2003年1月　重力波物理学研究会 @ 京都大学基礎物理学研究所：ポスター

# 数値相対論における定式化問題：数値計算報告

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## Outline:

- **数値相対論に適したEinstein方程式の定式化が，さまざまに提案されてきたが，それらを統一的に理解する方法を提唱．**
- **「時間発展と共に系が拘束面に漸近的に収束してゆく」発展システムが，既知の発展方程式に拘束条件を用いて Lagrange乗数補正することに得られることを示す．**
- **アイデアの中核となるのは，拘束条件式の発展を事前に計算し，Fourierモードに直して，固有値解析を行うこと．**
- **これまでに広く使われているADMや，Nakamura et alの方程式が，更に良い安定性を持つことを，数値的に実証．**

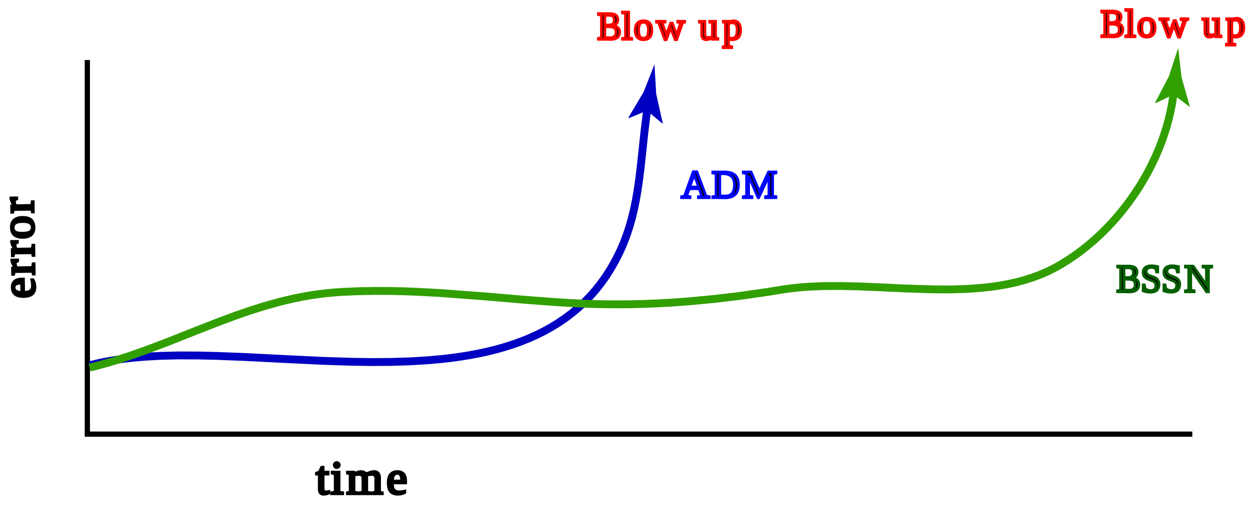
## Refs:

| | |
|---|---|
| ADM variables | PRD **63** (2001) 124019, CQG **19** (2002) 1027 |
| BSSN variables | PRD **66** (2002) 124003 |
| general | CQG **20** (2003) L31 |
| general | gr-qc/0209111 (review article, to be published from Nova Science Publ.) |

早稲田大数理　米田元　との共同研究，科研費 No. 14740179 (2002-2005) + 理研基礎科学特別研究員研究費

## Best formulation of the Einstein eqs. for long-term stable & accurate simulation?

Many (too many) trials and errors, not yet a definit recipe.



Mathematically equivalent formulations, but differ in its stability!

strategy 0: Arnowitt-Deser-Misner formulation

strategy 1: Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

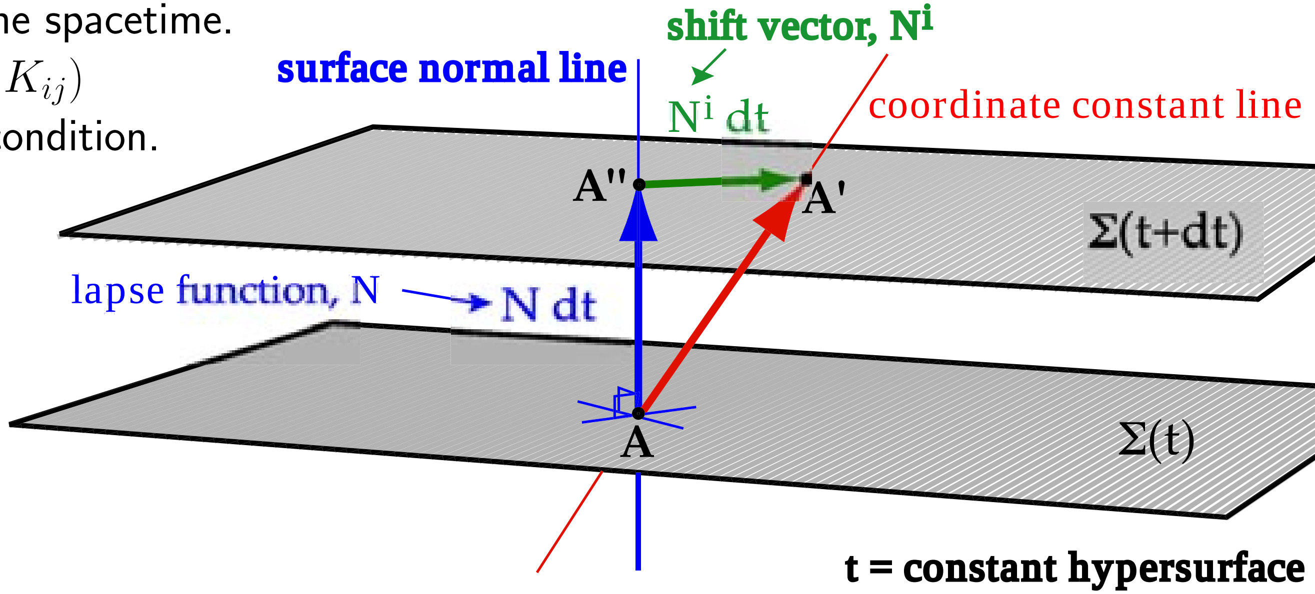
strategy 2: Apply a formulation which reveals a hyperbolicity explicitly

strategy 3: Formulate a system which is "asymptotically constrained" against a violation of constraints

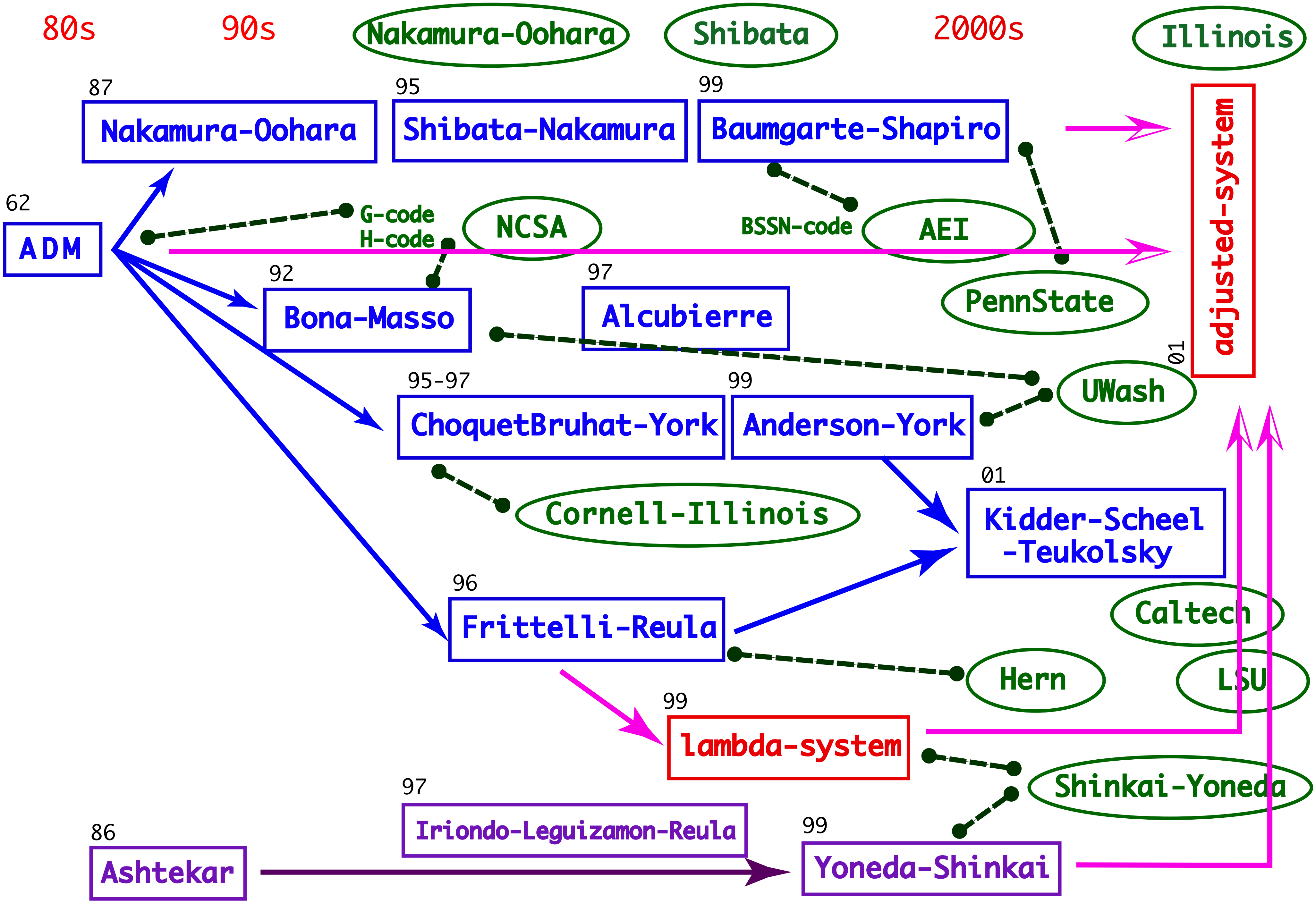
By adding constraints in RHS, we can kill error-growing modes
⇒ How can we understand the features systematically?

# strategy 0 The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

3+1 decomposition of the spacetime.
Evolve 12 variables $(\gamma_{ij}, K_{ij})$
with a choice of gauge condition.



| | Maxwell eqs. | ADM Einstein eq. |
|---|---|---|
| constraints | div $\mathbf{E} = 4\pi\rho$<br>div $\mathbf{B} = 0$ | $^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$<br>$D_jK^j{}_i - D_i\mathrm{tr}K = \kappa J_i$ |
| evolution eqs. | $\frac{1}{c}\partial_t\mathbf{E} = \text{rot } \mathbf{B} - \frac{4\pi}{c}\mathbf{j}$<br>$\frac{1}{c}\partial_t\mathbf{B} = -\text{rot } \mathbf{E}$ | $\partial_t\gamma_{ij} = -2NK_{ij} + D_jN_i + D_iN_j,$<br>$\partial_t K_{ij} = N(\,^{(3)}R_{ij} + \mathrm{tr}K\,K_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN$<br>$+ (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} - N\gamma_{ij}\Lambda$<br>$- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$ |


80s

90s

Nakamura-Oohara

Shibata

2000s

Illinois

87 Nakamura-Oohara

95 Shibata-Nakamura

99 Baumgarte-Shapiro

adjusted-system

01

62 ADM

G-code

H-code

NCSA

BSSN-code

AEI

PennState

92 Bona-Masso

97 Alcubierre

UWash

95-97 ChoquetBruhat-York

99 Anderson-York

Cornell-Illinois

01 Kidder-Scheel -Teukolsky

Caltech

96 Frittelli-Reula

Hern

LSU

99 lambda-system

Shinkai-Yoneda

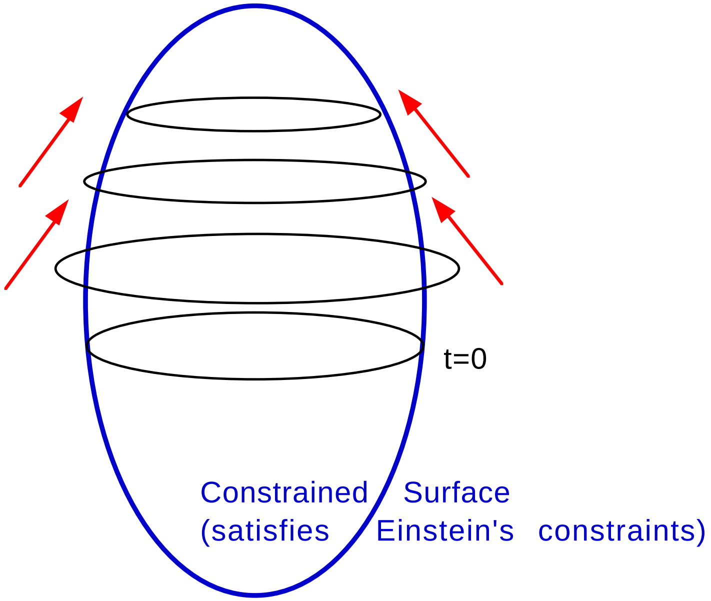
97 Iriondo-Leguizamon-Reula

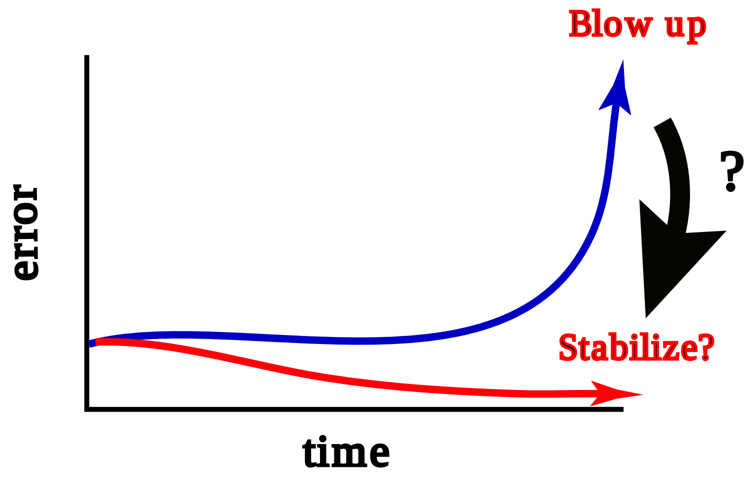
86 Ashtekar

99 Yoneda-Shinkai

## strategy 3 Formulate a system which is "asymptotically constrained" against a violation of constraints

### **"Asymptotically Constrained System"**– Constraint Surface as an Attractor





method 1: $\lambda$-system (Brodbeck et al, 2000)

- Add aritificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may prodict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. ⇒ for the ADM/BSSN formulation, too!!

## Idea of "Adjusted system" and Our Conjecture

CQG18 (2001) 441, PRD 63 (2001) 120419, CQG 19 (2002) 1027

### General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) \underbrace{+F(C^a, \partial_b C^a, \cdots)}$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$
   to make the system stable evolution

### How to specify $F(C^a, \partial_b C^a, \cdots)$ ?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) \underbrace{+G(C^a, \partial_b C^a, \cdots)}$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = \underbrace{A(\hat{C}^a)}\ \hat{C}^k$

**Conjecture:** Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs.
If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

# Example: the Maxwell equations

Yoneda HS, CQG 18 (2001) 441

Maxwell evolution equations.

$$\begin{aligned}
\partial_t E_i &= c\epsilon_i{}^{jk}\partial_j B_k + P_i\, C_E + Q_i\, C_B, \\
\partial_t B_i &= -c\epsilon_i{}^{jk}\partial_j E_k + R_i\, C_E + S_i\, C_B, \\
& C_E = \partial_i E^i \approx 0, \quad C_B = \partial_i B^i \approx 0,
\end{aligned}
\qquad
\begin{cases}
\text{sym. hyp} & \Leftrightarrow \quad P_i = Q_i = R_i = S_i = 0, \\
\text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\
\text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0
\end{cases}$$

Constraint propagation equations

$$\begin{aligned}
\partial_t C_E &= (\partial_i P^i) C_E + P^i(\partial_i C_E) + (\partial_i Q^i) C_B + Q^i(\partial_i C_B), \\
\partial_t C_B &= (\partial_i R^i) C_E + R^i(\partial_i C_E) + (\partial_i S^i) C_B + S^i(\partial_i C_B), \\
& \begin{cases}
\text{sym. hyp} & \Leftrightarrow \quad Q_i = R_i, \\
\text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i > 0, \\
\text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_iQ_i \geq 0
\end{cases}
\end{aligned}$$

CAFs?

$$\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} = \begin{pmatrix} \partial_i P^i + P^i k_i & \partial_i Q^i + Q^i k_i \\ \partial_i R^i + R^i k_i & \partial_i S^i + S^i k_i \end{pmatrix} \partial_l \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} \approx \begin{pmatrix} P^i k_i & Q^i k_i \\ R^i k_i & S^i k_i \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} =: T \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix}$$

$$\Rightarrow \quad \text{CAFs} = (P^i k_i + S^i k_i \pm \sqrt{(P^i k_i + S^i k_i)^2 + 4(Q^i k_i R^j k_j - P^i k_i S^j k_j)})/2$$

Therefore CAFs become negative-real when

$$P^i k_i + S^i k_i < 0, \qquad \text{and} \qquad Q^i k_i R^j k_j - P^i k_i S^j k_j < 0$$

## **HS original GR code** (2002/October)

Use the Cactus-code base structure (`http://cactuscode.org`)

-- parallelize, parameter control, I/O, PUGH, elliptic solvers, ...

-- original module for all GR part (initial data/ADMevolution)

`Cactus/arrangements/GR/`

## **PC cluster with 4** (2002/September)

Pentium 4, 2.53GHz, 2GB each, 80 GHD each, gigabit ether

-- about 1.2 x 10^6 yen for total parts

-- TurboLinux 7, Intel Fortran compiler, MPICH, ... all free

-- possible up to 120^3 grid full GR simulation

Grant-in-Aid for Scientific Research Fund of JSPS, No.14740179 (2002-2005)

# 3 Adjusted ADM systems

We adjust the standard ADM system using constraints as:

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, && (1)\\
&\quad + P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (2)\\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, && (3)\\
&\quad + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (4)
\end{aligned}$$

with constraint equations

$$\begin{aligned}
\mathcal{H} &:= R^{(3)} + K^2 - K_{ij} K^{ij}, && (5)\\
\mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K. && (6)
\end{aligned}$$

We can write the adjusted constraint propagation equations as

$$\begin{aligned}
\partial_t \mathcal{H} &= (\text{original terms}) + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], && (7)\\
\partial_t \mathcal{M}_i &= (\text{original terms}) + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. && (8)
\end{aligned}$$

# Example 1: standard ADM vs original ADM (in Schwarzschild coordinate)

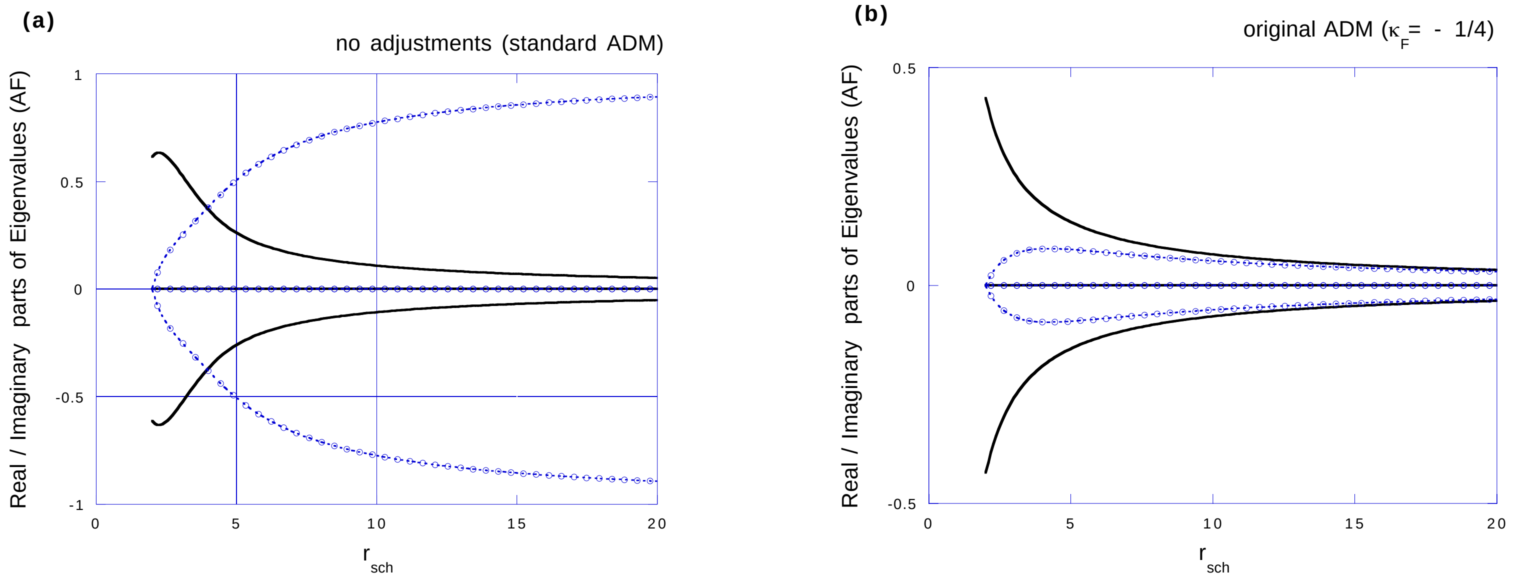


Figure 1: Amplification factors (AFs, eigenvalues of homogenized constraint propagation equations) are shown for the standard Schwarzschild coordinate, with (a) no adjustments, i.e., standard ADM, (b) original ADM ($\kappa_F = -1/4$). The solid lines and the dotted lines with circles are real parts and imaginary parts, respectively. They are four lines each, but actually the two eigenvalues are zero for all cases. Plotting range is $2 < r \leq 20$ using Schwarzschild radial coordinate. We set $k = 1, l = 2$, and $m = 2$ throughout the article.

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + \kappa_F \alpha \gamma_{ij} \mathcal{H},
\end{aligned}$$

# Example 2: Detweiler-type adjusted (in Schwarzschild coord.)

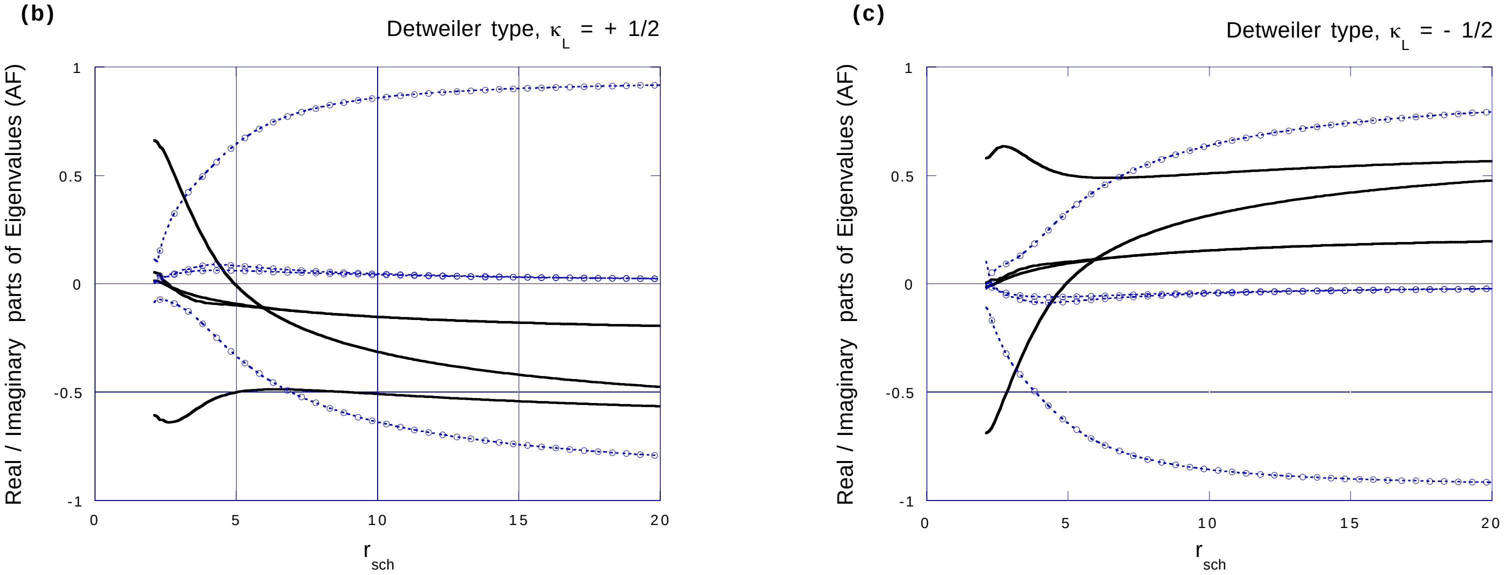


Figure 2: Amplification factors of the standard Schwarzschild coordinate, with Detweiler type adjustments. Multipliers used in the plot are (b) $\kappa_L = +1/2$, and (c) $\kappa_L = -1/2$.

$$\begin{aligned}
\partial_t \gamma_{ij} &= \text{(original terms)} + P_{ij}\mathcal{H}, \\
\partial_t K_{ij} &= \text{(original terms)} + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \\
& \text{where} \quad P_{ij} = -\kappa_L \alpha^3 \gamma_{ij}, \quad R_{ij} = \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}), \\
& \qquad\qquad S^k{}_{ij} = \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], \quad s^{kl}{}_{ij} = \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}],
\end{aligned}$$

# Comparisons of Adjusted ADM systems (Teukolsky wave)

## 3-dim, harmonic slice, periodic BC

**HS original Cactus/GR code**

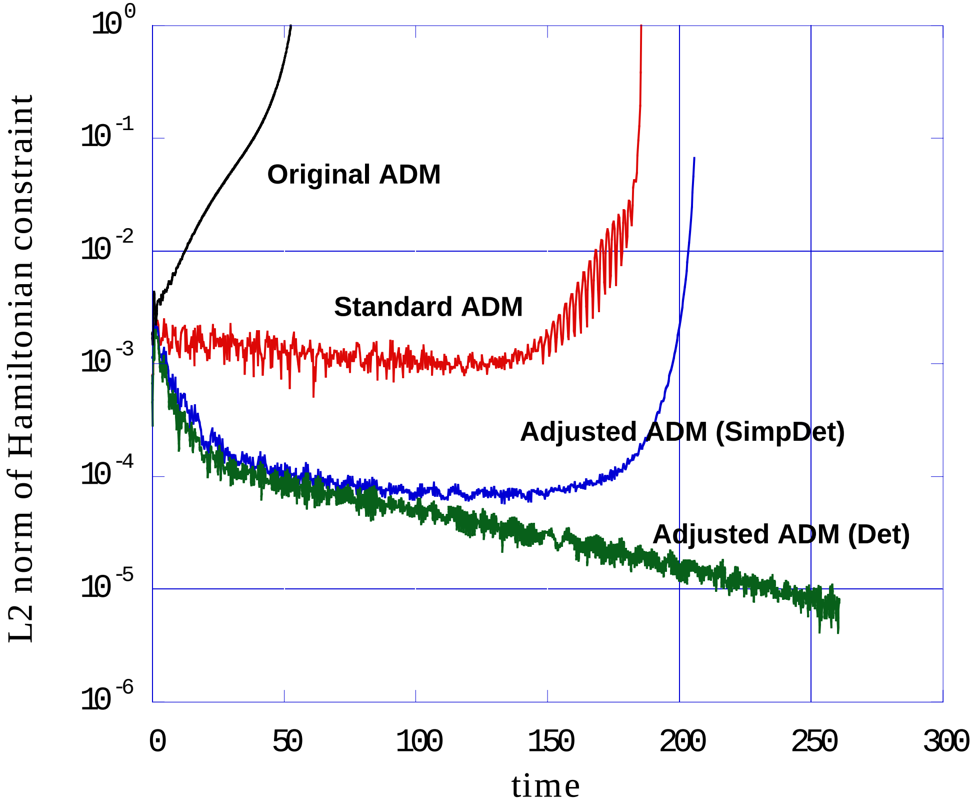


Figure 1: Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR/evolveADMeq code was used. Grid = $24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method.

## strategy 1 Shibata-Nakamura's (Baumgarte-Shapiro's) modifications to the standard ADM

– define new variables ($\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i$), instead of the ADM's ($\gamma_{ij}, K_{ij}$) where

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},$$

use momentum constraint in $\Gamma^i$-eq., and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

– The set of evolution equations become

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

$$\begin{aligned}
R_{ij} &= \partial_k\Gamma^k_{ij} - \partial_i\Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^\phi_{ij} \\
&\quad R^\phi_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi) \\
&\quad \tilde{R}_{ij} = -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}lm\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}
\end{aligned}$$

– No explicit explanations why this formulation works better.
AEI group (2000): the replacement by momentum constraint is essential.

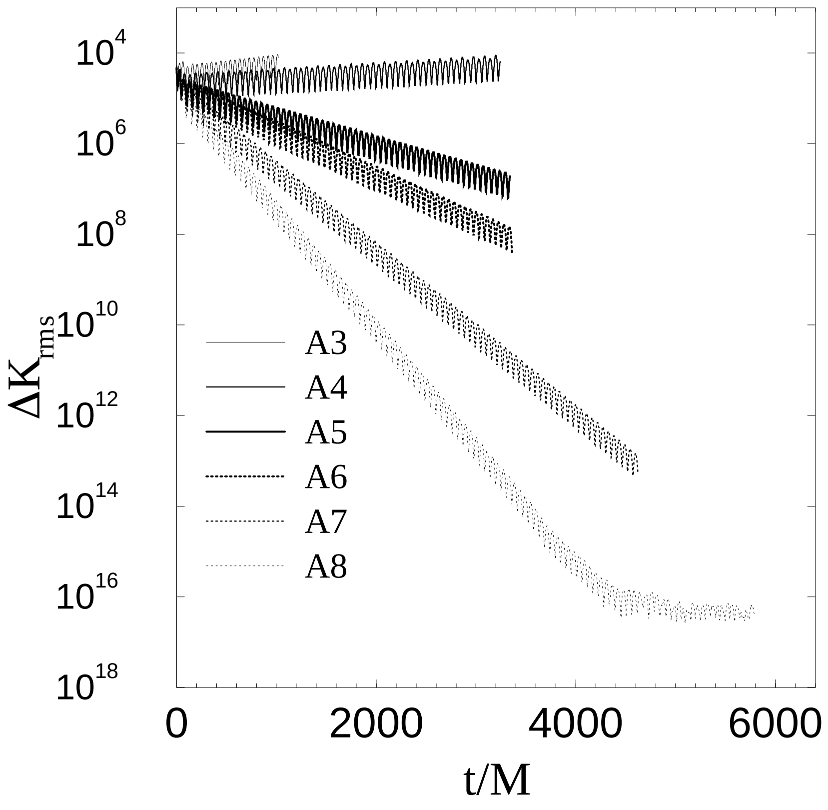
# Constraint Amplification Factors with each adjustment

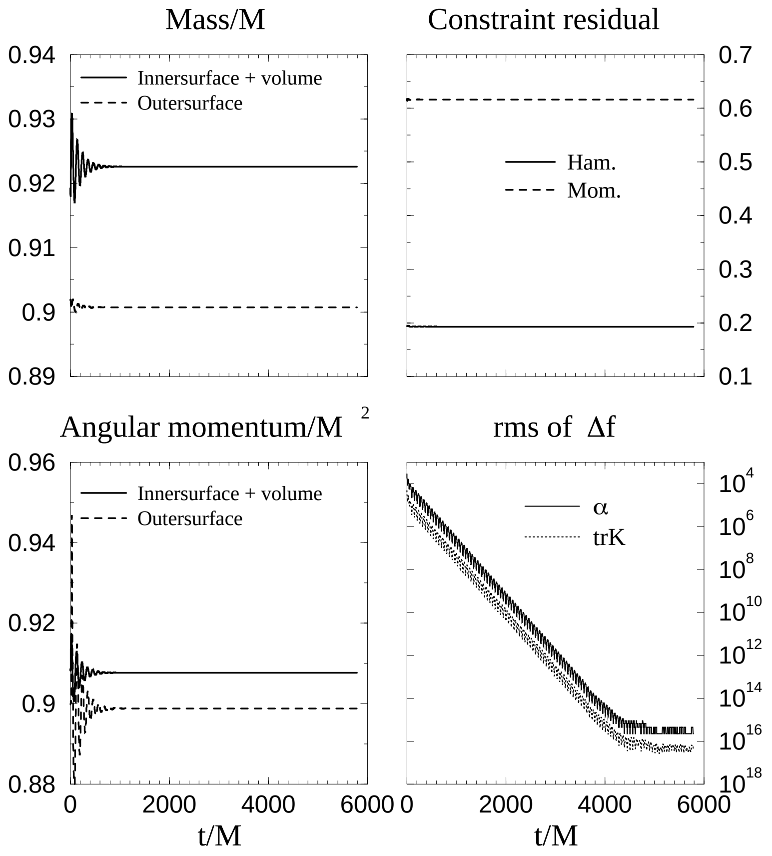
| adjustment | | CAFs | diag? | effect of the adjustment | |
|---|---|---|---|---|---|
| $\partial_t \phi$ | $\kappa_{\phi\mathcal{H}}\,\alpha\mathcal{H}$ | $(0,0,\pm\sqrt{-k^2}(*3), 8\kappa_{\phi\mathcal{H}}k^2)$ | no | $\kappa_{\phi\mathcal{H}} < 0$ makes 1 Neg. | |
| $\partial_t \phi$ | $\kappa_{\phi\mathcal{G}}\,\alpha\tilde{D}_k\mathcal{G}^k$ | $(0,0,\pm\sqrt{-k^2}(*2)$, long expressions) | yes | $\kappa_{\phi\mathcal{G}} < 0$ makes 2 Neg. 1 Pos. | |
| $\partial_t \tilde{\gamma}_{ij}$ | $\kappa_{SD}\,\alpha\tilde{\gamma}_{ij}\mathcal{H}$ | $(0,0,\pm\sqrt{-k^2}(*3),(3/2)\kappa_{SD}k^2)$ | yes | $\kappa_{SD} < 0$ makes 1 Neg. | Case (B) |
| $\partial_t \tilde{\gamma}_{ij}$ | $\kappa_{\tilde{\gamma}\mathcal{G}1}\,\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k$ | $(0,0,\pm\sqrt{-k^2}(*2)$, long expressions) | yes | $\kappa_{\tilde{\gamma}\mathcal{G}1} > 0$ makes 1 Neg. | |
| $\partial_t \tilde{\gamma}_{ij}$ | $\kappa_{\tilde{\gamma}\mathcal{G}2}\,\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k$ | (0,0, $(1/4)k^2\kappa_{\tilde{\gamma}\mathcal{G}2} \pm \sqrt{k^2(-1+k^2\kappa_{\tilde{\gamma}\mathcal{G}2}/16)}(*2)$,<br>long expressions) | yes | $\kappa_{\tilde{\gamma}\mathcal{G}2} < 0$ makes 6 Neg. 1 Pos. | Case (E1) |
| $\partial_t \tilde{\gamma}_{ij}$ | $\kappa_{\tilde{\gamma}\mathcal{S}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{S}$ | $(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{\tilde{\gamma}\mathcal{S}1})$ | no | $\kappa_{\tilde{\gamma}\mathcal{S}1} < 0$ makes 1 Neg. | |
| $\partial_t \tilde{\gamma}_{ij}$ | $\kappa_{\tilde{\gamma}\mathcal{S}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{S}$ | $(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{\tilde{\gamma}\mathcal{S}2}k^2)$ | no | $\kappa_{\tilde{\gamma}\mathcal{S}2} > 0$ makes 1 Neg. | |
| $\partial_t K$ | $\kappa_{K\mathcal{M}}\,\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k)$ | $(0,0,0,\pm\sqrt{-k^2}(*2),$<br>$(1/3)\kappa_{K\mathcal{M}}k^2 \pm (1/3)\sqrt{k^2(-9+k^2\kappa_{K\mathcal{M}}^2)})$ | no | $\kappa_{K\mathcal{M}} < 0$ makes 2 Neg. | |
| $\partial_t \tilde{A}_{ij}$ | $\kappa_{A\mathcal{M}1}\,\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k)$ | $(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{M}1}k^2)$ | yes | $\kappa_{A\mathcal{M}1} > 0$ makes 1 Neg. | |
| $\partial_t \tilde{A}_{ij}$ | $\kappa_{A\mathcal{M}2}\,\alpha(\tilde{D}_{(i}\mathcal{M}_{j)})$ | (0,0, $-k^2\kappa_{A\mathcal{M}2}/4 \pm \sqrt{k^2(-1+k^2\kappa_{A\mathcal{M}2}/16)}(*2)$ ,<br>long expressions) | yes | $\kappa_{A\mathcal{M}2} > 0$ makes 7 Neg | Case (D) |
| $\partial_t \tilde{A}_{ij}$ | $\kappa_{A\mathcal{A}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{A}$ | $(0,0,\pm\sqrt{-k^2}(*3), 3\kappa_{A\mathcal{A}1})$ | yes | $\kappa_{A\mathcal{A}1} < 0$ makes 1 Neg. | |
| $\partial_t \tilde{A}_{ij}$ | $\kappa_{A\mathcal{A}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{A}$ | $(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2}k^2)$ | yes | $\kappa_{A\mathcal{A}2} > 0$ makes 1 Neg. | |
| $\partial_t \tilde{\Gamma}^i$ | $\kappa_{\tilde{\Gamma}\mathcal{H}}\,\alpha\tilde{D}^i\mathcal{H}$ | $(0,0,\pm\sqrt{-k^2}(*3), -\kappa_{A\mathcal{A}2}k^2)$ | no | $\kappa_{\tilde{\Gamma}\mathcal{H}} > 0$ makes 1 Neg. | |
| $\partial_t \tilde{\Gamma}^i$ | $\kappa_{\tilde{\Gamma}\mathcal{G}1}\,\alpha\mathcal{G}^i$ | $(0,0,(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}1} \pm \sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}1}^2}(*2)$ , long.) | yes | $\kappa_{\tilde{\Gamma}\mathcal{G}1} < 0$ makes 6 Neg. 1 Pos. | Case (E2) |
| $\partial_t \tilde{\Gamma}^i$ | $\kappa_{\tilde{\Gamma}\mathcal{G}2}\,\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i$ | $(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}2} \pm \sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}2}^2}(*2)$ , long.) | yes | $\kappa_{\tilde{\Gamma}\mathcal{G}2} > 0$ makes 2 Neg. 1 Pos. | |
| $\partial_t \tilde{\Gamma}^i$ | $\kappa_{\tilde{\Gamma}\mathcal{G}3}\,\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j$ | $(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}3} \pm \sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}3}^2}(*2)$ , long.) | yes | $\kappa_{\tilde{\Gamma}\mathcal{G}3} > 0$ makes 2 Neg. 1 Pos. | |

Yoneda-HS, PRD66 (2002) 124003

# An Evolution of Adjusted BSSN Formulation

**by Yo-Baumgarte-Shapiro, PRD 66 (2002) 084026**





**Kerr-Schild BH (0.9 J/M), excision with cube, $1+\log$-lapse, $\Gamma$-driver shift.**

$$\partial_t \tilde{\Gamma}^i = (\cdots) + \frac{2}{3}\tilde{\Gamma}^i \beta^i{}_{,j} - (\chi + \frac{2}{3})\mathcal{G}^i \beta^j{}_{,j}$$

$\chi = 2/3$ **for (A4)-(A8)**

$$\partial_t \tilde{\gamma}_{ij} = (\cdots) - \kappa\alpha\tilde{\gamma}_{ij}\mathcal{H}$$

$\kappa = 0.1 \sim 0.2$ **for (A5), (A6) and (A8)**

## まとめ

**安定な数値計算を可能にするEinstein方程式の定式化は何か？**

### 今まで

- 数値的安定性が，定式化に依存することは事実として報告されている．
- 「初期値に解いた拘束条件は，時間発展でも保存する」==>> 数値的には必ずしも保証されない．
- ADM形式を改良するさまざまな試みが，過去10年にわたって続けられてきた．

### 我々の提案　これまでの試みを統一的に説明

- 「拘束条件式の発展方程式解析」 そして 「固有値解析による安定性への十分条件」
- 標準的なADM形式 は，BH近傍では，constraint violation modeが顕著に存在
- 「時間発展と共に，系が漸近的に拘束面へ収束していく」システムの構築がわずかな工夫で可能
- Adjusted ADM形式，Adjusted SN-BS形式を提案

### 数値的追認

- Adjusted ADM形式：これまで以上の数値的安定性が確認された．by PennState group / myself
- Adjusted SN-BS形式：これまで以上の数値的安定性が確認された．by Illinois/ AEI groups

**「双曲型偏微分方程式論」より有効で現実的な安定性予測！！**

## そのほか　最近の話題から

- **双曲性必要論に対する検討**
  - 「Strongly hyperbolicity 以上の双曲性がないと，長時間積分後，変数のノルムが発散することが証明できる」Calabrese-Pullin-Sarbach-Tiglio (PRD66(2002)041501)
  - 彼らの証明は，運動方程式の特性部分のみを平坦な背景時空で議論したもので，Einstein方程式にそのまま適用できるものではない．
  - 追試数値計算を行ったところ，full ADMでは長時間積分後，グリッド数に反比例してノルムが早期に発散することが確かに確認されたが，それらは，adjusted ADMにすることにより，延命可．
- **一般相対論的 MHD の定式化**
  - Baumgarte-Shapiro (astro-ph/0211340) がADM変数をベースにして，定式化．
  - Constraint Propagation 解析を進行中
- **adjusted systems 次の一手**
  - Lagrange multipliers の自動応答制御　試行錯誤中
- **「数値相対論における定式化問題　研究会」2002年5月 @ メキシコ**
  - 論文１「比較フォーマット」は，原稿作成進行中．(AEI/Pitt/Caltech/PennState/UNAM/RIKEN/...)
    `http://www.ApplesWithApples.org/` OPEN
  - 論文２「ゲージ波・線形重力波比較計算」は，データ回収中．
  - 今年「Mexico2　研究会」企画中．