Constraint Propagation of C^2-adjusted Equations — Another Recipe for Robust Evolution Systems —

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Abstract

We focus the formulation problem in numerical relativity, propose new sets of evolution equations, and demonstrate some numerical tests. The goal is to construct a robust evolution system against numerical instability during long-term numerical integration in strong gravitational field. One key idea is to adjust the evolution equations with constraint terms, as was systematically formulated by Yoneda and Shinkai. We here apply an adjusting method proposed by Fiske (2004) which uses the norm of constraints, C^2, and does not require the background metric for specifying effective Lagrange multipliers. We present sets of evolution equations both in ADM and BSSN formulations and show numerical tests using Gowdy wave propagation. Detail analyses are in progress, but we observe constraint damping effect as expected.

1 Introduction

In numerical relativity, it is essential to perform stable and accurate simulation. The standard way to integrate the Einstein equations is to split spacetime into three-dimensional space and time. Arnowitt-Deser-Misner (ADM) formulation[1] is the fundamental evolution system of spacetime decompositions. However, it is known that this formulation is not appropriate since the constraints are not satisfied during long-term numerical calculation and in strong gravitational field[2]. Several formulations which modified ADM formulation are suggested, Baumgarte-Shapiro-Shibata-Nakamura(BSSN) formulation[3] is widely used among them.

However, there exists more robust systems than the current standard BSSN system (e.g.[4, 5]) depending on problems. Therefore seeking a robust evolution system against the violation of constraints is still an important issue.

Yoneda and Shinkai[5] systematically investigated adjusted systems, which adds constraints to the evolution equations. With this method, we can predict the stability of numerical simulation by analyzing the eigenvalues of the coefficient matrix which is Fourier-transformed constraint propagation equations under assuming a fixed background metric.

Fiske[6] proposed an adjustment which uses the norm of constraints, C^2, and does not require the background metric for specifying effective Lagrange multipliers and applied this method to the Maxwell equations. A good point of his method is what the stability of the numerical simulation can be expected without depending on background metric. We apply his method to the ADM and BSSN formulations, and actually perform the effect of dumping by numerical simulation.

2 C^2-adjusted Systems

For variables u^i and constraint values C^i, evolution equations with constraint equations are generally written as

$$\partial_t u^i = f(u^i, \partial_j u^i, \cdots), \text{ and} \tag{1}$$

$$C^i(u^i, \partial_j u^i. \cdots) \approx 0. \tag{2}$$

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Suppose we adjust (1) with $C^2 \equiv C^i C_i$, and evaluate constraint propagation as

$$\partial_t C^2 = \frac{\delta C^2}{\delta u^i}(\partial_t u^i). \tag{3}$$

There exists various combinations of this adjustment. Fiske[6] proposed an adjusted term as

$$\partial_t u^i = [\textbf{Original} \quad \textbf{Terms}] - \kappa^{ij}\frac{\delta C^2}{\delta u^j}, \tag{4}$$

with κ^{ij} of positive definite. The constraint propagation, then, becomes

$$\partial_t C^2 = [\textbf{Original} \quad \textbf{Terms}] - \kappa^{ij}\frac{\delta C^2}{\delta u^i}\frac{\delta C^2}{\delta u^j}, \tag{5}$$

which clearly shows the dumping of constraints. If we set κ^{ij} so that the second term becomes more dominant of (5) than first term in evolution, then C^2 dumps because of $\partial_t C^2 < 0$. Fiske presented an numerical example in the Maxwell system.

3 Applications to the Einstein equations

3.1 For ADM Formulation

Now we apply Fiske's method to the ADM formulation[1], which can be written as

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta(\gamma_{ij}) - \kappa_{\gamma ijmn}\frac{\delta (C^A)^2}{\delta \gamma_{mn}}, \tag{6}$$

$$\partial_t K_{ij} = \alpha({}^{(3)}R_{ij} + KK_{ij} - 2K_{i\ell}K^\ell{}_j) - D_i D_j \alpha + \mathcal{L}_\beta(K_{ij}) - \kappa_{Kijmn}\frac{\delta (C^A)^2}{\delta K_{mn}}, \tag{7}$$

where $(C^A)^2$ is the norm of the constraints,

$$(C^A)^2 \equiv (\mathcal{H}^A)^2 + (\mathcal{M}^A)^i(\mathcal{M}^A)_i, \tag{8}$$

and both of $\kappa_{\gamma ijmn}, \kappa_{Kijmn}$ are positive definite.

For the modified ADM equations, (6)-(7), we confirm this system has better stablility than the standard ADM system by the method proposed by Yoneda and Shinkai[5]. That is, assuming the background metric to Minkowski metric, and setting $\kappa_{\gamma ijmn} = \kappa_{Kijmn} = \delta_{im}\delta_{jn}$, we analyzed the eigenvalues of the constraint propagation matrix . We found that all the real parts of eigenvalues are negative. Therefore the system is expected to dump the violation of constraints.

3.2 For BSSN Formulation

For the BSSN formulation[3, 5], evolution equations with Fiske-type adjustment are:

$$\partial_t \varphi = [\textbf{Original} \quad \textbf{Terms}] - \lambda_\varphi \frac{\delta (C^B)^2}{\delta \varphi}, \tag{9}$$

$$\partial_t K = [\textbf{Original} \quad \textbf{Terms}] - \lambda_K \frac{\delta (C^B)^2}{\delta K}, \tag{10}$$

$$\partial_t \widetilde{\gamma}_{ij} = [\textbf{Original} \quad \textbf{Terms}] - \lambda_{\widetilde{\gamma} ijmn} \frac{\delta (C^B)^2}{\delta \widetilde{\gamma}_{mn}}, \tag{11}$$

$$\partial_t \widetilde{A}_{ij} = [\textbf{Original} \quad \textbf{Terms}] - \lambda_{\widetilde{A} ijmn} \frac{\delta (C^B)^2}{\delta \widetilde{A}_{mn}}, \tag{12}$$

$$\partial_t \widetilde{\Gamma}^i = [\textbf{Original} \quad \textbf{Terms}] - \lambda_{\widetilde{\Gamma}}^{ij} \frac{\delta (C^B)^2}{\delta \widetilde{\Gamma}^j}, \tag{13}$$

where

$$(C^B)^2 \equiv (\mathcal{H}^B)^2 + (\mathcal{M}^B)^i(\mathcal{M}^B)_i + \mathcal{A}^2 + \mathcal{G}^i\mathcal{G}_i + \mathcal{S}^2, \tag{14}$$
$$\mathcal{A} \equiv \tilde{\gamma}^{ij}\tilde{A}_{ij}, \quad \mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\Gamma}^i{}_{mn}\tilde{\gamma}^{mn}, \quad \mathcal{S} \equiv -1 + \det(\tilde{\gamma}_{ij}),$$

and all of $\lambda_\varphi, \lambda_K, \lambda_{\tilde{\gamma}ijmn}, \lambda_{\tilde{A}ijmn}$ and $\lambda_{\tilde{\Gamma}}^{ij}$ are positive definite.

4 Numerical Examples

We demonstrate numerical simulations of above systems with polarized Gowdy wave:

$$ds^2 = t^{-1/2}\mathrm{e}^{\lambda/2}(-dt^2 + dx^2) + t(\mathrm{e}^P dy^2 + \mathrm{e}^{-P} dz^2). \tag{15}$$

which is one of the Apples-with-Apples tests [7], setting all of the numerical parameters to the same.

4.1 Adjusted ADM formulation

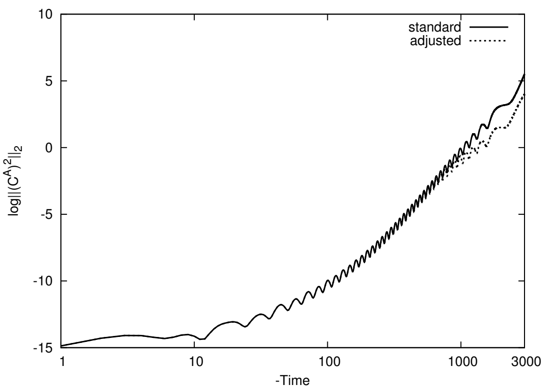

Figure 1: Polarized Gowdy-wave test with the adjusted ADM system. The vertical axis is $\log(||(C^A)^2||_2)$ and the horizontal axis is backward time. The dotted line is the one with (6)-(7) by setting $\kappa_{\gamma ijmn} = 1.0 \times 10^{-4.8}\alpha\gamma_{im}\gamma_{jn}$ and $\kappa_{Kijmn} = 1.0\times10^{-5.4}\alpha\gamma_{im}\gamma_{jn}$. The solid line is calculated with the standard ADM.

We see from Figure 1 that the adjusted ADM system, (6)-(7), has better stability than the standard ADM system. The norm $||(C^A)^2||_2$ of the adjusted ADM is 7.24×10^{-1} times of that of the standard ADM at time $t = -3000$.

4.2 Adjusted BSSN formulation

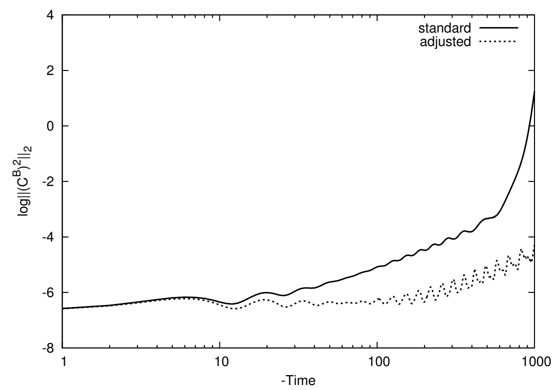

Figure 2: Polarized Gowdy-wave test with the adjusted BSSN system. The vertical axis is $\log(||(C^B)^2||_2)$ and the horizontal axis is backward time. The dotted line is with (9)-(13) by setting $\lambda_\varphi = 1.0 \times 10^{-2.9}\alpha$, $\lambda_K = 1.0 \times 10^{-3.3}\alpha$, $\lambda_{\tilde{\gamma}ijmn} = 1.0 \times 10^{-3.7}\alpha\tilde{\gamma}_{im}\tilde{\gamma}_{jn}$, $\lambda_{\tilde{A}ijmn} = 1.0 \times 10^{-4.4}\alpha\tilde{\gamma}_{im}\tilde{\gamma}_{jn}$, $\lambda_{\tilde{\Gamma}}^{ij} = 1.0 \times 10^{-0.2}\alpha\tilde{\gamma}^{ij}$. The solid line is calculated with the standard BSSN.

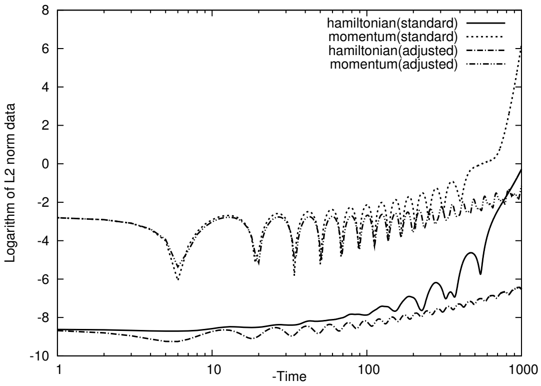

Figure 3: The same with Figure 2, but the result of $\log(||\mathcal{H}^B||_2)$ with the standard BSSN (solid line), $\log(||\mathcal{H}^B||_2)$ with the adjusted BSSN (dot-dashed line), $\log(||\mathcal{M}_i^B||_2)$ with the standard BSSN(dotted line), and $\log(||\mathcal{M}_i^B||_2)$ with the adjusted BSSN(two-dot-dashed line). The vertical axis is logarithm values of L2 norm of $\mathcal{H}^B, \mathcal{M}_i^B$ and the horizontal axis is backward time.

We see from Figure 2 that the adjusted BSSN system has better stability than the standard BSSN system. The norm $||(C^B)^2||_2$ of the adjusted BSSN is 3.95×10^{-3} times of that of the standard BSSN at time $t = -1000$. Kiuchi and Shinkai[4] performed the numerical simulation of polarlized Gowdy wave with other versions of adjusted BSSN systems[5]. We see that our result is better than theirs. Our result of $||\mathcal{H}^B||_2 \leq 2.5 \times 10^{-3}$ at $t = -1000$ but the result[4] of $||\mathcal{H}^B||_2 \geq 1.00 \times 10^1$ at $t = -1000$.

We think the stability of the adjusted BSSN formulation is explained by the dumping of $\mathcal{M}_i^B$ at the early time (about $t \leq -20$). As was argued by Kiuchi and Shinkai[4], the key of the stability of the evolution with BSSN system is to dump $\mathcal{M}_i^B$ earlier.

5 Summary

In this report, we applied the adjusting method suggested by Fiske to the ADM and BSSN formulations, and obtained the equations (6)-(7) and (9)-(13). We performed numerical tests with polarized Gowdy wave and showed that the adjusted ADM and BSSN systems have actually better stablility than the standard ADM and BSSN systems.

The advantage of the present systems to the previous adjusted systems [5, 8] is the way of specifying the Lagrange multipliers κ. In the present systems, κs are restricted as "positive definite" from the formulation independent on the background metric, while in the previous systems one needs to specify the signature of κs with eigenvalue analysis which depends on the background metric.

The detail numerical analysis on the range of effective parameters and the comparisons with other systems are underway.

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