Symmetric Hyperbolic System in the Ashtekar Formulation

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We present a first-order symmetric hyperbolic system in the Ashtekar formulation of general relativity for vacuum spacetime. We add terms from the constraint equations to the evolution equations with appropriate combinations, which is the same technique used by Iriondo, Leguizamón, and Reula [Phys. Rev. Lett. **79**, 4732 (1997)]. However, our system is different from theirs in that we primarily use Hermiticity of a characteristic matrix of the system to characterize our system *symmetric,* discuss the consistency of this system with a reality condition, and show the characteristic speeds of the system. [S0031-9007(98)08196-4]

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Hyperbolic formulation of the Einstein equation is one of the main research areas in general relativity [1]. This formulation is used in the proof of the existence, uniqueness, and stability (well-posedness) of the solutions of the Einstein equation by analytical methods [2]. Thus far, several first-order hyperbolic formulations are proposed; some of them are flux conservative [3], symmetrizable [4], or symmetric hyperbolic systems [5–7]. The recent interest in hyperbolic formulation arises from its application to numerical relativity. One of the expected advantages is the existence of the characteristic speeds of the system, with which we may treat the numerical boundary with the appropriate condition. Some numerical tests have been reported along this direction [8–10].

Recently, Iriondo, Leguizamón, and Reula (ILR) [11] discussed a symmetric hyperbolic system in the Ashtekar formulation [12] of general relativity. Ashtekar's formulation has many advantages in the treatment of gravity. By using his special pair of variables, the constraint equations which appear in the theory become low-order polynomials, and the theory has the correct form for gauge theoretical features. These suggest possibilities for treating a quantum description of gravity nonpertubatively. The classical applications of Ashtekar's formulation has also been discussed by several authors. For example, we [13] discussed the reality conditions for metric and triad, and proposed a new set of variables from the point of Lorentzian dynamics. We [14] also showed an example of passing a degenerate point in 3-space by losing the reality condition locally.

In this Letter, we present a new symmetric hyperbolic system in Ashtekar's formulation for Lorentzian vacuum spacetime. Iriondo, Leguizamón, and Reula [11] state that they have constructed a symmetric hyperbolic system. However, we think their discussion is not clear in the following three points. First, they used *anti*-Hermiticity as the principal symbol for defining their system *symmetric.* We, however, think that this does not derive the Hermiticity of the characteristic matrix (A, below in Eq. (14); B in [11]), since they do not define their vector k_a explicitly. We use rather the Hermiticity of the characteristic matrix primarily to construct a *symmetric hyperbolic* system. Second, they did not mention the consistency of their formulation with the reality conditions which are crucial in the study of the Lorentzian dynamics in Ashtekar variables. Third, they did not discuss the characteristic structure of the system, which should be shown in the normal hyperbolic formulations. Our discussion also covers these two.

The construction of this paper is as follows. After giving a brief review of Ashtekar's variables and reality conditions in Sect. I, we present our formulation in Sect. II. The discussion of characteristic speed and summary are in Sect. III.

I. *Ashtekar's formulation.*—The key feature of Ashtekar's formulation of general relativity [12] is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write the metric $g_{\mu\nu}$ using the tetrad e^I_μ, and define its inverse E^μ_I by giving $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$ and $E^\mu_I := e^J_\nu g^{\mu\nu}\eta_{IJ}$. [We use $\mu, \nu = 0, \ldots, 3$ and $i, j = 1, \ldots, 3$ as spacetime indices, while $I, J = (0), \ldots, (3)$ and $a, b = (1), \ldots, (3)$ are SO(1, 3), SO(3) indices, respectively. We raise and lower $\mu, \nu, \ldots$ by $g^{\mu\nu}$ and $g_{\mu\nu}$ (Lorentzian metric); $I, J, \ldots$ by $\eta^{IJ} = \mathrm{diag}(-1, 1, 1, 1)$ and η_{IJ}; $i, j, \ldots$ by γ^{ij} and γ_{ij} (3-metric). We use volume forms ϵ_{abc}; $\epsilon_{abc}\epsilon^{abc} = 3!$.] We define SO(3, C) self-dual and anti-self-dual connections ${}^{\pm}\mathcal{A}^a_\mu := \omega^{0a}_\mu \mp (i/2)\epsilon^a{}_{bc}\omega^{bc}_\mu$, where ω^{IJ}_μ is a spin connection 1-form (Ricci connection), $\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu e^J_\nu$. Ashtekar's plan is to use only a self-dual part of the connection ${}^{+}\mathcal{A}^a_\mu$ and to use its spatial part ${}^{+}\mathcal{A}^a_i$ as a dynamical variable. Hereafter, we simply denote ${}^{+}\mathcal{A}^a_\mu$ as $\mathcal{A}^a_\mu$.

The lapse function, N, and shift vector, N^i, are expressed as $E^\mu_0 = (1/N, -N^i/N)$. This allows one to think of E^μ_0 as a normal vector field to Σ spanned by the

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condition $t = x^0 = \text{const}$, which plays the same role as that of Arnowitt-Deser-Misner formulation. Ashtekar treated the set $(\mathcal{A}_i^a, \tilde{E}_a^i)$ as basic dynamical variables, where $\tilde{E}_a^i$ is an inverse of the densitized triad defined by $\tilde{E}_a^i := eE_a^i$, where $e := \det e_i^a$ is a density. This pair forms the canonical set.

In the case of pure gravitational spacetime, the Hilbert action takes the form

$$S = \int d^4x[(\partial_t \mathcal{A}_i^a)\tilde{E}_a^i + (i/2)\underset{\sim}{N}\tilde{E}_a^i\tilde{E}_b^j F_{ij}^c \epsilon^{ab}{}_c - \Lambda\underset{\sim}{N}\det\tilde{E} - N^i F_{ij}^a \tilde{E}_a^j + \mathcal{A}_0^a \mathcal{D}_i \tilde{E}_a^i], \tag{1}$$

where $\underset{\sim}{N} := e^{-1}N$, Λ is the cosmological constant, $\mathcal{D}_i\tilde{E}_a^i := \partial_i\tilde{E}_a^i - i\epsilon_{ab}{}^c\mathcal{A}_i^b\tilde{E}_c^i$, and $\det\tilde{E}$ is defined as $\det\tilde{E} = (1/6)\epsilon^{abc}\underset{\sim}{\epsilon}_{ijk}\tilde{E}_a^i\tilde{E}_b^j\tilde{E}_c^k$, where $\epsilon_{ijk} := \epsilon_{abc}e_i^a e_j^b e_k^c$ and $\underset{\sim}{\epsilon}_{ijk} := e^{-1}\epsilon_{ijk}$. ($\epsilon_{xyz} = e$, $\underset{\sim}{\epsilon}_{xyz} = 1$, $\epsilon^{xyz} = e^{-1}$, $\tilde{\epsilon}^{xyz} = 1$.)

Varying the action with respect to the nondynamical variables $\underset{\sim}{N}$, N^i, and $\mathcal{A}_0^a$ yields the constraint equations,

$$C_H := (i/2)\epsilon^{ab}{}_c\tilde{E}_a^i\tilde{E}_b^j F_{ij}^c - \Lambda\det\tilde{E} \approx 0, \tag{2}$$

$$C_{Mi} := -F_{ij}^a\tilde{E}_a^j \approx 0, \tag{3}$$

$$C_{Ga} := \mathcal{D}_i\tilde{E}_a^i \approx 0, \tag{4}$$

where $F_{\mu\nu}^a := (d\mathcal{A}^a)_{\mu\nu} - (i/2)\epsilon^a{}_{bc}(\mathcal{A}^b \wedge \mathcal{A}^c)_{\mu\nu}$ is the curvature 2-form.

The equations of motion for the dynamical variables ($\mathcal{A}_i^a$ and $\tilde{E}_a^i$) are

$$\partial_t\mathcal{A}_i^a = -i\epsilon^{ab}{}_c\underset{\sim}{N}\tilde{E}_b^j F_{ij}^c + N^j F_{ji}^a + \mathcal{D}_i\mathcal{A}_0^a + e\Lambda\underset{\sim}{N}e_i^a, \tag{5}$$

$$\partial_t\tilde{E}_a^i = -i\mathcal{D}_j(\epsilon^{cb}{}_a\underset{\sim}{N}\tilde{E}_c^j\tilde{E}_b^i) + 2\mathcal{D}_j(N^{[j}\tilde{E}_a^{i]}) + i\mathcal{A}_0^b\epsilon_{ab}{}^c\tilde{E}_c^i, \tag{6}$$

where $\mathcal{D}_j X_a^{ji} := \partial_j X_a^{ji} - i\epsilon_{ab}{}^c\mathcal{A}_j^b X_c^{ji}$, for $X_a^{ij} + X_a^{ji} = 0$.

In order to construct metric variables from the variables $(\mathcal{A}_i^a, \tilde{E}_a^i, \underset{\sim}{N}, N^i)$, we first prepare tetrad E_I^μ as $E_0^\mu = (1/e\underset{\sim}{N}, -N^i/e\underset{\sim}{N})$ and $E_a^\mu = (0, \tilde{E}_a^i/e)$. Using them, we obtain metric $g^{\mu\nu}$ such that

$$g^{\mu\nu} := E_I^\mu E_J^\nu \eta^{IJ}. \tag{7}$$

Notice that, in general, the metric (7) is not real. To ensure the metric is real valued, we need to impose real lapse and shift vectors together with two conditions (metric reality condition),

$$\mathrm{Im}(\tilde{E}_a^i\tilde{E}^{ja}) = 0, \tag{8}$$

$$\mathrm{Re}(\epsilon^{abc}\tilde{E}_a^k\tilde{E}_b^{(i}\mathcal{D}_k\tilde{E}_c^{j)}) = 0, \tag{9}$$

where the latter comes from the secondary condition of reality of the metric $\mathrm{Im}\{\partial_t(\tilde{E}_a^i\tilde{E}^{ja})\} = 0$ [15], and we assume $\det\tilde{E} > 0$ (see [13]).

For later convenience, we also prepare stronger reality conditions. These conditions are

$$\mathrm{Im}(\tilde{E}_a^i) = 0 \tag{10}$$

and

$$\mathrm{Im}(\partial_t\tilde{E}_a^i) = 0, \tag{11}$$

and we call them the "primary triad reality condition" and the "secondary triad reality condition," respectively. Using the equations of motion $\tilde{E}_a^i$, the gauge constraint (4), the metric reality conditions (8) and (9), and the primary condition (10), we see that (11) is equivalent to [13]

$$\mathrm{Re}(\mathcal{A}_0^a) = \partial_i(\underset{\sim}{N})\tilde{E}^{ia} + (1/2e)e_i^b\underset{\sim}{N}\tilde{E}^{ja}\partial_j\tilde{E}_b^i + N^i\,\mathrm{Re}(\mathcal{A}_i^a), \tag{12}$$

or with undensitized variables,

$$\mathrm{Re}(\mathcal{A}_0^a) = \partial_i(N)E^{ia} + N^i\,\mathrm{Re}(\mathcal{A}_i^a). \tag{13}$$

From this expression we see that the secondary triad reality condition restricts the three components of the "triad lapse" vector $\mathcal{A}_0^a$. Therefore (12) is not a restriction on the dynamical variables ($\mathcal{A}_i^a$ and $\tilde{E}_a^i$) but on the slicing, which we should impose on each hypersurface. Thus the secondary triad reality condition does not restrict the dynamical variables any farther than does the secondary metric condition.

II. *Hyperbolic formulation.*—We start by defining the hyperbolic system using Friedrichs' [16] method, which is first applied in general relativity by Fischer and Marsden [5]. That is, we say that the system is first-order (quasilinear) hyperbolic if a certain pair of variables u_i form a linear system as

$$\partial_t u_i = A^{lj}{}_i(u)\partial_l u_j + B_i(u), \tag{14}$$

where A is a characteristic matrix-valued function, of which the eigenvalues are all real, and B is a function. We further define that the system is symmetric when A is a Hermitian matrix [6,17].

The symmetric system gives us the energy integral inequalities, which are the primary tools for analyzing the well-posedness of the system. As was discussed by Geroch [18], most physical systems are expressed as symmetric hyperbolic systems.

Ashtekar's formulation itself is in the first-order hyperbolic form in the sense of (14), but not a symmetric hyperbolic form.

We start by writing the principal part of Ashtekar's evolution equations as

$$\partial_t\begin{bmatrix}\tilde{E}_a^i\\ \mathcal{A}_i^a\end{bmatrix} \cong \begin{bmatrix}A^l{}_a{}^{bi}{}_j & B^l{}_{ab}{}^{ij}\\ C^{lab}{}_{ij} & D^{la}{}_{bi}{}^j\end{bmatrix}\partial_l\begin{bmatrix}\tilde{E}_b^j\\ \mathcal{A}_j^b\end{bmatrix}, \tag{15}$$

where ($\cong$) means that we extracted only the terms which appear in the principal part of the system. The system is symmetric hyperbolic if

$$0 = A^{labij} - \overline{A}^{lbaji}, \tag{16}$$

$$0 = D^{labij} - \overline{D}^{lbaij}, \tag{17}$$

$$0 = B^{labij} - \overline{C}^{lbaji}, \tag{18}$$

where the overbar denotes the complex conjugate. [We think that the reader will not confuse A^{labij} and B^{labij} with matrix A and B in (14).]

We first prepare the constraints (2)–(4) as

$$\begin{aligned} C_H &\cong i\epsilon^{ab}{}_c\tilde{E}^i_a\tilde{E}^j_b\partial_i\mathcal{A}^c_j = i\epsilon^{dc}{}_b\tilde{E}^l_d\tilde{E}^j_c(\partial_l\mathcal{A}^b_j) \\ &= -i\epsilon_b{}^{cd}\tilde{E}^j_c\tilde{E}^l_d(\partial_l\mathcal{A}^b_j), \qquad (19) \\ C_{Mk} &= -F^a_{kj}\tilde{E}^j_a \cong -(\partial_k\mathcal{A}^a_j - \partial_j\mathcal{A}^a_k)\tilde{E}^j_a \\ &= [-\delta^l_k\tilde{E}^j_b + \delta^j_k\tilde{E}^l_b](\partial_l\mathcal{A}^b_j), \qquad (20) \\ C_{Ga} &= \mathcal{D}_i\tilde{E}^i_a \cong \partial_l\tilde{E}^l_a. \qquad (21) \end{aligned}$$

We apply the same technique with ILR to modify the equation of motion of $\tilde{E}^i_a$ and $\mathcal{A}^a_i$ by adding the constraints which weakly produce $C_H = 0$, $C_{Mk} = 0$, and $C_{Ga} = 0$. With a parametrization for triad lapse $\mathcal{A}^a_0$ with T and S as

$$\partial_i\mathcal{A}^a_0 \cong T^l{}_i{}^a{}_j{}^b\partial_l\tilde{E}^j_b + S^l{}_i{}^{aj}{}_b\partial_l\mathcal{A}^b_j, \qquad (22)$$

we write the principal parts of (5) and (6) as

$$\begin{aligned} \partial_t\tilde{E}^i_a &= -i\mathcal{D}_j(\epsilon^{cb}{}_a\underset{\sim}{N}\tilde{E}^j_c\tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) \\ &\quad + i\mathcal{A}^b_0\epsilon_{ab}{}^c\tilde{E}^i_c + P^i{}_{ab}C^b_G \\ &\cong -i\epsilon^{cb}{}_a\underset{\sim}{N}(\partial_j\tilde{E}^j_c)\tilde{E}^i_b - i\epsilon^{cb}{}_a\underset{\sim}{N}\tilde{E}^j_c(\partial_j\tilde{E}^i_b) \\ &\quad + \mathcal{D}_j(N^j\tilde{E}^i_a) - \mathcal{D}_j(N^i\tilde{E}^j_a) + P^i{}_a{}^b\partial_j\tilde{E}^j_b \\ &\cong [-i\epsilon^{bc}{}_a\underset{\sim}{N}\delta^l_j\tilde{E}^i_c - i\epsilon^{cb}{}_a\underset{\sim}{N}\tilde{E}^l_c\delta^i_j \\ &\quad + N^l\delta^i_j\delta^b_a - N^i\delta^l_j\delta^b_a + P^i{}_a{}^b\delta^l_j](\partial_l\tilde{E}^j_b), \qquad (23) \end{aligned}$$

$$\begin{aligned} \partial_t\mathcal{A}^a_i &= -i\epsilon^{ab}{}_c\underset{\sim}{N}\tilde{E}^j_bF^c_{ij} + N^jF^a_{ji} + \mathcal{D}_i\mathcal{A}^a_0 + e\Lambda\underset{\sim}{N}e^a_i + Q^a_iC_H + R_i{}^{ja}C_{Mj} \\ &\cong -i\epsilon^{ab}{}_c\underset{\sim}{N}\tilde{E}^j_b(\partial_i\mathcal{A}^c_j - \partial_j\mathcal{A}^c_i) + N^j(\partial_j\mathcal{A}^a_i - \partial_i\mathcal{A}^a_j) + T^l{}_i{}^a{}_j{}^b\partial_l\tilde{E}^j_b + S^l{}_i{}^{aj}{}_b\partial_l\mathcal{A}^b_j \\ &\quad - Q^a_ii\epsilon_b{}^{cd}\tilde{E}^j_c\tilde{E}^l_d(\partial_l\mathcal{A}^b_j) + R_i{}^{ka}[-\delta^l_k\tilde{E}^j_b + \delta^j_k\tilde{E}^l_b]\partial_l\mathcal{A}^b_j \\ &\cong [+i\epsilon^a{}_b{}^c\underset{\sim}{N}\tilde{E}^j_c\delta^l_i - i\epsilon^a{}_b{}^c\underset{\sim}{N}\tilde{E}^l_c\delta^j_i + N^l\delta^a_b\delta^j_i - N^j\delta^a_b\delta^l_i + S^l{}_i{}^{aj}{}_b \\ &\quad - iQ^a_i\epsilon_b{}^{cd}\tilde{E}^j_c\tilde{E}^l_d - R_i{}^{la}\tilde{E}^j_b + R_i{}^{ja}\tilde{E}^l_b](\partial_l\mathcal{A}^b_j) + T^l{}_i{}^a{}_j{}^b\partial_l\tilde{E}^j_b, \qquad (24) \end{aligned}$$

where P, Q, and R are parameters and will be fixed later. Note that we truncated $\mathcal{A}^a_0$ in (23), while it remains in (24), since only the derivative of $\mathcal{A}^a_0$ affects the principal part of the system. From these two equations, we get

$$\begin{aligned} A^{labij} &= -i\epsilon^{bca}\underset{\sim}{N}\gamma^{lj}\tilde{E}^i_c - i\epsilon^{cba}\underset{\sim}{N}\tilde{E}^l_c\gamma^{ij} \\ &\quad + N^l\gamma^{ij}\delta^{ab} - N^i\gamma^{lj}\delta^{ab} + P^{iab}\gamma^{lj}, \qquad (25) \\ B^{labij} &= 0, \qquad (26) \\ C^{labij} &= T^{liajb}, \qquad (27) \\ D^{labij} &= +i\epsilon^{abc}\underset{\sim}{N}\tilde{E}^j_c\gamma^{li} - i\epsilon^{abc}\underset{\sim}{N}\tilde{E}^l_c\gamma^{ji} \\ &\quad + N^l\delta^{ab}\gamma^{ji} - N^j\delta^{ab}\gamma^{li} + S^{liajb} \\ &\quad - iQ^{ai}\epsilon^{bcd}\tilde{E}^j_c\tilde{E}^l_d - R^{ila}\tilde{E}^{jb} + R^{ija}\tilde{E}^{lb}. \qquad (28) \end{aligned}$$

The condition (16) is written as

$$\begin{aligned} 0 &= -i\epsilon^{bca}\underset{\sim}{N}\gamma^{lj}\tilde{E}^i_c - i\epsilon^{acb}\underset{\sim}{N}\gamma^{li}\overline{\tilde{E}}^j_c \\ &\quad - 2i\epsilon^{cba}\underset{\sim}{N}\,\mathrm{Im}(\tilde{E}^l_c)\gamma^{ij} - N^i\gamma^{lj}\delta^{ab} \\ &\quad + N^j\gamma^{li}\delta^{ba} + P^{iab}\gamma^{lj} - \overline{P}^{jba}\gamma^{li}. \qquad (29) \end{aligned}$$

Because the third term in the right-hand side cannot be eliminated using P, we assume the triad reality condition $\mathrm{Im}(\tilde{E}^l_c) = 0$ hereafter. Then (17) and (18) become

$$0 = \overline{T}^{ljbia}, \qquad (30)$$

$$\begin{aligned} 0 &= i\epsilon^{abc}\underset{\sim}{N}\tilde{E}^j_c\gamma^{li} + i\epsilon^{bac}\underset{\sim}{N}\tilde{E}^i_c\gamma^{lj} - N^j\delta^{ab}\gamma^{li} + N^i\delta^{ba}\gamma^{lj} + S^{liajb} - \overline{S}^{ljbia} - iQ^{ai}\epsilon^{bcd}\tilde{E}^j_c\tilde{E}^l_d - i\overline{Q}^{bj}\epsilon^{acd}\tilde{E}^i_c\tilde{E}^l_d \\ &\quad - R^{ila}\tilde{E}^{jb} + R^{ija}\tilde{E}^{lb} + \overline{R}^{jlb}\tilde{E}^{ia} - \overline{R}^{jib}\tilde{E}^{la}. \qquad (31) \end{aligned}$$

The third and fourth terms in (31) cannot be eliminated using Q or R, so $S^{liajb} = \gamma^{li}\delta^{ab}N^j$ is determined. Thus S and $T^{ljbia} = 0$ [Eq. (30)] decides the form of the triad lapse as

$$\mathcal{A}^a_0 = \mathcal{A}^a_jN^j + \text{nondynamical terms} \qquad (32)$$

in the result. In order to be consistent with the triad reality condition (13), we need to specify the lapse as $\partial_iN = 0$. This lapse condition is also supported by the fact that, if we do not assume $\partial_iN = 0$, then the secondary triad reality condition (12) makes the system second order. ILR do not discuss the consistency of the system with the reality condition (especially with the secondary reality condition). However, since ILR assume $\mathcal{A}^a_0 = \mathcal{A}^a_jN^j$, we think that ILR also need to impose a similar restricted lapse condition in order to preserve the reality of the system.

The rest of our effort is finished when we specify parameters P, Q, and R. P is given by decomposing (29) into real/imaginary parts;

$$\begin{aligned} 0 &= -N^i\gamma^{lj}\delta^{ab} + N^j\gamma^{li}\delta^{ba} \\ &\quad + \mathrm{Re}(P)^{iab}\gamma^{lj} - \mathrm{Re}(P)^{jba}\gamma^{li}, \qquad (33) \\ 0 &= -\epsilon^{bca}\underset{\sim}{N}\gamma^{lj}\tilde{E}^i_c - \epsilon^{acb}\underset{\sim}{N}\gamma^{li}\tilde{E}^j_c \\ &\quad + \mathrm{Im}(P)^{iab}\gamma^{lj} + \mathrm{Im}(P)^{jba}\gamma^{li}. \qquad (34) \end{aligned}$$

By multiplying γ_{li} into these two equations and taking the symmetric and antisymmetric operations to the index ab,

we obtain

$$P^{iab} = N^i \delta^{ab} + i\underset{\sim}{N}\epsilon^{abc}\tilde{E}^i_c . \quad (35)$$

For Q and R, we found that the combination of the choice

$$Q^{ai} = e^{-2}\underset{\sim}{N}\tilde{E}^{ia} \quad (36)$$

and

$$R^{ila} = ie^{-2}\underset{\sim}{N}\epsilon^{acd}\tilde{E}^i_d\tilde{E}^l_c \quad (37)$$

satisfies the condition (31).

III. *Discussion.*—In summary, by adding constraint terms with appropriate coefficients, we succeeded in constructing a symmetric hyperbolic formulation of Ashtekar's system. This formulation is consistent with the secondary triad reality condition, which requires one to impose a constant lapse function for the evolving system.

The characteristic speeds of this system are given by finding eigenvalues of the characteristic matrix A of (14). Since A is a Hermitian, eigenvalues of A are all real. Then it is again clear that this system is symmetric hyperbolic. Actually, the eigenvalues of the 18×18 matrix A^l for the x^l direction are N^l (multiplicity $= 6$), $N^l \pm \sqrt{\gamma^{ll}}N$ (five each), and $N^l \pm 3\sqrt{\gamma^{ll}}N$ (one each), where we do not take the sum in γ^{ll} here. These speeds are independent from the way of taking a triad. We omit showing the related eigenvectors because of limited space.

As we denoted in Sect. II, our formulation requires a triad reality condition. In order to make the system first order, the lapse function is assumed to be constant. Shift vectors and triad lapse $\mathcal{A}^a_0$ should have a relation (32). This can be interpreted that the shift is free and the triad lapse is determined. This gauge restriction sounds tight, but this arises from our general assumption of (22). There might be a possibility of improving the situation by renormalizing the shift and triad lapse terms into the left-hand side of equations of motion such as the case of general relativity [4]. Or this might be because our system constitutes Ashtekar's original variables. We are now trying to release this gauge restriction and/or to simplify the characteristic speeds by other gauge possibilities and also by introducing new dynamical variables. This effort will be reported elsewhere.

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Note added.—After we submitted this paper, we found that ILR submitted a second paper as a preprint [19]. In the second paper, they discuss how to treat the reality condition in a different way than we do.

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