Constraint Propagation of C^2-adjusted BSSN Equations

— Another Recipe for Robust Evolution Systems —

Takuya Tsuchiya[1], Gen Yoneda[1] and Hisa-aki Shinkai[2]

[1] Department of Mathematical Sciences, Waseda University, Tokyo, Japan

[2] Faculty of Information Science and Technology, Osaka Institute of Technology, Osaka, Japan

Abstract

In order to construct a robust evolution system against numerical instability for integrating the Einstein equations, we propose a new set of evolution equations by adjusting BSSN evolution equations with constraints. We apply an adjustment proposed by Fiske (2004) which uses the norm of the constraints, C^2. The advantage of this method is that the signature of the effective Lagrange multipliers are determined in advance. We show this feature by eigenvalue-analysis of constraint propagations and perform numerical tests using Gowdy wave propagation which indicates robust evolutions against the violation of the constraints than existing formulations.

Background and present problem

- The ADM formulation is not appriciate to perform numerical simulation for strong gravitational field and long term calculation.
- Is the current standard BSSN evolution equation is the best formulation?
- An unified treatment, called *adjusted system*, was proposed by Yoneda and Shinkai (Phys. Rev. D **63** 124019(2001)).
- An adjustment using the norm of constraint was proposed by Fiske (Phys. Rev. D **69**, 047501 (2004)), we apply this method to the BSSN system.

Main Idea: Adjusted Systems

- Suppose a time evolution system with constraints:

$$\partial_t u^i = f(u^i, \partial_j u^i, \cdots), \quad C^i(u^i, \partial_j u^i, \cdots) \approx 0, \qquad (1)$$

where u^i are variables and C^i are constraints. The propagation equation of the constraints,

$$\partial_t C^2 = \frac{\delta C^2}{\delta u^i}\partial_t u^i = h(C^i, \partial_j C^i, \cdots) \approx 0, \qquad (2)$$
$$\text{where } C^2 \equiv C^i C_i,$$

expresses the violation of the system.

- If we adjust evlolution equations by adding the constraint terms:

$$\partial_t u^i = [\textbf{Original} \quad \textbf{Terms}] - \kappa^{ij}\frac{\delta C^2}{\delta u^j}, \qquad (3)$$

where κ^{ij} is a parameter, then the constraint propagation is also changed as

$$\partial_t C^2 = [\textbf{Original} \quad \textbf{Terms}] - \kappa^{ij}\frac{\delta C^2}{\delta u^i}\frac{\delta C^2}{\delta u^j}. \qquad (4)$$

The last term is positive definite. Therefore we can control the violation of constraints by specifying appropriate adjustments.

CAFs $\simeq$ eigenvalues of (4)

CAFs (constraint amplification facters) is a tool for predicting the violation of constraints.

- The CAFs are the eigenvalues of the coefficient matrix of the constraiant propagation equations, (4).
- Negative real parts, or non-zero imaginary-parts of CAFs are preferable for stable evolutions.

Application to the BSSN formulation

The widely used BSSN evolution equations are,

$$\partial_t\varphi = -(1/6)\alpha K + (1/6)(\partial_i\beta^i) + \beta^i(\partial_i\varphi), \qquad (5)$$
$$\partial_t K = \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - D_iD^i\alpha + \beta^i(\partial_i K), \qquad (6)$$
$$\partial_t\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} - (2/3)\tilde{\gamma}_{ij}(\partial_\ell\beta^\ell) + \tilde{\gamma}_{j\ell}(\partial_i\beta^\ell) + \tilde{\gamma}_{i\ell}(\partial_j\beta^\ell) + \beta^\ell(\partial_\ell\tilde{\gamma}_{ij}), \qquad (7)$$
$$\partial_t\tilde{A}_{ij} = \alpha K\tilde{A}_{ij} - 2\alpha\tilde{A}_{i\ell}\tilde{A}^\ell{}_j + \alpha e^{-4\varphi}R_{ij}{}^{TF} - e^{-4\varphi}(D_iD_j\alpha)^{TF} - (2/3)\tilde{A}_{ij}(\partial_\ell\beta^\ell) + (\partial_i\beta^\ell)\tilde{A}_{j\ell} + (\partial_j\beta^\ell)\tilde{A}_{i\ell} + \beta^\ell(\partial_\ell\tilde{A}_{ij}), \qquad (8)$$
$$\partial_t\tilde{\Gamma}^i = 2\alpha\{6(\partial_j\varphi)\tilde{A}^{ij} + \tilde{\Gamma}^i{}_{j\ell}\tilde{A}^{j\ell} - (2/3)\tilde{\gamma}^{ij}(\partial_j K)\} - 2(\partial_j\alpha)\tilde{A}^{ij} + (2/3)\tilde{\Gamma}^i(\partial_j\beta^j) + (1/3)\tilde{\gamma}^{ij}(\partial_\ell\partial_j\beta^\ell) + \beta^\ell(\partial_\ell\tilde{\Gamma}^i) - \tilde{\Gamma}^j(\partial_j\beta^i) + \tilde{\gamma}^{j\ell}(\partial_j\partial_\ell\beta^i), \qquad (9)$$

The BSSN system has 5 constraint equations; both "kinetic" and "algebraic" constraint equations:

$$\mathcal{H}^B \equiv e^{-4\varphi}\tilde{R} - 8e^{-4\varphi}(\tilde{D}_i\tilde{D}^i\varphi + (\tilde{D}^m\varphi)(\tilde{D}_m\varphi)) + (2/3)K^2 - \tilde{A}_{ij}\tilde{A}^{ij} - (2/3)\mathcal{A}K \approx 0, \qquad (10)$$
$$(\mathcal{M}^B)_i \equiv -(2/3)\tilde{D}_iK + 6(\tilde{D}_j\varphi)\tilde{A}^j{}_i + \tilde{D}_j\tilde{A}^j{}_i - 2(\tilde{D}_i\varphi)\mathcal{A} \approx 0, \qquad (11)$$
$$\mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{j\ell}\tilde{\Gamma}^i{}_{j\ell} \approx 0, \qquad (12)$$
$$\mathcal{A} \equiv \tilde{A}^{ij}\tilde{\gamma}_{ij} \approx 0, \qquad (13)$$
$$\mathcal{S} \equiv \det(\tilde{\gamma}_{ij}) - 1 \approx 0. \qquad (14)$$

The C^2-adjusted BSSN evolution equations are written as

$$\partial_t\varphi = (5) - \lambda_\varphi\left(\frac{\delta(C^B)^2}{\delta\varphi}\right), \qquad (15)$$
$$\partial_t K = (6) - \lambda_K\left(\frac{\delta(C^B)^2}{\delta K}\right), \qquad (16)$$
$$\partial_t\tilde{\gamma}_{ij} = (7) - \lambda_{\tilde{\gamma}ijmn}\left(\frac{\delta(C^B)^2}{\delta\tilde{\gamma}_{mn}}\right), \qquad (17)$$
$$\partial_t\tilde{A}_{ij} = (8) - \lambda_{\tilde{A}ijmn}\left(\frac{\delta(C^B)^2}{\delta\tilde{A}_{mn}}\right), \qquad (18)$$
$$\partial_t\tilde{\Gamma}^i = (9) - \lambda^{ij}_{\tilde{\Gamma}}\left(\frac{\delta(C^B)^2}{\delta\tilde{\Gamma}^j}\right), \qquad (19)$$

where $(C^B)^2$ is the norm of the constraints,

$$(C^B)^2 \equiv (\mathcal{H}^B)^2 + (\mathcal{M}^B)^i(\mathcal{M}^B)_i + \mathcal{G}^i\mathcal{G}_i + \mathcal{A}^2 + \mathcal{S}^2,$$

and all of the coefficients, $\lambda_\varphi, \lambda_K, \lambda_{\tilde{\gamma}ijmn}, \lambda_{\tilde{A}ijmn}$ and $\lambda^{ij}_{\tilde{\Gamma}}$ are supposed to be positive definite.

CAFs of the C^2-adjusted BSSN system

CAFs of the system

$$\begin{pmatrix}\hat{\mathcal{H}}^B \\ \hat{\mathcal{M}}^B \\ \hat{\mathcal{G}}_i \\ \hat{\mathcal{A}} \\ \hat{\mathcal{S}}\end{pmatrix} = \begin{pmatrix} & & \\ \cdots & \cdots & \cdots \\ & & \end{pmatrix}\begin{pmatrix}\hat{\mathcal{H}}^B \\ \hat{\mathcal{M}}^B \\ \hat{\mathcal{G}}_i \\ \hat{\mathcal{A}} \\ \hat{\mathcal{S}}\end{pmatrix} \qquad (20)$$

are confirmed to be

- three negative real numbers, and
- six complex numbers with negative real part,

if we fix the background metric is Minkowskii metric and set $\lambda_\varphi = \lambda_K = \lambda$, $\lambda_{\tilde{\gamma}ijmn} = \lambda_{\tilde{A}ijmn} = \lambda\delta_{im}\delta_{jn}$ and $\lambda^{ij}_{\tilde{\Gamma}} = \lambda\delta^{ij}$ for simplicity, where $\lambda > 0$.

Numerical Test : Polarized Gowdy Wave

We show damping of constraint in numerical evolutions using polarized Gowdy wave evolution, which is one of the standard tests for comparisons of formulations in numerical relativity as is known to the Apples-with-Apples testbeds (Class. Quantum Grav. 21 (2004) 589).

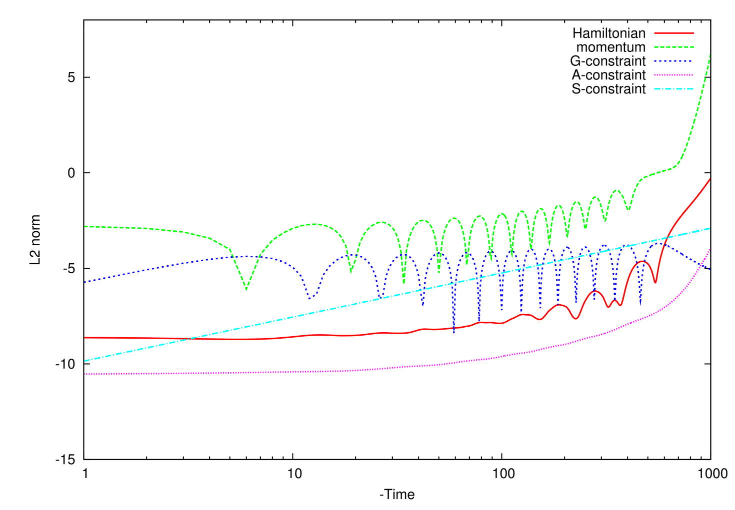
The metric of polarized Gowdy wave is

$$ds^2 = t^{-1/2}e^{\lambda/2}(-dt^2 + dx^2) + t(e^P dy^2 + e^{-P}dz^2), \qquad (21)$$

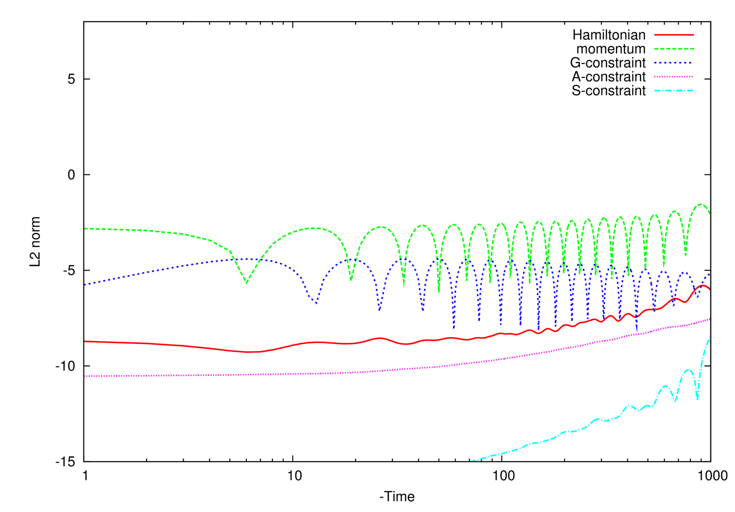
where P and λ are functions of x and t. The time coordinate t is chosen such that time increases as the universe expands, this metric is singular at $t = 0$ which corresponds to the cosmological singularity.

Results

- Constraint violations of the standard BSSN system.



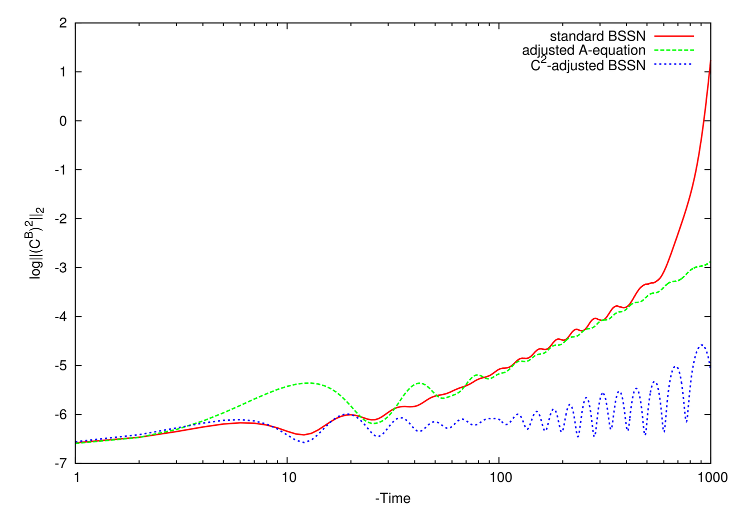
- Constraint violations of the C^2-adjusted BSSN system. Better performance than the standard system. ($\lambda_\varphi = 0, \lambda_K = 10^{-2.7}, \lambda_{\tilde{\gamma}} = 10^{-5.0}\delta_{im}\delta_{jn}, \lambda_{\tilde{A}} = 0, \lambda_{\tilde{\Gamma}} = 10^{-1.4}\delta_{ij}$.)



- The L2 norm of the $(C^B)^2$ of three systems, including another type of adjustment,

$$\partial_t\tilde{A}_{ij} = (8) + \kappa_A\alpha\tilde{D}_{(i}\mathcal{M}_{j)}$$

with $\kappa_A = 10^{-2.4}$. C^2-adjusted BSSN system keeps the violation of constraint lowest.



Summary

- The violation of the momentum constraints (green line) is alwayes the largest. Therefore, it is important to control the violation of the momentum constraints.
- The C^2-adjusted BSSN system has the feature of the constraint damping.
- We confirmed that this new adjustment also works better than a previous BSSN adjustment.