Wormhole evolutions in higher-dimensional gravity

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Abstract

We know numerically the four-dimensional Ellis wormhole solution (the so-called Morris-Thorne's traversable wormhole) is unstable against an input of scalar-pulse from one side. We investigate this feature for higher-dimensional versions, both in n-dimensional general relativity and in 5-dimensional Gauss-Bonnet gravity. We derived Ellis wormhole solutions in n-dimensional general relativity, and evolved it numerically in dual-null coordinate with/without Gauss-Bonnet corrections. Preliminary results show that those are also unstable. We also find that the throat of wormhole in Gauss-Bonnet gravity tends to expand (or shrink) after an input of ghost-scalar pulse if the coupling constant α is positive (negative).

1 Introduction

Wormholes are popular tools in science fictions as a way for rapid interstellar travel, time machines and warp drives. However, wormholes are also a scientific topic after the influential study of traversable wormholes by Morris & Thorne [1]. They considered "traversable conditions" for human travel through wormholes responding to Carl Sagan's idea for his novel (*Contact*), and concluded that such a wormhole solution is available if we allow "exotic matter" (negative-energy matter).

The introduction of exotic matter sounds to be unusual for the first time, but such matter appears in quantum field theory and in alternative gravitational theories such as scalar-tensor theories. The Morris-Thorne solution is constructed with a massless Klein-Gordon field whose gravitational coupling takes the opposite sign to normal, which appears in Ellis's earlier work [2], who called it a drainhole.

Ellis (Morris-Thorne) wormhole solution was studied in many contexts. Among them, we focus its dynamical features. The first numerical simulation on its stability behavior was reported by one of the authors [3]. They use a dual-null formulation for spherically symmetric space-time integration, and observed that the wormhole is unstable against Gaussian pulses in either exotic or normal massless Klein-Gordon fields. The wormhole throat suffers a bifurcation of horizons and either explodes to form an inflationary universe or collapses to a black hole, if the total input energy is negative or positive, respectively. These basic behaviors were repeatedly confirmed by other groups [4, 5]. [3]

The changes of wormhole either to a black hole or an expanding throat supports an unified understanding of black holes and traversable wormholes proposed by Hayward [8]. His proposal is that the two are dynamically interconvertible, and that traversable wormholes are understandable as black holes under negative energy density.

In this article, we introduce our extensional works of [3]; (a) constructing Ellis solutions in higher-dimensional general relativity, (b) dynamical effects of Gauss-Bonnet coupling constant in 5-dimensional wormhole solution.

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[3]Note that Armendariz-Picon[6] reports that the Ellis wormhole is stable using perturbation analysis. However, the conclusion is obtained with fixing the throat of wormhole and not the same situation with above numerical works[7].

2 Wormhole solutions in higher-dimensional general relativity

2.1 Field equations

We consider the following n-dimensional Einstein-Klein-Gordon system

$$S = \int d^n x\sqrt{-g}\Big[\frac{1}{2\kappa_n^2}R - \frac{1}{2}\epsilon(\partial\phi)^2 - V(\phi)\Big], \tag{1}$$

where κ_n^2 is a n-dimensional gravitational constant, $\epsilon = 1$ (or -1) for the normal (ghost) field.

We consider the static and spherically symmetric space-time with the metric

$$ds_D^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + R(r)^2 h_{ij}dx^i dx^j \tag{2}$$

where $h_{ij}dx^i dx^j$ represents the line element of a $(n-2)$-dimensional constant curvature space $S^{(n-2)}$ with curvature $k = \pm 1,\ 0$.

The Einstein tensor becomes

$$G_{tt} = -\frac{n-2}{2}f^2\left[\frac{2R''}{R} + \frac{f'R'}{fR} + (n-3)\frac{R'^2}{R^2}\right] + \frac{f}{2R^2}{}^{(n-2)}\mathcal{R}, \tag{3}$$

$$G_{rr} = \frac{n-2}{2}\frac{R'}{R}\left[\frac{f'}{f} + (n-3)\frac{R'}{R}\right] - \frac{1}{2fR^2}{}^{(n-2)}\mathcal{R}, \tag{4}$$

$$G_{ij} = \left[\frac{f''}{2} + (n-3)f\left(\frac{R''}{R} + \frac{f'R'}{fR} + \frac{n-4}{2}\frac{R'^2}{R^2}\right)\right]g_{ij} + {}^{(n-2)}\mathcal{R}_{ij} - \frac{1}{2R^2}{}^{(n-2)}\mathcal{R}g_{ij}, \tag{5}$$

where ${}^{(n-2)}\mathcal{R}$ is the scalar curvature of $S^{(n-2)}$ and is obtained as

$${}^{(n-2)}\mathcal{R}_{ijkl} = k(h_{ik}h_{jl} - h_{il}h_{jk}), \tag{6}$$

$${}^{(n-2)}\mathcal{R}_{ij} = (n-3)kh_{ij}, \tag{7}$$

$${}^{(n-2)}\mathcal{R} = (n-2)(n-3)k. \tag{8}$$

The non-zero components of the energy-momentum tensors

$$T_{\mu\nu} = \epsilon\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\Big[\frac{1}{2}\epsilon(\nabla\phi)^2 + V(\phi)\Big], \tag{9}$$

are

$$T_{tt} = f\Big[\frac{1}{2}\epsilon f\phi'^2 + V(\phi)\Big], \tag{10}$$

$$T_{rr} = f^{-1}\Big[\frac{1}{2}\epsilon f\phi'^2 - V(\phi)\Big], \tag{11}$$

$$T_{ij} = \Big[\frac{1}{2}\epsilon f\phi'^2 + V(\phi)\Big]R^2 h_{ij}. \tag{12}$$

The Einstein equation, $G_{\mu\nu} = \kappa_n^2 T_{\mu\nu}$, becomes

$$(t,t): \quad -\frac{n-2}{2}f^2\left[\frac{2R''}{R} + \frac{f'R'}{fR} + (n-3)\frac{R'^2}{R^2}\right] + \frac{(n-2)(n-3)kf}{2R^2} = \kappa_n^2 f\Big[\frac{1}{2}\epsilon f\phi'^2 + V(\phi)\Big], \tag{13}$$

$$(r,r): \quad \frac{n-2}{2}\frac{R'}{R}\left[\frac{f'}{f} + (n-3)\frac{R'}{R}\right] - \frac{(n-2)(n-3)k}{2fR^2} = \frac{\kappa_n^2}{f}\Big[\frac{1}{2}\epsilon f\phi'^2 - V(\phi)\Big], \tag{14}$$

$$(i,j): \quad \left[\frac{f''}{2} + (n-3)f\left(\frac{R''}{R} + \frac{f'R'}{fR} + \frac{n-4}{2}\frac{R'^2}{R^2}\right)\right] - \frac{(n-3)(n-4)k}{2R^2} = \kappa_n^2\Big[\frac{1}{2}\epsilon f\phi'^2 + V(\phi)\Big] \tag{15}$$

The Klein-Gordon equation

$$\Box\phi = -\epsilon \frac{dV}{d\phi} \tag{16}$$

becomes

$$\frac{1}{R^{n-2}} \left(R^{n-2} f \phi' \right)' = -\epsilon \frac{dV}{d\phi}. \tag{17}$$

Hereafter, we assume that the scalar field is ghost, $\epsilon = -1$.

2.2 Wormhole solution with massless scalar field in spacetime $k = 1$

We show the simplest solution, under the assumptions of the massless scalar field, $V(\phi) = 0$, in the closed universe, $k = 1$. Other cases are presented elsewhere [7]. The Klein-Gordon equation (17) is integrated as

$$\phi' = \frac{C}{f R^{n-2}}, \tag{18}$$

where C is an integration constant. The Einstein equations, then, are reduced to

$$\frac{(n-2)R'}{R} \left[\frac{(n-3)R'}{R} + \frac{f'}{f} \right] - \frac{(n-2)(n-3)}{f R^2} = -\frac{\kappa_n^2 C^2}{f^2 R^{2(n-2)}} \tag{19}$$

$$\frac{(n-2)R''}{R} = \frac{\kappa_n^2 C^2}{f^2 R^{2(n-2)}} \tag{20}$$

We impose that the wormhole has a throat radius, a, at the coordinate $r = 0$. Then the regularity conditions are

$$R = a, \quad R' = 0, \quad f = f_0 \quad f' = 0, \tag{21}$$

where f_0 is a constant. We can assume $a = 1$ and $f_0 = 1$ without loss of generality, but we keep a in the equations for a while. Eq. (19) gives

$$\kappa_n^2 C^2 = (n-2)(n-3) a^{2(n-3)} \tag{22}$$

The solution of Eqs. (18)-(20) is

$$f \equiv 1, \tag{23}$$

$$R' = \sqrt{1 - \left(\frac{a}{R} \right)^{2(n-3)}}, \tag{24}$$

$$\phi = \frac{\sqrt{(n-2)(n-3)}}{\kappa_n} a^{n-3} \int \frac{1}{R^{n-2}} dr. \tag{25}$$

The Eq. (24) is integrated to give

$$r(R) = -m B_{R^m} \left(-m, \frac{1}{2} \right) - \frac{\sqrt{\pi}\Gamma[1-m]}{\Gamma[m(n-4)]}, \tag{26}$$

where $m = \frac{1}{2(n-3)}$, and $B_z(p, q)$ is an incomplete beta function defined by

$$B_z(p, q) := \int_0^z t^{p-1} (1-t)^{q-1} dt \tag{27}$$

which can be expressed with the hypergeometric function $F(\alpha, \beta, \gamma; z)$ as

$$B_z(p, q) = \frac{z^p}{p} F(p, 1-q, p+1; z). \tag{28}$$

Although Eq. (26) is implicit with respect to R, it is rewritten in the explicit form by using the inverse incomplete beta function.

For $n = 4$, this solution reduces to Ellis's wormhole solution. For $n \to \infty$, the function becomes $R = r + a$ and ϕ behaves like a step-function approaching $\phi \to \pi/2$.

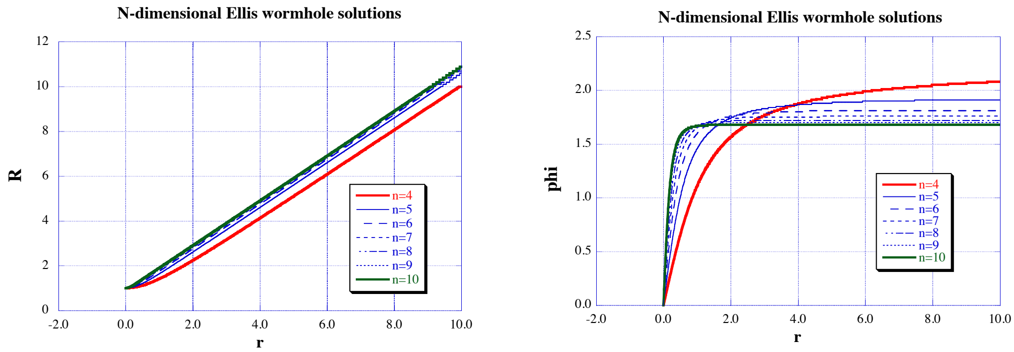

Figure 1: The n-dimensional wormhole solution. The circumference radius R (left panel) and the scalar field ϕ (right panel) are plotted as a function of radial coordinate r. The cases of $n = 4$–10 are shown.

3 Effects of Gauss-Bonnet coupling constant in 5-dimensional wormhole evolution

3.1 Gauss-Bonnet gravity

Gauss-Bonnet gravity is derived from the superstring theory, with additional higher-order curvature correction terms to general relativity. Such higher-order corrections can be treated as an expansion of $\mathcal{R}$ in the action, but the Gauss-Bonnet term,

$$\mathcal{L}_{\rm GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}, \tag{29}$$

has nice properties such that it is ghost-free combinations[9] and does not give higher derivative equations but an ordinary set of equations with up to the second derivative in spite of the higher curvature combinations.

The Einstein-Gauss-Bonnet action in $(N+1)$-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$ is described as

$$S = \int_{\mathcal{M}} d^{N+1}X\sqrt{-g}\left[\frac{1}{2\kappa^2}\left\{\alpha_{\rm GR}\left(\mathcal{R}-2\Lambda\right)+\alpha_{\rm GB}\mathcal{L}_{\rm GB}\right\}+\mathcal{L}_{\rm matter}\right], \tag{30}$$

where κ^2 is the $(N+1)$-dimensional gravitational constant, $\mathcal{R}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}_{\mu\nu\rho\sigma}$ and $\mathcal{L}_{\rm matter}$ are the $(N+1)$-dimensional scalar curvature, Ricci tensor, Riemann curvature and the matter Lagrangian, respectively. This action reproduces the standard $(N+1)$-dimensional Einstein gravity, if we set the coupling constant $\alpha_{\rm GB}$ (≥ 0) equals to zero.

The action (30) gives the gravitational equation as

$$\alpha_{\rm GR}\mathcal{G}_{\mu\nu} + \alpha_{\rm GB}\mathcal{H}_{\mu\nu} = \kappa^2\,\mathcal{T}_{\mu\nu}\,, \tag{31}$$

where

$$\begin{aligned}
\mathcal{G}_{\mu\nu} &= \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + g_{\mu\nu}\Lambda, & (32)\\
\mathcal{H}_{\mu\nu} &= 2\Big[\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}_{\ \nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}^{\ \alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}\Big] - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{\rm GB}, & (33)\\
\mathcal{T}_{\mu\nu} &= -2\frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\rm matter}. & (34)
\end{aligned}$$

The higher-order curvature terms are considered as correction terms from string theory. These terms are known to produce two solution branches normally, only one of which has general-relativity limit. The theory is expected to have singularity-avoidance features in the context of gravitational collapses and/or cosmology, but as far as we know there is no studies so far using fully numerical evolutions. (Numerical studies on critical phenomena are recently reported for small $\alpha_{\rm GB}$ [10, 11, 12]).

Studies on wormholes in Gauss-Bonnet gravity have long histories. Several solutions and their classifications are reported in [13, 14], while their energy conditions are considered in [15]. Similar researches are extended to the Lovelock gravity[16], and also to the Dilatonic Gauss-Bonnet system [17]. Our aim is to investigate their dynamical features.

3.2 Dual-null evolution system

In this article, we report our two initial numerical results. One is the evolution of 5D wormhole in general relativity. The other is its evolution in Gauss-Bonnet gravity.

We implemented our 4D dual-null evolution code [3] to 5D evolution code with Gauss-Bonnet terms. The system we consider is spherical symmetry, and expressed using dual-null coordinate

$$ds^2 = -2e^{-f(x^+,x^-)}dx^+\,dx^- + r^2(x^+,x^-)d\Omega_3^2. \tag{35}$$

Wormhole is constructed with ghost scalar field $\phi(x^+,x^-)$, but we also include normal scalar field $\psi(x^+,x^-)$ contribution. The energy-momentum tensor is written as

$$\begin{aligned} T_{\mu\nu} &= T^{\psi}_{\mu\nu} + T^{\phi}_{\mu\nu} \qquad (36)\\ &= \left[\psi_{,\mu}\psi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}(\nabla\psi)^2 + V_1(\psi)\right)\right] + \left[\epsilon\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\left(\frac{1}{2}\epsilon(\nabla\phi)^2 + V_2(\phi)\right)\right], \end{aligned}$$

where $\epsilon = -1$. This derives Klein-Gordon equations

$$\Box\psi = \frac{dV_1}{d\psi}, \qquad \Box\phi = -\epsilon\frac{dV_2}{d\phi}. \tag{37}$$

Following [3], we introduce the conformal factor Ω, expansion $\vartheta_\pm$, in-affinity $\nu_\pm$, and scalar momentum $\pi_\pm, p_\pm$ as

$$\Omega = \frac{1}{r}, \tag{38}$$
$$\vartheta_\pm \equiv 3\partial_\pm r, \tag{39}$$
$$\nu_\pm \equiv \partial_\pm f, \tag{40}$$
$$\pi_\pm \equiv r\partial_\pm\psi = \frac{1}{\Omega}\partial_\pm\psi, \tag{41}$$
$$p_\pm \equiv r\partial_\pm\phi = \frac{1}{\Omega}\partial_\pm\phi. \tag{42}$$

We also define

$$\eta = \Omega^2\left(e^{-f} + \frac{2}{9}\vartheta_+\vartheta_-\right), \tag{43}$$
$$\tilde{A} = \alpha_{\rm GR} + 4\alpha_{\rm GB}\eta e^f, \tag{44}$$
$$B = \kappa^2 T_{+-} + e^{-f}\Lambda. \tag{45}$$

The set of evolution equations (x^+ and x^--directions), then, are

$$\partial_\pm\Omega = -\frac{1}{3}\vartheta_\pm\Omega^2 \tag{46}$$
$$\partial_\pm\vartheta_\pm = -\nu_\pm\vartheta_\pm - \frac{1}{\tilde{A}\Omega}\kappa^2 T_{\pm\pm} \tag{47}$$
$$\partial_\pm\vartheta_\mp = \frac{1}{\tilde{A}\Omega}(-3\alpha_{\rm GR}\eta + B) \tag{48}$$
$$\partial_\pm f = \nu_\pm \tag{49}$$
$$\begin{aligned} \partial_\pm\nu_\mp &= \frac{\alpha_{\rm GR}}{\tilde{A}}\left\{\eta - \frac{4\,(3\alpha_{\rm GR}\eta - B)}{3\tilde{A}}\right\} + \frac{(\kappa^2 T_{zz}\Omega^2 - \Lambda)}{\tilde{A}e^f} \\ &\quad + \frac{8\alpha_{\rm GB}}{9\tilde{A}^3}\left\{e^f\,(3\alpha_{\rm GR}\eta - B)^2 - \kappa^4 T_{++}T_{--}\right\} \qquad (50) \end{aligned}$$

together with Klein-Gordon equations

$$\partial_\pm \psi = \Omega \pi_\pm, \tag{51}$$
$$\partial_\pm \phi = \Omega p_\pm, \tag{52}$$
$$\partial_+ \pi_- = -\frac{1}{6}\Omega\vartheta_+\pi_- - \frac{1}{2}\Omega\vartheta_-\pi_+ - \frac{1}{2e^f\Omega}\frac{dV_1}{d\psi}, \tag{53}$$
$$\partial_+ p_- = -\frac{1}{6}\Omega\vartheta_+ p_- - \frac{1}{2}\Omega\vartheta_- p_+ - \frac{1}{2e^f\Omega}\frac{dV_2}{d\phi}, \tag{54}$$
$$\partial_- \pi_+ = -\frac{1}{2}\Omega\vartheta_+\pi_- - \frac{1}{6}\Omega\vartheta_-\pi_+ - \frac{1}{2e^f\Omega}\frac{dV_1}{d\psi}, \tag{55}$$
$$\partial_- p_+ = -\frac{1}{2}\Omega\vartheta_+ p_- - \frac{1}{6}\Omega\vartheta_- p_+ - \frac{1}{2e^f\Omega}\frac{dV_2}{d\phi}, \tag{56}$$

where the energy momentum tensor is written as

$$T_{++} = \Omega^2(\pi_+^2 - p_+^2), \tag{57}$$
$$T_{--} = \Omega^2(\pi_-^2 - p_-^2), \tag{58}$$
$$T_{+-} = e^{-f}\left(V_1(\psi) + V_2(\phi)\right), \tag{59}$$
$$T_{zz} = e^f(\pi_+\pi_- - p_+p_-) - \frac{1}{\Omega^2}\left(V_1(\psi) + V_2(\phi)\right). \tag{60}$$

3.3 Preliminal results

We prepare the solution obtained in §2.2 as our initial data on $(x^+, x^-) = (x^+, 0)$ hypersurface, and integrate along to x^--direction using the set of equations above. Numerical integration techniques are the same with [3].

The preliminal results show that the wormhole throat is unstable, the expansion $\vartheta_\pm$ go splitting soon after the evolution begins. Fig. 2 shows their locations in (x^+, x^-) plane. If the location of ϑ_+ is outer (in x^+-direction) than that of ϑ_-, then the region $\vartheta_- < x < \vartheta_+$ is judged as a black-hole. Otherwise the region $\vartheta_+ < x < \vartheta_-$ can be judged as an expanding throat. Fig. 2 indicates the throat begins expanding, then turns to be a black hole. We also evolve the same initial data with Gauss-Bonnet terms $\alpha_{\rm GB} \neq 0$ and study their effects to the evolutions. We see if $\alpha_{\rm GB} > 0$ the throat expansion becomes slower. On the contrary, if $\alpha_{\rm GB} < 0$, then the throat expansion is accelerated in the initial stage.

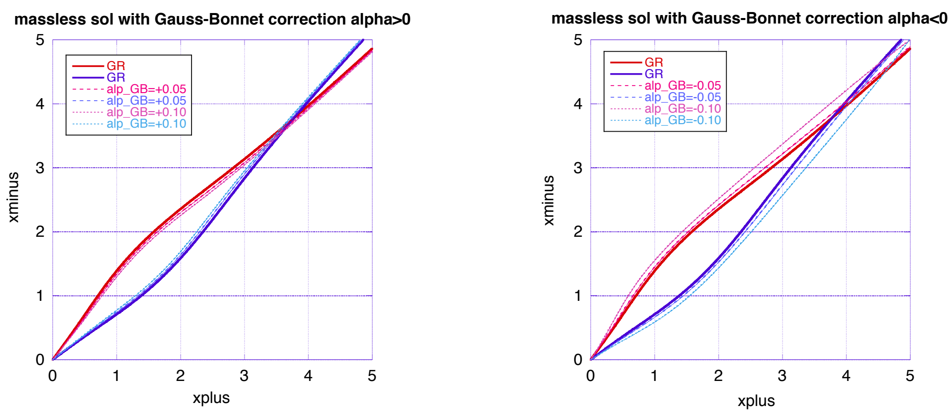

Figure 2: Location of the expansion ϑ_+ (red lines) and ϑ_- (blue lines) for evolutions of a solution in §2.2 as a function of (x_+, x_-). The throat begins expanding, then turns to be a black hole. When $\alpha_{\rm GB} > 0$ (left panel), expansions are slightly slowing down and $\alpha_{\rm GB} > 0$ affects to black hole formation earlier, while when $\alpha_{\rm GB} < 0$ (right panel) we see Gauss-Bonnet term accelerates expansion.

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