\psi)^2\hat{\gamma}_{ij}, \tag{31}$$

$$R_{ijkl} = \psi^{2m}\{\hat{R}_{ijkl} + m\psi^{-1}\hat{\gamma}_{il}[\hat{D}_j\hat{D}_k\psi - (m+1)\psi^{-1}\hat{D}_j\psi\hat{D}_k\psi] - m\psi^{-1}\hat{\gamma}_{ik}[\hat{D}_j\hat{D}_l\psi - (m+1)\psi^{-1}\hat{D}_j\psi\hat{D}_l\psi] + m\psi^{-1}\hat{\gamma}_{jk}[\hat{D}_i\hat{D}_l\psi - (m+1)\psi^{-1}\hat{D}_i\psi\hat{D}_l\psi] - m\psi^{-1}\hat{\gamma}_{jl}[\hat{D}_i\hat{D}_k\psi - (m+1)\psi^{-1}\hat{D}_i\psi\hat{D}_k\psi] + m^2\psi^{-2}(\hat{D}\psi)^2(\hat{\gamma}_{il}\hat{\gamma}_{jk} - \hat{\gamma}_{ik}\hat{\gamma}_{jl})\}. \tag{32}$$

Regarding the extrinsic curvature, we decompose K_{ij} into its trace part $K = \gamma^{ij}K_{ij}$ and the traceless part $A_{ij} = K_{ij} - \frac{1}{N}\gamma_{ij}K$ and assume the conformal transformation [25] as

$$A_{ij} = \psi^{\ell}\hat{A}_{ij}, \qquad A^{ij} = \psi^{\ell-4m}\hat{A}^{ij}, \tag{33}$$

$$K = \psi^{\tau}\hat{K}, \tag{34}$$

where ℓ and τ are constants. For the matter terms, we also assume the relations $\rho = \psi^{-p}\hat{\rho}$ and $J^i = \psi^{-q}\hat{J}^i$, where p and q are constants, while we regard the cosmological constant as common to both flames, $\Lambda = \hat{\Lambda}$.

Up to here, the powers of conformal transformation ℓ, m, τ, p, and q are not yet specified. Note that, in the standard three-dimensional initial-data construction cases, the combination of $m = 2$, $\ell = -2$, $\tau = 0$, $p = 5$, and $q = 10$ is preferred since this simplifies the equations. We also remark that if we choose $\tau = \ell - 2m$, then the extrinsic curvature can be transformed as $K_{ij} = \psi^{\ell}\hat{K}_{ij}$ and $K^{ij} = \psi^{\ell-4m}\hat{K}^{ij}$.

B. Hamiltonian constraint

Using these equations, the Hamiltonian constraint equation (19) turns to be

$$2(N-1)m\hat{D}_a\hat{D}^a\psi - (N-1)[2-(N-2)m]m(\hat{D}\psi)^2\psi^{-1} = \hat{R}\psi - \frac{N-1}{N}\varepsilon\psi^{2m+2\tau+1}\hat{K}^2 + \varepsilon\psi^{-2m+2\ell+1}\hat{A}_{ab}\hat{A}^{ab} + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} + \alpha_{\rm GB}\hat{\Theta}\psi^{2m+1}. \tag{35}$$

We will show the explicit form of the GB part $\hat{\Theta} = M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}$ in Appendix A. At this moment, we observe that (35) can be simplified in the following two ways.

(A) If we specify $\tau = \ell - 2m$ and $m = 2/(N-2)$, then (35) becomes

$$\frac{4(N-1)}{N-2}\hat{D}_a\hat{D}^a\psi = \hat{R}\psi - \varepsilon\psi^{2\ell+1-4/(N-2)} \times (\hat{K}^2 - \hat{K}_{ab}\hat{K}^{ab}) + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} + \alpha_{\rm GB}\hat{\Theta}\psi^{1+4/(N-2)}. \tag{36}$$

In the case of the Einstein gravity ($\alpha_{\rm GB} = 0$) with $\Lambda = 0$, the combination $\ell = 2/(N-2)$ and $p = -1$ makes the right-hand side of (36) linear. If we choose $\ell = -2$, which will make the momentum constraint simpler as we see later, (36) also remains as a simple equation.

(B) If we specify $\tau = 0$ and $m = 2/(N-2)$, then (35) becomes

$$\frac{4(N-1)}{N-2}\hat{D}_a\hat{D}^a\psi = \hat{R}\psi - \varepsilon\frac{N-1}{N}\psi^{1+4/(N-2)}\hat{K}^2 + \varepsilon\psi^{2\ell+1-4/(N-2)}\hat{A}_{ab}\hat{A}^{ab} + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} + \alpha_{\rm GB}\hat{\Theta}\psi^{1+4/(N-2)}. \quad (37)$$

C. Momentum constraint

In order to express the momentum constraint equation in a tractable form, additionally to the variables in Sec. III A, we introduce the transverse traceless part and the longitudinal part of $\hat{A}^{ij}$ as

$$\hat{D}_j\hat{A}^{ij}_{TT} = 0, \quad (38)$$

$$\hat{A}^{ij}_L = \hat{A}^{ij} - \hat{A}^{ij}_{TT}, \quad (39)$$

respectively. Since the conformal transformation of the divergence $D_jA_i{}^j$ becomes

$$D_jA_i{}^j = \psi^{\ell-2m}\{\hat{D}_j\hat{A}_i{}^j + \psi^{-1}[\ell + m(N-2)]\hat{A}_i{}^j\hat{D}_j\psi\}, \quad (40)$$

the momentum constraint (20) can be written as

$$\psi^{\ell-2m}\hat{D}_a\hat{A}_{iL}{}^a + [\ell + (N-2)m]\psi^{\ell-2m-1}\hat{A}_{iL}{}^a\hat{D}_a\psi - \frac{N-1}{N}\hat{D}_i(\psi^\tau\hat{K}) + 2\alpha_{\rm GB}\hat{\Xi}_i = \kappa^2\psi^{2m-q}\hat{J}_i. \quad (41)$$

We will show the explicit form of the GB part $\hat{\Xi}_i$ in Appendix B; meanwhile, we proceed with the standard procedure, that is, introducing a vector potential W^i for $\hat{A}^{ij}_L$ as

$$\hat{A}^{ij}_L = \hat{D}^iW^j + \hat{D}^jW^i - \frac{2}{N}\hat{\gamma}^{ij}\hat{D}_kW^k. \quad (42)$$

Since the divergence of $\hat{A}^{ij}_L$ becomes

$$\hat{D}_j\hat{A}^{ij}_L = \hat{D}_a\hat{D}^aW^i + \frac{N-2}{N}\hat{D}^i\hat{D}_kW^k + \hat{R}^i{}_kW^k, \quad (43)$$

the momentum constraint (41) becomes

$$\hat{D}_a\hat{D}^aW_i + \frac{N-2}{N}\hat{D}_i\hat{D}_kW^k + \hat{R}_{ik}W^k + \psi^{-1}[\ell + (N-2)m] \times\left(\hat{D}^aW^b + \hat{D}^bW^a - \frac{2}{N}\hat{\gamma}^{ab}\hat{D}_kW^k\right)\hat{\gamma}_{bi}\hat{D}_a\psi - \psi^{2m-\ell}\frac{N-1}{N}\hat{D}_i(\psi^\tau\hat{K}) + \psi^{2m-\ell}2\alpha_{\rm GB}\hat{\Xi}_i = \kappa^2\psi^{4m-\ell-q}\hat{J}_i. \quad (44)$$

Similarly to the case of the Hamiltonian constraint equation, we consider the following two cases.

(A) If we specify $\tau = \ell - 2m$ and $m = 2/(N-2)$, then (44) becomes

$$\hat{D}_a\hat{D}^aW_i + \frac{N-2}{N}\hat{D}_i\hat{D}_kW^k + \hat{R}_{ik}W^k + \psi^{-1}(\ell+2)\left(\hat{D}^aW^b + \hat{D}^bW^a - \frac{2}{N}\hat{\gamma}^{ab}\hat{D}_kW^k\right) \times\hat{\gamma}_{bi}\hat{D}_a\psi - \frac{N-1}{N}\left[\left(\ell - \frac{4}{N-2}\right) \times(\hat{D}_i\psi)\hat{K} + \hat{D}_i\hat{K}\right] + \psi^{-\ell+4/(N-2)}2\alpha_{\rm GB}\hat{\Xi}_i = \kappa^2\psi^{8/(N-2)-\ell-q}\hat{J}_i. \quad (45)$$

In the case of the Einstein gravity ($\alpha_{\rm GB} = 0$), the choice of $\ell = -2$ cancels the mixing term between ψ and W^i. Further, when $\alpha_{\rm GB} = 0$, we have a chance to make two constraint equations, (36) and (45), decouple by assuming $\hat{K} = 0$ and $q = 8/(N-2) + 2$. However, when $\alpha_{\rm GB} \neq 0$, this decoupling feature is no longer available, since the term $\hat{\Xi}_i$ includes ψ-related terms as we see in Eq. (B7) in Appendix B.

(B) If we specify $\tau = 0$ and $m = 2/(N-2)$, then (44) becomes

$$\hat{D}_a\hat{D}^aW_i + \frac{N-2}{N}\hat{D}_i\hat{D}_kW^k + \hat{R}_{ik}W^k + \psi^{-1}(\ell+2) \times\left[\hat{D}^aW^b + \hat{D}^bW^a - \frac{2}{N}\hat{\gamma}^{ab}\hat{D}_kW^k\right]\hat{\gamma}_{bi}\hat{D}_a\psi - \psi^{4/(N-2)-\ell}\frac{N-1}{N}\hat{D}_i\hat{K} + \psi^{4/(N-2)-\ell}2\alpha_{\rm GB}\hat{\Xi}_i = \kappa^2\psi^{8/(N-2)-\ell-q}\hat{J}_i. \quad (46)$$

For the Einstein gravity ($\alpha_{\rm GB} = 0$), the choice of $\ell = -2$ again cancels the mixing term between ψ and W^i. The decoupling feature between (37) and (46) is available when $\alpha_{\rm GB} = 0$, $\hat{K} = \text{const}$, and $q = 8/(N-2) + 2$. However, when $\alpha_{\rm GB} \neq 0$, this decoupling feature is no longer available, since the term $\hat{\Xi}_i$ includes ψ-related terms as we see in Eq. (B13) in Appendix B.

D. Procedures

For the readers' convenience, we summarize the above procedure briefly. The initial data $(\gamma_{ij}, K_{ij}, \rho, J^i)$ can be constructed by solving the Hamiltonian constraint (19) and the momentum constraint equations (20). This can be done by the following steps.

(1) Give the initial assumption (trial values) for $\hat{\gamma}_{ij}$, K, $\hat{A}^{TT}_{ij}$, and $\hat{\rho}$, $\hat{J}$.

(2) Solve (35) and (44) for ψ and W^i by fixing the exponents ℓ, m, τ, p, and q.

(3) By the following inverse conformal transformations:

$$\gamma_{ij} = \psi^{2m}\hat{\gamma}_{ij}, \quad (47)$$

$$K_{ij} = \psi^{\ell}\Big[\hat{A}^{TT}_{ij} + \hat{D}_i W_j + \hat{D}_j W_i - \frac{2}{N}\hat{\gamma}_{ij}\hat{D}_k W^k\Big] + \frac{1}{N}\psi^{2m+\tau}\hat{\gamma}_{ij}\hat{K}, \quad (48)$$

$$\rho = \psi^{-p}\hat{\rho}, \quad (49)$$

$$J^i = \psi^{-q}\hat{J}^i, \quad (50)$$

we obtain the solutions γ_{ij}, K_{ij}, ρ, and J^i, which satisfy the constraints.

E. Momentarily static case

The easiest construction of the initial data may be under the assumption of the momentarily static (or time-symmetric) situation $K_{ij} = J_i = 0$. In such a case, the momentum constraint becomes trivial, and the Hamiltonian constraint (35) can be reduced as

$$2(N-1)m\hat{D}_a\hat{D}^a\psi - (N-1)[2-(N-2)m]m(\hat{D}\psi)^2\psi^{-1} = \hat{R}\psi + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} + \alpha_{\rm GB}\hat{\Theta}\psi^{2m+1}, \quad (51)$$

where

$$\begin{aligned}\hat{\Theta} = {}& (N-3)m\psi^{-4m}\{4(N-2)m\psi^{-2}[(\hat{D}_a\hat{D}^a\psi)^2 - (\hat{D}_a\hat{D}_b\psi)(\hat{D}^a\hat{D}^b\psi)] - 4\psi^{-1}[\hat{R} - (N-2)\\ & \times[(N-3)m-2]m\psi^{-2}(\hat{D}\psi)^2]\hat{D}_a\hat{D}^a\psi + 8\psi^{-1}[\hat{R}^{ab} + (N-2)m(m+1)\psi^{-2}\hat{D}^a\psi\hat{D}^b\psi]\\ & \times\hat{D}_a\hat{D}_b\psi + (N-1)_2 m^2[(N-4)m-4]\psi^{-4}(\hat{D}\psi)^4 - 2\psi^{-2}[(N-4)m-2](\hat{D}\psi)^2\hat{R}\\ & - 8(m+1)\psi^{-2}\hat{R}^{ab}\hat{D}_a\psi\hat{D}_b\psi\} + \psi^{-4m}\hat{R}_{\rm GB}, \end{aligned} \quad (52)$$

where

$$(D-n)_m = \frac{(D-n)!}{(D-m-1)!} = (D-n)(D-n-1)\ldots(D-m), \quad (53)$$

and $\hat{R}_{\rm GB} = \hat{R}^2 - 4\hat{R}_{ab}\hat{R}^{ab} + \hat{R}_{abcd}\hat{R}^{abcd}$.

When $N = 3$, $\hat{\Theta}$ simply becomes $\hat{\Theta} = \psi^{-4m}\hat{R}_{\rm GB}$ so that (51) will be reduced as

$$8\hat{D}_a\hat{D}^a\psi = \hat{R}\psi + 2\varepsilon\kappa^2\hat{\rho}\psi^{-p} - 2\hat{\Lambda} + \alpha_{\rm GB}\hat{R}_{\rm GB}\psi^{-3}, \quad (54)$$

for the choice of $m = 2$.

However, in general N, it is hard to find an appropriate m which simplifies the equation, even if $\hat{\gamma}_{ij}$ is taken to be the flat space-time that reduces $\hat{\Theta}$ as

$$\begin{aligned}\hat{\Theta} = {}& (N-3)m^2\psi^{-4m-2}\{4(N-2)[(\hat{D}_a\hat{D}^a\psi)^2 - (\hat{D}_a\hat{D}_b\psi)(\hat{D}^a\hat{D}^b\psi)] + 4(N-2)\\ & \times[(N-3)m-2]\psi^{-1}(\hat{D}\psi)^2\hat{D}_a\hat{D}^a\psi + 8(N-2)\\ & \times(m+1)\psi^{-1}(\hat{D}^a\psi)(\hat{D}^b\psi)\hat{D}_a\hat{D}_b\psi + (N-1)_2 m\\ & \times[(N-4)m-4]\psi^{-2}(\hat{D}\psi)^4\}. \end{aligned} \quad (55)$$

Roughly speaking, Eq. (55) is a quadratic equation with respect to the second-order derivative of ψ, which means that there are two roots in general when a set of trial values $\hat{\gamma}_{ij}$ and $\hat{\rho}$ on the hypersurface is given. The meaning of the existence of two solutions is more clearly understood by assuming $p = 0$ and $q = 0$ in the conformal transformation of the matter field. In this case, two different γ_{ij}'s are obtained through the different conformal transformations even for the same matter field distributions.

The nonuniqueness of the solution is due to the higher curvature combination of the GB terms. The existence of two solutions can also be seen for the black hole solutions in the Einstein-GB theory [14,16]. In the black hole case, the gravitational equations reduce to a quadratic equation under the suitable choice of a metric function. The two roots of the equation correspond to the different black hole solutions. In the $\alpha_{\rm GB} \to 0$ limit, one of the black hole solutions approaches to the solutions in GR, while the metric component of the other solution diverges and there is no counterpart in GR. Hence the former is classified into the GR branch and the latter into the GB branch. In some mass parameter regions of the black hole, the metric components of these branches coincide at the certain radius. It is a multiple root of the field equation. Solving the field equations beyond this radius, we find that the metric becomes imaginary, and this region is physically irrelevant. The space-time becomes singular at this radius, and it is called a branch singularity, which is a new ingredient of GB gravity.

When we solve the Hamiltonian constraint equation (55), a similar situation may occur. The conformal factor ψ becomes imaginary, and the solution γ_{ij} is also imaginary in some regions on the hypersurface. The boundary of this region corresponds to the singularity. The situation is the same also in the non-time-symmetric case.

IV. DYNAMICAL EQUATIONS

A. Dynamical equations in general relativity

Equations in GR are obtained by putting $\alpha_{\rm GB} = 0$. The evolution equations of $(N+1)$-dimensional ADM formulation are presented in Ref. [26], but here we generalize them to the set of equations for both spacelike and timelike foliations.

First, we rewrite the dynamical equations (21) by introducing the metric components as

$$\begin{aligned} ds^2 &= \varepsilon\alpha^2(dx^0)^2 + \gamma_{ij}(dx^i + \beta^i dx^0)(dx^j + \beta^j dx^0) \\ &= (\varepsilon\alpha^2 + \beta_a\beta^a)(dx^0)^2 + 2\beta_i dx^i dx^0 + \gamma_{ij}dx^i dx^j, \end{aligned} \tag{56}$$

where α and β^i are the lapse and shift functions, respectively. The components of the normal vector, then, are

$$n_\mu = (\varepsilon\alpha, 0, \dots, 0), \qquad n^\mu = \frac{1}{\alpha}(1, -\beta^i). \tag{57}$$

In the matrix from, the $(N+1)$ and N metrics are expressed as

$$g_{\mu\nu} = \begin{pmatrix} \varepsilon\alpha^2 + \beta_a\beta^a & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \tag{58}$$

$$g^{\mu\nu} = \frac{\varepsilon}{\alpha^2}\begin{pmatrix} 1 & -\beta^i \\ -\beta^j & \varepsilon\alpha^2\gamma^{ij} + \beta^i\beta^j \end{pmatrix}, \tag{59}$$

and

$$\gamma_{\mu\nu} = \begin{pmatrix} \beta_a\beta^a & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad \gamma^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}. \tag{60}$$

The trace of Eq. (21) minus half of Eq. (19) gives

$$\gamma^{ab}\mathcal{L}_n K_{ab} = -\frac{\varepsilon}{2}M - K_{ab}K^{ab} - \frac{\varepsilon\kappa^2}{N-1}\mathcal{T} + \varepsilon\Lambda. \tag{61}$$

This is equivalent to the trace of Einstein equation (3) with the projection (14). Substituting Eq. (61) into Eq. (21), we find

$$\begin{aligned} \mathcal{L}_n K_{ij} &= -\varepsilon M_{ij} - K_{ia}K^a{}_j + \varepsilon\kappa^2\Big(S_{ij} - \frac{1}{N-1}\mathcal{T}\gamma_{ij}\Big) \\ &\quad + \frac{2\varepsilon}{N-1}\gamma_{ij}\Lambda. \end{aligned} \tag{62}$$

The extrinsic curvature and its Lie derivative are expressed as

$$K_{ij} = \frac{1}{2\alpha}(-\partial_0\gamma_{ij} + D_j\beta_i + D_i\beta_j), \tag{63}$$

$$\begin{aligned} \mathcal{L}_n K_{ij} &= \frac{1}{\alpha}(\partial_0 K_{ij} + D_iD_j\alpha - \beta^a D_a K_{ij} - K_{aj}D_i\beta^a \\ &\quad - K_{ai}D_j\beta^a), \end{aligned} \tag{64}$$

respectively.

With the metric components (56), Eqs. (62) and (63) are

$$\partial_0\gamma_{ij} = -2\alpha K_{ij} + D_j\beta_i + D_i\beta_j, \tag{65}$$

$$\begin{aligned} \partial_0 K_{ij} &= -\alpha\varepsilon M_{ij} - \alpha K_{ia}K^a{}_j - D_iD_j\alpha + \beta^a(D_aK_{ij}) \\ &\quad + (D_j\beta^a)K_{ia} + (D_i\beta^a)K_{aj} \\ &\quad - \alpha\varepsilon\kappa^2\Big(S_{ij} - \frac{1}{N-1}\mathcal{T}\gamma_{ij}\Big) + \frac{2\alpha\varepsilon}{N-1}\gamma_{ij}\Lambda. \end{aligned} \tag{66}$$

If we have matter, we need to evolve them together with metric. The dynamical equations for matter terms can be derived from the conservation equation $\nabla^\mu\mathcal{T}_{\mu\nu} = 0$.

B. Dynamical equations in Gauss-Bonnet gravity

With the Gauss-Bonnet terms, the evolution equation (21) cannot be expressed explicitly for each $\mathcal{L}_n K_{ij}$. That is, (21) is rewritten as

$$\begin{aligned} &(1 + 2\alpha_{\rm GB}M)\mathcal{L}_n K_{ij} - (\gamma_{ij}\gamma^{ab} + 2\alpha_{\rm GB}W_{ij}{}^{ab})\mathcal{L}_n K_{ab} \\ &\quad - 8\alpha_{\rm GB}M_{(i}{}^a\mathcal{L}_n K_{|a|j)} \\ &= -\varepsilon(M_{ij} - \tfrac{1}{2}M\gamma_{ij}) - K_{ia}K^a{}_j + \gamma_{ij}K_{ab}K^{ab} \\ &\quad + \varepsilon\kappa^2 S_{ij} - \varepsilon\gamma_{ij}\Lambda - 2\varepsilon\alpha_{\rm GB}H_{ij}, \end{aligned} \tag{67}$$

and the second and third terms on the right-hand side include the linearly coupled terms between $\mathcal{L}_n K_{ij}$. Therefore, in an actual simulation, we have to extract each evolution equation of K_{ij} using a matrix form of Eq. (67) such as

$$\mathbf{k} = A\mathbf{k} + \mathbf{b}, \tag{68}$$

where $\mathbf{k} = (\mathcal{L}_n K_{11}, \mathcal{L}_n K_{12}, \dots, \mathcal{L}_n K_{NN})^T$ and A and $\mathbf{b}$ are the appropriate matrix and vector, respectively, derived from Eq. (67).

The procedure of inverting the matrix $(1 - A)$ is technically available, but the invertibility of the matrix is not generally guaranteed at this moment. In the case of the standard ADM foliation in four-dimensional Einstein equations, the continuity of the time evolutions depends on the models and the choice of gauge conditions for the lapse function and shift vectors. If the combination is not appropriate, then the foliation hits the singularity which stops the evolution. The similar obstacle may exist also for the Gauss-Bonnet gravity. We expect that in the most cases Eq. (68) is invertible for K_{ij}, but we cannot deny the pathological cases which depend on the models and gauge conditions. Such a study must be done together with actual numerical integrations in the future.

We obtain, however, a similar form of the equation by introducing the $(N+1)$-dimensional Weyl curvature [27,28]. It is useful for the discussion of the brane-world cosmology although it is not written by only the

N-dimensional quantities, and we cannot adopt it as the evolution equation directly. The trace of Eq. (21) minus half of Eq. (19) gives

$$(N-1)\left[\frac{\varepsilon}{2}M + K_{ab}K^{ab} + h^{ab}\mathcal{L}_n K_{ab}\right] + 2(n-3)\alpha_{\rm GB}\left[\frac{\varepsilon}{4}(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}) + MK_{ab}K^{ab} - 2K^i{}_jK^j{}_kM^k{}_i - 2N_aN^a + N_{abc}N^{abc} + Mh^{ab}\mathcal{L}_nK_{ab} - 2M^{ab}\mathcal{L}_nK_{ab}\right] = -\varepsilon\kappa^2\mathcal{T} + \varepsilon\Lambda. \quad (69)$$

By the last term on the left-hand side of this equation, the term of the Lie derivative cannot be expressed by the other term explicitly.

Let us rewrite the dynamical equation (21) in a different form. From Eqs. (14), (17), and (18) with the decomposition of the Riemann tensor as

$$\mathcal{R}_{\mu\nu\rho\sigma} = \frac{2}{N-1}(g_{\mu[\rho}\mathcal{R}_{\sigma]\nu} - g_{\nu[\rho}\mathcal{R}_{\sigma]\mu}) - \frac{2}{N(N-1)}g_{\mu[\rho}g_{\sigma]\nu}\mathcal{R} + \mathcal{C}_{\mu\nu\rho\sigma}, \quad (70)$$

where $\mathcal{C}_{\mu\nu\rho\sigma}$ is the $(N+1)$-dimensional Weyl curvature, we find

$$\mathcal{L}_nK_{ij} = \varepsilon\frac{N-1}{N-2}E_{ij} + \frac{\varepsilon}{N-2}\left(M_{ij} - \frac{1}{N}M\gamma_{ij}\right) - K_{ia}K^a{}_j + \frac{1}{N}\gamma_{ij}K_{ab}K^{ab} + \frac{1}{N}\gamma_{ij}\gamma^{ab}\mathcal{L}_nK_{ab}, \quad (71)$$

where

$$E_{ij} := \mathcal{C}_{\mu\nu\rho\sigma}n^\mu n^\rho\gamma^\nu{}_i\gamma^\sigma{}_j. \quad (72)$$

However, because Eq. (71) is a trace-free equation, $\mathcal{L}_nK_{\mu\nu}$ cannot be fixed by Eq. (71). Inserting Eq. (71) into Eq. (69), we find

$$h^{ab}\mathcal{L}_nK_{ab} = -\frac{\varepsilon}{2}M - K_{ab}K^{ab} - \frac{\varepsilon}{U}(\kappa^2\mathcal{T} - \Lambda) + \frac{2\varepsilon(N-3)\alpha_{\rm GB}}{U}I, \quad (73)$$

where

$$U = N - 1 + \frac{2(N-2)(N-3)}{N}\alpha_{\rm GB}M, \quad (74)$$

$$I = \frac{N-6}{4(N-2)}M^2 + \frac{N}{N-2}M_{ab}M^{ab} + \frac{1}{4}M_{abcd}M^{abcd} - \varepsilon\left[N_aN^a + 4N_{abc}N^{abc} - \frac{2(N-1)}{N-2}M_{ab}E^{ab}\right]. \quad (75)$$

From Eq. (71) with Eq. (73), we then find

$$\mathcal{L}_nK_{ij} = \frac{N-1}{N-2}E_{ij} + \frac{\varepsilon}{N-2}\left(M_{ij} - \frac{1}{2}M\gamma_{ij}\right) - K_{ia}K^a{}_j - \frac{\varepsilon}{NU}(\kappa^2\mathcal{T} - \Lambda)\gamma_{ij} - \frac{2\varepsilon(N-3)\alpha_{\rm GB}}{NU}I\gamma_{ij}. \quad (76)$$

Since E_{ij} is a $(N+1)$-dimensional quantity, $\mathcal{L}_nK_{ij}$ cannot be evaluated by the valuables on N-dimensional hypersurface with this equation. This means that Eq. (76) cannot be used for the full dynamics as we mentioned. For example, however, for the Friedmann brane-world model, where the constant-time slice of the timelike hypersurface is homogeneous and isotropic, E_{ij} can be written using quantities on the hypersurface. For such limited cases where the term E_{ij} can be evaluated on the hypersurface, Eq. (76) is useful for simplifying the situation.

V. DISCUSSION

With the aim of numerical investigations of space-time dynamics in higher-dimensional and/or higher-curvature gravity models, we presented the basic equations of the Einstein-Gauss-Bonnet gravity theory.

We show the $(N+1)$-dimensional decomposition of the basic equations, in order to treat the space-time as a Cauchy problem. With the aim of investigations of bulk space-time in recent brane-world models, we also prepared the equations for both timelike and spacelike foliations. The equations can be separated into the constraints (the Hamiltonian constraint and the momentum constraint) and the evolution equations.

Two constraints should be solved for constructing the initial data. By showing the conformally transformed constraint equations, we discussed how the constraints can be simplified by tuning the powers of conformal factors. If we have Gauss-Bonnet terms, however, the equations still remain in a complicated style.

For the evolution equations, we find that $\mathcal{L}_nK_{ij}$ components are coupled. However, this mixture is only up to the linear order due to the quasilinear property of the Gauss-Bonnet terms, so that the equations can be in a treatable form in numerics.

We are now developing our numerical code and hope to present some results elsewhere in the near future.

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Note added in proof.—Regarding the invertibility of Eq. (68), Deruelle and Madore [29] gave an explicit example in a simple cosmological model where the equation corresponding to Eq. (68) is not invertible.

APPENDIX A: GB PART OF HAMILTONIAN CONSTRAINT EQUATION

In Eq. (35), the GB part of the Hamiltonian constraint equation $\hat{\Theta} = M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}$ is not written explicitly. It becomes

$$\begin{aligned}\hat{\Theta} = &(N-3)m\psi^{-4m}\{4(N-2)m\psi^{-2}[(\hat{D}_a\hat{D}^a\psi)^2 - \hat{D}_a\hat{D}_b\psi\hat{D}^a\hat{D}^b\psi]\\ &- 4\psi^{-1}[\hat{M} - (N-2)[(N-3)m-2]m\psi^{-2}\hat{D}_a\psi\hat{D}^a\psi]\hat{D}_a\hat{D}^a\psi\\ &+ 8\psi^{-1}[\hat{M}^{ab} + (N-2)m(m+1)\psi^{-2}\hat{D}^a\psi\hat{D}^b\psi]\hat{D}_a\hat{D}_b\psi + (N-1)_2 m^2[(N-4)m-4]\psi^{-4}(\hat{D}_a\psi\hat{D}^a\psi)^2\\ &- 2\psi^{-2}[(N-4)m-2]\hat{M}\hat{D}_c\psi\hat{D}^c\psi - 8(m+1)\psi^{-2}\hat{M}^{ab}\hat{D}_a\psi\hat{D}_b\psi\} + \psi^{-4m}(\hat{\Upsilon}^2 - 4\hat{\Upsilon}_{ab}\hat{\Upsilon}^{ab} + \hat{\Upsilon}_{abcd}\hat{\Upsilon}^{abcd}),\end{aligned} \tag{A1}$$

where

$$\hat{\Upsilon} = \hat{R} - \varepsilon\left[\frac{N-1}{N}\psi^{2m+2\tau}\hat{K}^2 - \psi^{2\ell-2m}\hat{A}_{ab}\hat{A}^{ab}\right], \tag{A2}$$

$$\begin{aligned}\hat{\Upsilon}_{ij} = \hat{R}_{ij} - \varepsilon\bigg[&\frac{N-1}{N^2}\psi^{2m+2\tau}\hat{\gamma}_{ij}\hat{K}^2 + \frac{N-2}{N}\psi^{\ell+\tau}\hat{K}\hat{A}_{ij}\\ &- \psi^{2\ell-2m}\hat{A}_{ia}\hat{A}^a{}_j\bigg],\end{aligned} \tag{A3}$$

$$\begin{aligned}\hat{\Upsilon}_{ijkl} = \hat{R}_{ijkl} - \varepsilon\bigg[&\frac{1}{N^2}\psi^{2m+2\tau}(\hat{\gamma}_{ik}\hat{\gamma}_{jl} - \hat{\gamma}_{il}\hat{\gamma}_{jk})\hat{K}^2\\ &+ \frac{1}{N}\psi^{\ell+\tau}(\hat{A}_{ik}\hat{\gamma}_{jl} - \hat{A}_{il}\hat{\gamma}_{jk} + \hat{A}_{jl}\hat{\gamma}_{ik} - \hat{A}_{jk}\hat{\gamma}_{il})\\ &+ \psi^{2\ell-2m}(\hat{A}_{ik}\hat{A}_{jl} - \hat{A}_{il}\hat{A}_{jk})\bigg].\end{aligned} \tag{A4}$$

When $\tau = \ell - 2m$ and $m = 2/(N-2)$, which corresponds to case A [Eq. (36)],

$$\begin{aligned}\hat{\Theta} = &\frac{2(N-3)}{N-2}\psi^{-8/(N-2)}\bigg\{8\psi^{-2}[(\hat{D}_a\hat{D}^a\psi)^2 - \hat{D}_a\hat{D}_b\psi\hat{D}^a\hat{D}^b\psi] - 4\psi^{-1}(\hat{M} + 2\psi^{-2}\hat{D}_a\psi\hat{D}^a\psi)\hat{D}_a\hat{D}^a\psi\\ &+ 8\psi^{-1}\left(\hat{M}^{ab} + \frac{2N}{N-2}\psi^{-2}\hat{D}^a\psi\hat{D}^b\psi\right)\hat{D}_a\hat{D}_b\psi - \frac{8N(N-1)}{(N-2)^2}\psi^{-4}(\hat{D}_a\psi\hat{D}^a\psi)^2 + \frac{8}{N-2}\psi^{-2}\hat{M}\hat{D}_c\psi\hat{D}^c\psi\\ &- \frac{8N}{N-2}\psi^{-2}\hat{M}^{ab}\hat{D}_a\psi\hat{D}_b\psi\bigg\} + \psi^{-8/(N-2)}(\hat{\Upsilon}^2 - 4\hat{\Upsilon}_{ab}\hat{\Upsilon}^{ab} + \hat{\Upsilon}_{abcd}\hat{\Upsilon}^{abcd}),\end{aligned} \tag{A5}$$

where

$$\hat{\Upsilon} = \hat{R} - \varepsilon\psi^{2\ell-4/(N-2)}(\hat{K}^2 - \hat{K}_{ab}\hat{K}^{ab}), \tag{A6}$$

$$\hat{\Upsilon}_{ij} = \hat{R}_{ij} - \varepsilon\psi^{2\ell-4/(N-2)}(\hat{K}\hat{K}_{ij} - \hat{K}_{ia}\hat{K}^a{}_j), \tag{A7}$$

$$\hat{\Upsilon}_{ijkl} = \hat{R}_{ijkl} - \varepsilon\psi^{2\ell-4/(N-2)}(\hat{K}_{ik}\hat{K}_{jl} - \hat{K}_{il}\hat{K}_{jk}). \tag{A8}$$

When $\tau = 0$, $m = 2/(N-2)$, which corresponds to case B [Eq. (37)], Θ is expressed as Eq. (A5) and

$$\hat{\Upsilon} = \hat{R} - \varepsilon\left[\frac{N-1}{N}\psi^{4/(N-2)}\hat{K}^2 - \psi^{2\ell-4/(N-2)}\hat{A}_{ab}\hat{A}^{ab}\right], \tag{A9}$$

$$\begin{aligned}\hat{\Upsilon}_{ij} = \hat{R}_{ij} - \varepsilon\bigg[&\frac{N-1}{N^2}\psi^{4/(N-2)}\hat{\gamma}_{ij}\hat{K}^2 + \frac{N-2}{N}\psi^{\ell}\hat{K}\hat{A}_{ij}\\ &- \psi^{2\ell-4/(N-2)}\hat{A}_{ia}\hat{A}^a{}_j\bigg],\end{aligned} \tag{A10}$$

$$\begin{aligned}\hat{\Upsilon}_{ijkl} = \hat{R}_{ijkl} - \varepsilon\bigg[&\frac{1}{N^2}\psi^{4/(N-2)}(\hat{\gamma}_{ik}\hat{\gamma}_{jl} - \hat{\gamma}_{il}\hat{\gamma}_{jk})\hat{K}^2\\ &+ \frac{1}{N}\psi^{\ell}(\hat{A}_{ik}\hat{\gamma}_{jl} - \hat{A}_{il}\hat{\gamma}_{jk} + \hat{A}_{jl}\hat{\gamma}_{ik} - \hat{A}_{jk}\hat{\gamma}_{il})\\ &+ \psi^{2\ell-4/(N-2)}(\hat{A}_{ik}\hat{A}_{jl} - \hat{A}_{il}\hat{A}_{jk})\bigg].\end{aligned} \tag{A11}$$

APPENDIX B: GB PART OF MOMENTUM CONSTRAINT EQUATION

In Eq. (44), the GB part of the Hamiltonian constraint equation $\hat{\Xi}_i$ is not written explicitly. It becomes

$$\begin{aligned}\Xi_i = {}& \psi^{\ell-4m}\bigg\{\hat{R} - 2(N-3)m\psi^{-1}\hat{D}_b\hat{D}^b\psi - (N-3)m[(N-4)m+2]\psi^{-2}\hat{D}_b\psi\hat{D}^b\psi - \frac{N^2-3N+4}{N^2}\varepsilon\psi^{+2m+2\tau}\hat{K}^2 \\ & - \varepsilon\psi^{2\ell-2m}\hat{A}_{bc}\hat{A}^{bc}\bigg\}\hat{D}_a\hat{A}^a{}_i + \psi^{\ell-4m}\bigg\{-2\hat{R}^b{}_i + 2(N-3)m\psi^{-1}\hat{D}^b\hat{D}_i\psi - 2(N-3)m(m+1)\psi^{-2}\hat{D}_i\psi\hat{D}^b\psi \\ & + \frac{2(N-3)}{N}\varepsilon\psi^{\ell+\tau}\hat{K}\hat{A}_i{}^b - 2\varepsilon\psi^{2\ell-2m}\hat{A}_i{}^c\hat{A}_c{}^b\bigg\}\hat{D}_a\hat{A}^a{}_b \\ & + \psi^{\ell-4m}\bigg\{2\hat{R}^{ab} - 2(N-3)m\psi^{-1}\hat{D}^b\hat{D}^a\psi - 2(N-1)m(m+1)\psi^{-2}\hat{D}^a\psi\hat{D}^b\psi - \frac{2(N-3)}{N}\varepsilon\psi^{\ell+\tau}\hat{K}\hat{A}^{ab} \\ & + 2\varepsilon\psi^{2\ell-2m}\hat{A}^a{}_c\hat{A}^{cb}\bigg\}(\hat{D}_i\hat{A}_{ab} - \hat{D}_a\hat{A}_{ib}) + 2\varepsilon\psi^{3\ell-6m}\hat{A}_i{}^a\hat{A}^{bc}(\hat{D}_a\hat{A}_{bc} - \hat{D}_b\hat{A}_{ac}) + \mathcal{R}_i + \mathcal{D}_i + \mathcal{A}^{(1)}\hat{D}_i\psi + \mathcal{A}^{(2)}\hat{D}_i\hat{K} \\ & + \mathcal{A}^{(3)}\hat{D}_a\psi\hat{A}^a{}_i - \frac{2(N-2)_3}{N^2}\varepsilon\psi^{\ell-2m+2\tau}\hat{K}(\hat{D}_a\hat{K} + \tau\psi^{-1}\hat{K}\hat{D}_a\psi)\hat{A}^a{}_i \\ & + \frac{2(N-3)}{N}[(N-4)m+2\ell+\tau]\varepsilon\psi^{2\ell-4m+\tau-1}\hat{K}\hat{D}_b\psi\hat{A}^b{}_a\hat{A}^a{}_i + \frac{2(N-3)}{N}\varepsilon\psi^{2\ell-4m+\tau}\hat{D}_b\hat{K}\hat{A}^b{}_a\hat{A}^a{}_i \\ & - 2[(N-6)m+3\ell]\varepsilon\psi^{3\ell-6m-1}\hat{D}_c\psi\hat{A}^c{}_b\hat{A}^b{}_a\hat{A}^a{}_i, \end{aligned} \tag{B1}$$

where

$$\begin{aligned}\mathcal{R}_i = {}& \bigg\{[(N-3)m+\ell]\psi^{\ell-4m-1}\hat{A}_i{}^a\hat{D}_a\psi - \frac{N-3}{N}\psi^{-2m+\tau}(\hat{D}_i\hat{K} + \tau\psi^{-1}\hat{K}\hat{D}_i\psi)\bigg\}\hat{R} \\ & + \bigg\{\frac{2(N-3)}{N}\tau\psi^{-2m+\tau-1}\hat{K}\hat{D}_a\psi + \frac{2(N-3)}{N}\psi^{-2m+\tau}\hat{D}_a\hat{K} - 2[(N-3)m+\ell]\psi^{\ell-4m-1}\hat{A}_a{}^b\hat{D}_b\psi\bigg\}\hat{R}_i{}^a \\ & - 2(m-\ell)\psi^{\ell-4m-1}(\hat{A}_{ab}\hat{D}_i\psi - \hat{A}_{ib}\hat{D}_a\psi)\hat{R}^{ab} + 2(m-\ell)\psi^{\ell-4m-1}\hat{D}_a\psi\hat{A}_{bc}\hat{R}_i{}^{cab}, \end{aligned} \tag{B2}$$

$$\begin{aligned}\mathcal{D}_i = {}& \bigg\{\frac{N^2-8N+11}{N}m\psi^{-2m+\tau-1}(\hat{D}_i\hat{K} + \tau\psi^{-1}\hat{K}\hat{D}_i\psi) - 2m[(N^2-6N+7)m + (N-3)\ell]\psi^{\ell-4m-2}\hat{D}_b\psi\hat{A}^b{}_i\bigg\} \\ & \times \hat{D}_a\hat{D}^a\psi - \bigg\{\frac{2(N-2)_3}{N}m\psi^{-2m+\tau-1}(\hat{D}_a\hat{K} + \tau\psi^{-1}\hat{K}\hat{D}_a\psi) - 2m[(N^2-4N+5)m + (N-2)(\ell-2)] \\ & \times \psi^{\ell-4m-2}\hat{D}_b\psi\hat{A}^b{}_a\bigg\}\hat{D}^a\hat{D}_i\psi + 2(N-3)m(m-\ell)\psi^{\ell-4m-2}(\hat{A}_{ab}\hat{D}_i\psi - \hat{A}_{ia}\hat{D}_b\psi)\hat{D}^b\hat{D}^a\psi, \end{aligned} \tag{B3}$$

$$\begin{aligned}\mathcal{A}^{(1)} = 2\bigg\{ & -\frac{N-2}{N}m(m+1)\psi^{-2m+2\tau-2}(\hat{D}_a\hat{K} + \tau\psi^{-1}\hat{K}\hat{D}_a\psi)(\hat{D}^a\hat{K} + \tau\psi^{-1}\hat{K}\hat{D}^a\psi) \\ & + \frac{(N-2)^2}{N}m(m+1)\psi^{-2m+\tau-2}\hat{D}_a\hat{K}\hat{D}^a\psi + \frac{N-2}{2N}m[(N^2-4N+5)m+2]\tau\psi^{-2m+\tau-3}\hat{K}\hat{D}_a\psi\hat{D}^a\psi \\ & - (N-2)_3m^2(m+1)\psi^{\ell-4m-3}\hat{D}^a\psi\hat{D}^b\psi\hat{A}_{ab} + \frac{N-3}{N}(m-\ell-\tau)\varepsilon\psi^{2\ell-4m+\tau-1}\hat{K}\hat{A}_{ab}\hat{A}^{ab} \\ & - (m-\ell)\varepsilon\psi^{3\ell-6m-1}\hat{A}_a{}^b\hat{A}_b{}^c\hat{A}_c{}^a\bigg\}, \end{aligned} \tag{B4}$$

$$\mathcal{A}^{(2)} = \frac{1}{N}\bigg\{(N-2)_3m[(N-3)m-2]\psi^{-2m+\tau-2}\hat{D}_a\psi\hat{D}^a\psi - \frac{(N-1)_2(N+1)}{N^2}\varepsilon\psi^{3\tau}\hat{K}^2 - (N-3)\varepsilon\psi^{2\ell-4m+\tau}\hat{A}_{ab}\hat{A}^{ab}\bigg\}, \tag{B5}$$

$$\begin{aligned}\mathcal{A}^{(3)} = {}& -m[(N-2)^2(N-5)m^2 + (N-2)_3(\ell-2)m + (N-1)(3\ell-2)]\psi^{\ell-4m-3}\hat{D}_a\psi\hat{D}^a\psi \\ & - \frac{1}{N^2}[(N-1)(N^2-8)m + (N^2-N+2)\ell]\varepsilon\psi^{\ell-2m+2\tau-1}\hat{K}^2 + [(N-6)m+3\ell]\varepsilon\psi^{3\ell-6m-1}\hat{A}_{ab}\hat{A}^{ab}. \end{aligned} \tag{B6}$$

When $\tau = \ell - 2m$ and $m = 2/(N-2)$, which corresponds to case A [Eq. (45)],

$$\begin{aligned}\Xi_i &= \psi^{\ell-8/(N-2)}\bigg\{\hat{R} - \frac{4(N-3)}{N-2}\psi^{-1}\hat{D}_b\hat{D}^b\psi - \frac{4(N-3)^2}{N-2}\psi^{-2}\hat{D}_b\psi\hat{D}^b\psi \\ &\quad - \frac{N^2-3N+4}{N^2}\varepsilon\psi^{2\ell-4/(N-2)}\hat{K}^2 - \varepsilon\psi^{2\ell-4/(N-2)}\hat{A}_{bc}\hat{A}^{bc}\bigg\}\hat{D}_a\hat{A}^a{}_i \\ &\quad + \psi^{\ell-8/(N-2)}\bigg\{-2\hat{R}^b{}_i + \frac{4(N-3)}{N-2}\psi^{-1}\hat{D}^b\hat{D}_i\psi - \frac{4N(N-3)}{(N-2)^2}\psi^{-2}\hat{D}_i\psi\hat{D}^b\psi + \frac{2(N-3)}{N}\varepsilon\psi^{2\ell-4/(N-2)}\hat{K}\hat{A}_i{}^b \\ &\quad - 2\varepsilon\psi^{2\ell-4/(N-2)}\hat{A}_i{}^c\hat{A}_c{}^b\bigg\}\hat{D}_a\hat{A}^a{}_b + \psi^{\ell-8/(N-2)}\bigg\{2\hat{R}^{ab} - \frac{4(N-3)}{N-2}\psi^{-1}\hat{D}^b\hat{D}^a\psi - \frac{4N(N-1)}{(N-2)^2}\psi^{-2}\hat{D}^a\psi\hat{D}^b\psi \\ &\quad - \frac{2(N-3)}{N}\varepsilon\psi^{2\ell-4/(N-2)}\hat{K}\hat{A}^{ab} + 2\varepsilon\psi^{2\ell-4/(N-2)}\hat{A}^a{}_c\hat{A}^{cb}\bigg\}(\hat{D}_i\hat{A}_{ab} - \hat{D}_a\hat{A}_{ib}) \\ &\quad + 2\varepsilon\psi^{3\ell-12/(N-2)}\hat{A}_i{}^a\hat{A}^{bc}(\hat{D}_a\hat{A}_{bc} - \hat{D}_b\hat{A}_{ac}) + \mathcal{R}_i + \mathcal{D}_i + \mathcal{A}^{(1)}\hat{D}_i\psi + \mathcal{A}^{(2)}\hat{D}_i\hat{K} + \mathcal{A}^{(3)}\hat{D}_a\psi\hat{A}^a{}_i \\ &\quad - \frac{2(N-2)_3}{N^2}\varepsilon\psi^{3\ell-12/(N-2)}\hat{K}\bigg[\hat{D}_a\hat{K} + \bigg(\ell - \frac{4}{N-2}\bigg)\psi^{-1}\hat{K}\hat{D}_a\psi\bigg]\hat{A}^a{}_i \\ &\quad + \frac{2(N-3)}{N}\bigg[3\ell + \frac{2(N-6)}{N-2}\bigg]\varepsilon\psi^{3\ell-12/(N-2)-1}\hat{K}\hat{D}_b\psi\hat{A}^b{}_a\hat{A}^a{}_i + \frac{2(N-3)}{N}\varepsilon\psi^{3\ell-12/(N-2)}\hat{D}_b\hat{K}\hat{A}^b{}_a\hat{A}^a{}_i \\ &\quad - 2\bigg[3\ell + \frac{2(N-6)}{N-2}\bigg]\varepsilon\psi^{3\ell-12/(N-2)-1}\hat{D}_c\psi\hat{A}^c{}_b\hat{A}^b{}_a\hat{A}^a{}_i,\end{aligned} \tag{B7}$$

where

$$\begin{aligned}\mathcal{R}_i &= \psi^{\ell-8/(N-2)}\bigg\{\bigg[\ell + \frac{2(N-3)}{N-2}\bigg]\psi^{-1}\hat{A}_i{}^a\hat{D}_a\psi - \frac{N-3}{N}\bigg[\hat{D}_i\hat{K} + \bigg(\ell - \frac{4}{N-2}\bigg)\psi^{-1}\hat{K}\hat{D}_i\psi\bigg]\bigg\}\hat{R} \\ &\quad + \psi^{\ell-8/(N-2)}\bigg\{\frac{2(N-3)}{N}\bigg(\ell - \frac{4}{N-2}\bigg)\psi^{-1}\hat{K}\hat{D}_a\psi + \frac{2(N-3)}{N}\hat{D}_a\hat{K} - 2\bigg[\ell + \frac{2(N-3)}{N-2}\bigg]\psi^{-1}\hat{A}_a{}^b\hat{D}_b\psi\bigg\}\hat{R}_i{}^a \\ &\quad + 2\bigg(\ell - \frac{2}{N-2}\bigg)\psi^{\ell-8/(N-2)-1}(\hat{A}_{ab}\hat{D}_i\psi - \hat{A}_{ib}\hat{D}_a\psi)\hat{R}^{ab} - 2\bigg(\ell - \frac{2}{N-2}\bigg)\psi^{\ell-8/(N-2)-1}\hat{D}_a\psi\hat{A}_{bc}\hat{R}_i{}^{cab},\end{aligned} \tag{B8}$$

$$\begin{aligned}\mathcal{D}_i &= \psi^{\ell-8/(N-2)}\bigg\{\frac{2(N^2-8N+11)}{N(N-2)}\psi^{-1}\bigg[\hat{D}_i\hat{K} + \bigg(\ell - \frac{4}{N-2}\bigg)\psi^{-1}\hat{K}\hat{D}_i\psi\bigg] \\ &\quad - \frac{4}{N-2}\bigg[(N-3)\ell + \frac{2(N^2-6N+7)}{N-2}\bigg]\psi^{-2}\hat{D}_b\psi\hat{A}^b{}_i\bigg\}\hat{D}_a\hat{D}^a\psi \\ &\quad - \psi^{\ell-8/(N-2)}\bigg\{\frac{4(N-3)}{N}\psi^{-1}\bigg[\hat{D}_a\hat{K} + \bigg(\ell - \frac{4}{N-2}\bigg)\psi^{-1}\hat{K}\hat{D}_a\psi\bigg] - 4\bigg[\ell + \frac{6}{(N-2)^2}\bigg]\psi^{-2}\hat{D}_b\psi\hat{A}^b{}_a\bigg\}\hat{D}^a\hat{D}_i\psi \\ &\quad - \frac{4(N-3)}{N-2}\bigg(\ell - \frac{2}{N-2}\bigg)\psi^{\ell-8/(N-2)-2}(\hat{A}_{ab}\hat{D}_i\psi - \hat{A}_{ia}\hat{D}_b\psi)\hat{D}^b\hat{D}^a\psi,\end{aligned} \tag{B9}$$

$$\begin{aligned}\mathcal{A}^{(1)} &= 2\psi^{\ell-8}\bigg\{-\frac{2}{N-2}\psi^{\ell-4/(N-2)-2}\bigg[\hat{D}_a\hat{K} + \bigg(\ell - \frac{4}{N-2}\bigg)\psi^{-1}\hat{K}\hat{D}_a\psi\bigg]\bigg[\hat{D}^a\hat{K} + \bigg(\ell - \frac{4}{N-2}\bigg)\psi^{-1}\hat{K}\hat{D}^a\psi\bigg] \\ &\quad + 2\psi^{-2}\hat{D}_a\hat{K}\hat{D}^a\psi + \frac{2(N^2-3n+3)}{N(N-2)}\bigg(\ell - \frac{4}{N-2}\bigg)\psi^{-3}\hat{K}\hat{D}_a\psi\hat{D}^a\psi - \frac{4N(N-3)}{(N-2)^2}\psi^{-3}\hat{D}^a\psi\hat{D}^b\psi\hat{A}_{ab} \\ &\quad + \frac{2(N-3)}{N}\bigg(\ell - \frac{1}{N-2}\bigg)\varepsilon\psi^{\ell-4/(N-2)-1}\hat{K}\hat{A}_{ab}\hat{A}^{ab} + \bigg(\ell - \frac{2}{N-2}\bigg)\varepsilon\psi^{\ell-4/(N-2)-1}\hat{A}_a{}^b\hat{A}_b{}^c\hat{A}_c{}^a\bigg\},\end{aligned} \tag{B10}$$

$$\mathcal{A}^{(2)} = \frac{1}{N}\psi^{\ell-8/(N-2)}\bigg\{-\frac{4(N-3)}{N-2}\psi^{-2}\hat{D}_a\psi\hat{D}^a\psi - \frac{(N-1)_2(N+1)}{N^2}\varepsilon\psi^{2\ell-4/(N-2)}\hat{K}^2 - (N-3)\varepsilon\psi^{2\ell-4/(N-2)}\hat{A}_{ab}\hat{A}^{ab}\bigg\}, \tag{B11}$$

$$\mathcal{A}^{(3)} = -\psi^{\ell-8/(N-2)}\bigg\{\frac{4}{N-2}[(N-3)\ell-6]\psi^{-3}\hat{D}_a\psi\hat{D}^a\psi + \frac{1}{N^2}\bigg[(N^2-N+2)\ell + \frac{2(N-1)}{(N-2)(N^2-8)}\bigg]\varepsilon\psi^{2\ell-4-1}\hat{K}^2 - \bigg[3\ell+\frac{2(N-6)}{N-2}\bigg]\varepsilon\psi^{2\ell-4/(N-2)-1}\hat{A}_{ab}\hat{A}^{ab}\bigg\}. \quad \text{(B12)}$$

When $\tau = 0$ and $m = 2/(N-2)$, which corresponds to case B [Eq. (46)],

$$\begin{aligned}\hat{\Xi}_i = {} & \psi^{\ell-8/(N-2)}\bigg\{\hat{R} - \frac{4(N-3)}{N-2}\psi^{-1}\hat{D}_b\hat{D}^b\psi - \frac{8(N-3)^2}{(N-2)^2}\psi^{-2}\hat{D}_b\psi\hat{D}^b\psi - \frac{N^2-3N+4}{N^2}\varepsilon\psi^{4/(N-2)}\hat{K}^2 \\ & - \varepsilon\psi^{2\ell-4/(N-2)}\hat{A}_{bc}\hat{A}^{bc}\bigg\}\hat{D}_a\hat{A}^a{}_i + \psi^{\ell-8/(N-2)}\bigg\{-2\hat{R}^b{}_i + \frac{4(N-3)}{N-2}\psi^{-1}\hat{D}^b\hat{D}_i\psi - \frac{4N(N-3)}{(N-2)^2}\psi^{-2}\hat{D}_i\psi\hat{D}^b\psi \\ & + \frac{2(N-3)}{N}\varepsilon\psi^{\ell}\hat{K}\hat{A}_i{}^b - 2\varepsilon\psi^{2\ell-4/(N-2)}\hat{A}_i{}^c\hat{A}_c{}^b\bigg\}\hat{D}_a\hat{A}^a{}_b + \psi^{\ell-8/(N-2)}\bigg\{2\hat{R}^{ab} - \frac{4(N-3)}{N-2}\psi^{-1}\hat{D}^b\hat{D}^a\psi \\ & - \frac{4N(N-1)}{(N-2)^2}\psi^{-2}\hat{D}^a\psi\hat{D}^b\psi - \frac{2(N-3)}{N}\varepsilon\psi^{\ell}\hat{K}\hat{A}^{ab} + 2\varepsilon\psi^{2\ell-4/(N-2)}\hat{A}^a{}_c\hat{A}^{cb}\bigg\}(\hat{D}_i\hat{A}_{ab} - \hat{D}_a\hat{A}_{ib}) \\ & + 2\varepsilon\psi^{3\ell-12/(N-2)}\hat{A}_i{}^a\hat{A}^{bc}(\hat{D}_a\hat{A}_{bc} - \hat{D}_b\hat{A}_{ac}) + \mathcal{R}_i + \mathcal{D}_i + \mathcal{A}^{(1)}\hat{D}_i\psi + \mathcal{A}^{(2)}\hat{D}_i\hat{K} + \mathcal{A}^{(3)}\hat{D}_a\psi\hat{A}^a{}_i \\ & - \frac{2(N-2)_3}{N^2}\varepsilon\psi^{\ell-4/(N-2)}\hat{K}\hat{D}_a\hat{K}\hat{A}^a{}_i + \frac{4(N-3)}{N}\bigg(\ell+\frac{N-4}{N-2}\bigg)\varepsilon\psi^{2\ell-8/(N-2)-1}\hat{K}\hat{D}_b\psi\hat{A}^b{}_a\hat{A}^a{}_i \\ & + \frac{2(N-3)}{N}\varepsilon\psi^{2\ell-8/(N-2)}\hat{D}_b\hat{K}\hat{A}^b{}_a\hat{A}^a{}_i - 2\bigg[3\ell+\frac{2(N-6)}{N-2}\bigg]\varepsilon\psi^{3\ell-12/(N-2)-1}\hat{D}_c\psi\hat{A}^c{}_b\hat{A}^b{}_a\hat{A}^a{}_i,\end{aligned} \quad \text{(B13)}$$

where

$$\begin{aligned}\mathcal{R}_i = {} & \bigg\{\bigg[\ell+\frac{2(N-3)}{N-2}\bigg]\psi^{\ell-8/(N-2)-1}\hat{A}_i{}^a\hat{D}_a\psi - \frac{N-3}{N}\psi^{-4/(N-2)}\hat{D}_i\hat{K}\bigg\}\hat{R} \\ & + 2\bigg\{\frac{(N-3)}{N}\psi^{-4/(N-2)}\hat{D}_a\hat{K} - \bigg[\ell+\frac{2(N-3)}{N-2}\bigg]\psi^{\ell-8/(N-2)-1}\hat{A}_a{}^b\hat{D}_b\psi\bigg\}\hat{R}_i{}^a \\ & + 2\bigg(\ell-\frac{2}{N-2}\bigg)\psi^{\ell-8/(N-2)-1}\{(\hat{A}_{ab}\hat{D}_i\psi - \hat{A}_{ib}\hat{D}_a\psi)\hat{R}^{ab} - \hat{D}_a\psi\hat{A}_{bc}\hat{R}_i{}^{cab}\},\end{aligned} \quad \text{(B14)}$$

$$\begin{aligned}\mathcal{D}_i = {} & \frac{2}{N-2}\bigg\{\frac{N^2-8N+11}{N}\psi^{-4/(N-2)-1}\hat{D}_i\hat{K} - 2\bigg[(N-3)\ell + \frac{2(N^2-6N+7)}{N-2}\bigg]\psi^{\ell-8/(N-2)-2}\hat{D}_b\psi\hat{A}^b{}_i\bigg\}\hat{D}_a\hat{D}^a\psi \\ & - 4\bigg\{\frac{N-3}{N}\psi^{-4/(N-2)-1}\hat{D}_a\hat{K} - \bigg[\ell+\frac{6}{(N-2)^2}\bigg]\psi^{\ell-8/(N-2)-2}\hat{D}_b\psi\hat{A}^b{}_a\bigg\}\hat{D}^a\hat{D}_i\psi \\ & - \frac{4(N-3)}{N-2}\bigg(\ell-\frac{2}{N-2}\bigg)\psi^{\ell-8/(N-2)-2}(\hat{A}_{ab}\hat{D}_i\psi - \hat{A}_{ia}\hat{D}_b\psi)\hat{D}^b\hat{D}^a\psi,\end{aligned} \quad \text{(B15)}$$

$$\begin{aligned}\mathcal{A}^{(1)} = {} & -\frac{4}{N-2}\psi^{-4/(N-2)-2}\hat{D}_a\hat{K}\hat{D}^a\hat{K} + 4\psi^{-4/(N-2)-2}\hat{D}_a\hat{K}\hat{D}^a\psi - \frac{8N(N-3)}{(N-2)^2}\psi^{\ell-8/(N-2)-3}\hat{D}^a\psi\hat{D}^b\psi\hat{A}_{ab} \\ & - \frac{2(N-3)}{N}\bigg(\ell-\frac{2}{N-2}\bigg)\varepsilon\psi^{2\ell-8/(N-2)-1}\hat{K}\hat{A}_{ab}\hat{A}^{ab} + 2\bigg(\ell-\frac{2}{N-2}\bigg)\varepsilon\psi^{3\ell-12/(N-2)-1}\hat{A}_a{}^b\hat{A}_b{}^c\hat{A}_c{}^a,\end{aligned} \quad \text{(B16)}$$

$$\mathcal{A}^{(2)} = -\frac{1}{N}\bigg\{2(N-3)\psi^{-4/(N-2)-2}\hat{D}_a\psi\hat{D}^a\psi + \frac{(N-1)_2(N+1)}{N^2}\varepsilon\hat{K}^2 + (N-3)\varepsilon\psi^{2\ell-8/(N-2)}\hat{A}_{ab}\hat{A}^{ab}\bigg\}, \quad \text{(B17)}$$

$$\begin{aligned}\mathcal{A}^{(3)} = {} & -\frac{4}{N-2}[(N-3)\ell-6]\psi^{\ell-8/(N-2)-3}\hat{D}_a\psi\hat{D}^a\psi - \frac{1}{N^2}\bigg[(N^2-N+2)\ell + \frac{2(N-1)(N^2-8)}{N-2}\bigg]\varepsilon\psi^{\ell-4/(N-2)-1}\hat{K}^2 \\ & + \bigg[3\ell+\frac{2(N-6)}{N-2}\bigg]\varepsilon\psi^{3\ell-12/(N-2)-1}\hat{A}_{ab}\hat{A}^{ab}.\end{aligned} \quad \text{(B18)}$$

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