Charged Brane-World Black Holes

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We study charged brane-world black holes in the model of Randall and Sundrum in which our universe is viewed as a domain wall in asymptotically anti-de Sitter space. Such black holes can carry two types of "charge", one arising from the bulk Weyl tensor and one from a gauge field trapped on the wall. We use a combination of analytical and numerical techniques to study how these black holes behave in the bulk. It has been shown that a Reissner-Nordstrom geometry is induced on the wall when only Weyl charge is present. However, we show that such solutions exhibit pathological features in the bulk. For more general charged black holes, our results suggest that the extent of the horizon in the fifth dimension is usually less than for an uncharged black hole that has the same mass or the same horizon radius on the wall.

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I. INTRODUCTION

In many of the brane-world scenarios, the matter fields which we observe are trapped on the brane [1–4] (see also [5] for older proposals). If matter trapped on a brane undergoes gravitational collapse then a black hole will form. Such a black hole will have a horizon that extends into the dimensions transverse to the brane: it will be a higher dimensional object.

Within the context of the second Randall-Sundrum (RS) scenario [4], it is important that the induced metric on the domain wall[*] is, to a good approximation, the solution predicted by standard General Relativity in four dimensions. Otherwise the usual astrophysical properties of black holes and stars would not be recovered.

In a recent paper [6], the gravitational collapse of *uncharged*, non-rotating matter in the second model of RS was investigated. There it was proposed that what would appear to be a four-dimensional black hole from the point of view of an observer in the brane-world, is really a five-dimensional "black cigar", which extends into the extra fifth dimension. If this cigar extends all the way down to the anti-de Sitter (AdS) horizon, then we recover the metric for a black string in AdS. However, such a black string is unstable near the AdS horizon. This instability, known as the "Gregory-Laflamme" instability [7], implies that the string will fragment in the region near the AdS horizon. However, the solution is stable far from the AdS horizon. Thus, one may conclude that the late time solution describes an object that looks like the black string far from the AdS horizon (so the metric on the domain wall is close to Schwarzschild) but has a horizon that closes off before reaching the AdS horizon. A similar effect occurs when there is more than one extra dimension transverse to the brane [8]. These conclusions are supported by an exact calculation for a three dimensional RS model [9].

In this paper, we consider black holes charged under gauge fields which are *trapped on the brane*. The flux lines of such gauge fields can pierce the horizon only where it

[*]In this paper, we use the terms "domain wall" and "brane" interchangeably.

actually intersects the brane. The bulk theory is the same as for the uncharged case so one might expect that the black cigar solution would still describe the bulk metric of such a charged brane-world black hole. The effect of the charge might simply be to modify the position of the brane in the bulk spacetime. If this were the case, then we might be able to repeat the analysis of [6] by starting with the black string metric and solving the Israel equations appropriate for the presence of a gauge field on the brane. However, in the Appendix we prove that this is not possible. It is still conceivable that the bulk metric is the same as that of the black *cigar*, but unfortunately the form of the cigar metric is not known. We are therefore forced to study charged brane-world black holes numerically.

A recent paper [10] has claimed to give a solution describing a non-charged black hole in the RS scenario. By using the brane-world Einstein equations derived in [11], it was shown that a Reissner-Nordstrom (RN) geometry could arise on the domain wall provided that the bulk Weyl tensor take a particular form at the wall. We regard this solution as unsatisfactory for two reasons. First, there is no Maxwell field on the domain wall so the black hole cannot be regarded as charged[†]. Secondly, only the induced metric on the domain wall was given – the bulk metric was not discussed. The solution is simply a solution to the Hamiltonian constraint of general relativity and gives appropriate initial data for evolution into the bulk. Until this evolution is performed and boundary conditions in the bulk are imposed, it is not clear what this solution represents. For example, it might give rise to some pathology such as a naked curvature singularity. We would then not regard it as a brane-world black hole, which should have a regular horizon [6,9]. One aim of the present paper is to evolve the initial data of [10] in order to understand what this "solution" really describes.

The second aim of this paper is to study brane-world black holes that are charged with respect to a Maxwell field on the brane. We start by solving the Hamiltonian constraint on the brane to give an induced metric that is close to, but not exactly, Reissner-Nordstrom. The "charge" of [10] arises as an integration constant in the metric. We then evolve this "initial" data away from the domain wall in order to study the resulting bulk spacetime. Our solution to the Hamiltonian constraint is based on a metric ansatz that is almost certainly not obeyed by the true solution describing a charged brane-world black hole. However, we expect our ansatz to be sufficiently close to the true solution that useful results can be obtained without a knowledge of the exact metric, just as in [6].

[†]In the AdS/CFT interpretation [12] of the RS model, this black hole must be charged with respect to a $U(1)$ subgroup of the dual CFT.

Our results suggest that it is more natural to take the "charge squared" parameter of [10] to be negative than positive since the latter gives an apparent horizon that grows relative to the black string as one moves away from the brane. For black holes charged with respect to a Maxwell field, we find that the horizon shrinks in the fifth dimension. In both cases (and for black holes carrying both charges), we obtain a numerical upper bound on the length of the horizon in the fifth dimension. We find that increasing either type of charge tends to decrease this length, even if the horizon radius on the brane is held fixed.

It is worth emphasizing that this paper is quite distinct from recent papers which have appeared on the subject of charged black holes in brane-world scenarios [13–15]. This is because these papers all study the effects of *bulk* charges on the brane-world geometry, whereas our analysis deals with gauge degrees of freedom that are truly localized on the brane. One consistent interpretation of the RN solution of [10] would be as the induced metric on the brane in the (bulk) charged black string solution of [14,15]. However, in this paper we will study whether sense can be made of this solution without introducing bulk gauge fields.

Related numerical work on uncharged brane-world black holes has recently appeared in [16]. The difference between that paper and the present work is that we will prescribe "initial" data on the brane and evolve it in the spacelike direction transverse to the brane, whereas in [16], initial data was prescribed on a spacelike hypersurface and evolved in a timelike direction.

The outline of this paper is as follows. First, we set up the basic notation and formalism for a covariant treatment of the second brane-world scenario of Randall and Sundrum. Next we solve the Hamiltonian constraint for "initial" data on the brane and obtain a RN solution with small corrections. We then numerically evolve the solution into the bulk subject to the constraint that the metric solve the vacuum Einstein equations with a negative cosmological constant. Finally we discuss the properties of the resulting bulk spacetime.

II. FORMULATION AND STRATEGY

A. Covariant formulation of brane-world gravity

We shall be discussing a thin domain wall in a five dimensional bulk spacetime. We shall assume that the spacetime is symmetric under reflections in the wall. The 5-dimensional Einstein equation is

$$^{(5)}R_{\mu\nu} - \frac{1}{2}{}^{(5)}g_{\mu\nu}{}^{(5)}R = \kappa_5^2\,{}^{(5)}T_{\mu\nu}, \tag{2.1}$$

where $\kappa_5^2 = 8\pi G_5$ and G_5 is the five dimensional Newton constant. The energy-momentum tensor has the form

$$^{(5)}T^{\mu\nu} = -\Lambda_5{}^{(5)}g_{\mu\nu} + \delta(\chi)[-\lambda h_{\mu\nu} + T_{\mu\nu}]. \tag{2.2}$$

In the above, the brane is assumed to be located at $\chi = 0$ where χ is a Gaussian normal coordinate to the domain wall. $\chi = 0$ is the fixed point of the Z_2 reflection symmetry. Λ and λ denote the bulk cosmological constant and the domain wall tension respectively. $h_{\mu\nu}$ is the induced metric on the wall, given by $h_{\mu\nu} = {}^{(5)}g_{\mu\nu} - n_\mu n_\nu$ where n_μ is the unit normal to the wall. The effect of the singular source at $\chi = 0$ is described by Israel's junction condition [17]

$$K_{\mu\nu}|_{\chi=0} = -\frac{1}{6}\kappa_5^2 \lambda h_{\mu\nu} - \frac{1}{2}\kappa_5^2\Big(T_{\mu\nu} - \frac{1}{3}h_{\mu\nu}T\Big). \qquad (2.3)$$

Here, $K_{\mu\nu}$ denotes the extrinsic curvature of the domain wall, defined by $K_{\mu\nu} = h_\mu^\rho h_\nu^\sigma \nabla_\rho n_\sigma$. In equation (2.3), we are calculating the extrinsic curvature on the side of the domain wall that the normal point *into*. This is because we want to evolve initial date prescribed on the wall in the direction of this normal. Using the Gauss equation and the junction condition, we recover the Einstein equation on the brane [11]:

$${}^{(4)}G_{\mu\nu} = -\Lambda_4 h_{\mu\nu} + 8\pi G_4 T_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu}, \qquad (2.4)$$

where

$$\Lambda_4 = \frac{1}{2}\kappa_5^2\Big(\Lambda_5 + \frac{1}{6}\kappa_5^2\lambda^2\Big) \qquad (2.5)$$

$$G_4 = \frac{\kappa_5^4 \lambda}{48\pi} \qquad (2.6)$$

$$\pi_{\mu\nu} = \frac{1}{12}TT_{\mu\nu} - \frac{1}{4}T_{\mu\alpha}T_\nu^\alpha + \frac{1}{8}h_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{24}h_{\mu\nu}T^2 \qquad (2.7)$$

and $E_{\mu\nu}$ is the 'electric' part of the 5-dimensional Weyl tensor:

$$E_{\mu\nu} = {}^{(5)}C_{\mu\alpha\nu\beta}n^\alpha n^\beta \qquad (2.8)$$

We shall now specialize to the RS model. This has

$$\Lambda_5 = -\frac{6}{\kappa_5^2 \ell^2}, \qquad \lambda = \frac{6}{\kappa_5^2 \ell}, \qquad (2.9)$$

which implies

$$\Lambda_4 = 0, \qquad G_4 = \frac{G_5}{\ell}. \qquad (2.10)$$

The matter on the domain wall will be assumed to be a Maxwell field. This implies $T = 0$, so we can rewrite the Einstein equation as

$${}^{(4)}R_{\mu\nu} = 8\pi G_4 T_{\mu\nu} - \frac{\kappa_5^4}{4}T_{\mu\rho}T_\nu^\rho - E_{\mu\nu}. \qquad (2.11)$$

The Israel equation gives the extrinsic curvature of the wall:

$$K_{\mu\nu}|_{\chi=0} = -\frac{1}{\ell}h_{\mu\nu} - \frac{\kappa_5^2}{2}T_{\mu\nu}. \qquad (2.12)$$

B. Strategy

We adopt the following procedure: We take a certain charged black hole geometry for the brane. When we solve for the bulk, we Wick rotate twice. This gives a Kaluza-Klein bubble spacetime [21,22] from which we obtain boundary conditions at the condition on the bubble surface. Wick rotating back gives boundary conditions at the bulk horizon for our problem. The Kaluza-Klein bubble spacetime is reviewed in Appendix B.

C. Metric and field equations

We assume that the induced metric on the brane takes the form

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2 d\Omega_2^2, \qquad (2.13)$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Note that this is a *guess.* It is unlikely that the exact metric describing a brane-world black hole would have precisely this form - in general one would expect the coefficients of dt^2 and dr^2 to be independent (when the coefficient of $d\Omega_2^2$ is fixed as r^2). However, we know that the induced metric describing a charged black hole should be close to Reissner-Nordstrom, which *can* be written in this form, so our ansatz is probably quite a good guess. We expect that deviations from the exact metric will give rise to pathologies when this initial data is evolved into the bulk. Even so, the analysis of [6] shows that it is possible to extract quite a lot of information from a pathological solution. The function $U(r)$ will be determined from the Hamiltonian constraint equation below. The bulk metric is assumed to take the form

$$ds^2 = N(\chi, r)^2 d\chi^2 - e^{2a(\chi,r)}U(r)dt^2 + \frac{e^{2b(\chi,r)}dr^2}{U(r)} + e^{2c(\chi,r)}r^2 d\Omega_2^2, \qquad (2.14)$$

N is the lapse function which describes the embedding geometry of the hypersurface spanned by (t, r, θ, φ) during the evolution in the bulk spacetime.

The extrinsic curvature of a hypersurface of constant χ (with unit normal $n = Nd\chi$) is given by

$$K_t^t = \frac{\dot{a}}{N}, \quad K_r^r = \frac{\dot{b}}{N} \quad \text{and} \quad K_\theta^\theta = K_\varphi^\varphi = \frac{\dot{c}}{N}, \qquad (2.15)$$

where a dot denotes ∂_χ. The spacetime is described by the evolution equation,

$$\dot{K}_\nu^\mu = N\Big({}^{(4)}R_\nu^\mu - KK_\nu^\mu + \frac{4}{\ell^2}\delta_\nu^\mu\Big) - D^\mu D_\nu N, \qquad (2.16)$$

the Hamiltonian constraint equation,

$${}^{(4)}R - K^2 + K_{\mu\nu}K^{\mu\nu} = -\frac{12}{\ell^2}, \qquad (2.17)$$

and the momentum constraint equation,

$$D_\mu K^\mu_\nu - D_\nu K = 0. \tag{2.18}$$

Here ${}^{(4)}R^\mu_\nu$ and ${}^{(4)}R$ are the Ricci tensor and Ricci scalar on hypersurfaces of constant χ.

III. BRANE AND BULK GEOMETRY

A. Brane Geometry : Charged black hole "initial data"

The action for the Maxwell field on the brane is taken to be

$$S = -\frac{1}{16\pi G_4}\int d^4x\sqrt{-h}F_{\mu\nu}F^{\mu\nu}, \tag{3.1}$$

giving energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi G_4}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}h_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right). \tag{3.2}$$

The field strength F is related to a potential A by $F = dA$. The equations of motion are satisfied if we take $A = -\Phi(r)dt$ with $\Phi(r) = Q/r$. This gives

$$T_{\mu\nu} = \frac{1}{8\pi G_4}\frac{Q^2}{r^4}\mathrm{diag}\left(\mathrm{U}, -\mathrm{U}^{-1}, \mathrm{r}^2, \mathrm{r}^2\sin^2\theta\right). \tag{3.3}$$

This can be substituted into the right hand side of the Israel equation (2.12) to give an expression for the extrinsic curvature. This can then be substituted into the Hamiltonian equation (2.17), along with our metric ansatz to give an equation for $U(r)$. Solving this equation gives[‡]

$$U(r) = 1 - \frac{2G_4M}{r} + \frac{\beta + Q^2}{r^2} + \frac{l^2Q^4}{20r^6}, \tag{3.4}$$

where M and β are arbitrary constants of integration. Substituting into the Einstein equation on the domain wall gives

$$-E_{\mu\nu} = \left(\frac{\beta}{r^4} + \frac{l^2Q^4}{2r^8}\right)\mathrm{diag}\left(\mathrm{U}, -\mathrm{U}^{-1}, \mathrm{r}^2, \mathrm{r}^2\sin^2\theta\right). \tag{3.5}$$

It is interesting to compare $-E_{\mu\nu}$ with $8\pi G_4T_{\mu\nu}$ since these quantities appear on an equal footing in the effective Einstein equation (2.4). It is clear that the constant of integration β is in some sense analogous to Q^2, which is why the authors of [10] obtained a RN solution. However, since their solution did not have a Maxwell field, it cannot really be regarded as a charged black hole in the usual sense. Rather it carries "tidal" charge associated with the bulk Weyl tensor. β might be regarded as a *five* dimensional mass parameter.

We shall only consider initial data that corresponds to an object with an event horizon (in the four dimensional sense) on the domain wall. In some cases there may be more than one horizon. We shall use r_+ to denote the position of the outermost horizon, i.e., the largest solution of $U(r) = 0$. This has to be found numerically except when $Q = 0$.

Our "initial data" is given by

$$\left.\frac{\dot{a}}{N}\right|_{\chi=0} = \left.\frac{\dot{b}}{N}\right|_{\chi=0} = -\frac{1}{\ell} + \frac{\ell}{2}\frac{Q^2}{r^4}, \tag{3.6}$$

$$\left.\frac{\dot{c}}{N}\right|_{\chi=0} = -\frac{1}{\ell} - \frac{\ell}{2}\frac{Q^2}{r^4}, \tag{3.7}$$

$$a(\chi = 0, r) = b(0, r) = c(0, r) = 0. \tag{3.8}$$

We shall study the following cases:
(i) No electromagnetic charge, i.e., $Q = 0$. In this case, the induced metric on the domain wall is exactly RN [10]. The horizon radius is

$$r_+ = M + \sqrt{M^2 - \beta}. \tag{3.9}$$

The induced metric has a regular horizon if $\beta \leq M^2$. Note that there is nothing to stop us choosing β to be *negative*, which emphasizes the difference between the solution of [10] and a charged black hole. If we take β to be negative then the induced metric has only one horizon, instead of the two horizons of a non-extreme RN black hole.
(ii) No tidal charge, i.e., $\beta = 0$. In this case, the induced metric on the domain wall is Reissner-Nordstrom with a correction term. Note that $-E_{\mu\nu}$ is non-zero but is of order $1/r^8$, which suggests that the total "tidal energy" on the wall is zero.

We shall also consider the general case **(iii) Both charges non-zero**, i.e., $\beta \neq 0$, $Q \neq 0$.

B. Bulk Geometry

The bulk geometry is obtained by integrating (2.15) and (2.16) in the χ-direction numerically. We use the

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[‡]It is interesting to compare this form for $U(r)$ with the behaviour expected from the linear perturbation analysis of the second RS model [4,18–20]. In linearized theory, $U(r) = 1 - \phi(r)$ where $\phi(r)$ is the Newtonian potential. For $r \gg \ell$, the leading order corrections to $\phi(r)$ are expected to be proportional to $G_4M\ell^2/r^3$ and ℓ^2Q^2/r^4. Such terms are not present in our expression for $U(r)$. However, we shall be interested in black holes for which $G_4M \gg \ell$, so these correction terms will be small compared with terms like $(G_4M/r)^3$ and $(G_4MQ/r^2)^2$, which would be neglected in linearized theory. In other words, the RS correction are dominated by post-Newtonian corrections [20] so it is not appropriate to compare $U(r)$ with the linearized results beyond leading order.

standard 'free-evolution' method, that is we do not solve the constraint equations (2.17) and (2.18) during the evolution, but instead use them to monitor the accuracy of the simulation.

We obtain the solution numerically in the region $r_+ < r < r_e$ with $r_e \sim 5r_+$. Boundary conditions at $r = r_+$ are specified by first Wick rotating $\chi = iT$, $t = i\tau$, which takes the metric to a Kaluza-Klein bubble metric (see Appendix B). Therefore we can apply the numerical techniques that are used in the study of Kaluza-Klein bubbles [24], although the physics of Kaluza-Klein bubbles is unrelated to the physics of black holes. It was shown in the Appendix of [24] that at the inner boundary $r = r_+$, a and b evolve synchronously, that is, $a(T, r_+) = b(T, r_+)$. Analytically continuing back to our original spacetime yields the boundary condition $a(\chi, r_+) = b(\chi, r_+)$. The evolution equation for the trace of $K_{\mu\nu}$ and the momentum constraint are also used at $r = r_+$. At the outer boundary $r = r_e$, we assume the components of the extrinsic curvature [equation (2.15)] fall off like $-1/\ell + \mathcal{O}\left(r^{-4}\right)$ [cf. (3.7)]. We apply the geodesic gauge condition (slicing condition), $N = 1$.

We use the Crank-Nicholson integrating scheme with two iterations [25]. The numerical code passed convergence tests, and the results shown in this paper are all obtained to acceptable accuracy.

We were only able to solve numerically in a region near the domain wall with a maximum value for χ of $\mathcal{O}(1)$. This is because the volume element of surfaces of constant χ decreases exponentially as one moves away from the wall, just as in pure AdS. The evolution was stopped when $\sqrt{-g}$ became too small to monitor accurately.

We are interested in how charge affects the shape of the horizon, in particular how far it extends into the fifth dimension. This will be measured by the ratio of the physical size of the apparent horizon $r_+ e^{c(\chi, r_+)}$, to that of a black string [6] with the same horizon radius r_+ on the wall[§]. The size of the black string apparent horizon in the bulk is $r_+ e^{-\chi/\ell}$, so the ratio is

$$R(\chi) = e^{c(\chi, r_+) + \chi/\ell}. \tag{3.10}$$

We remark that the only apparent horizon that appears during the χ-evolution is at $r = r_+$. Here we define apparent horizon as the outermost region of negative expansion of the outgoing null geodesic congruences, where we define the expansion rate, θ_+, as

$$\theta_+ = {}^{(3)}\nabla_a s^a + {}^{(3)}K - s^a s^b \, {}^{(3)}K_{ab}, \tag{3.11}$$

where $s^a = (1/\sqrt{g_{rr}})\partial_r$ is an outwards pointing unit vector in the 3-dimensional metric. We checked (3.11) during the evolution and confirmed its positivity for $r > r_+$.

Our initial conditions give the behaviour of the ratio $R(\chi)$ near the brane:

$$\dot{R}(\chi)|_{\chi=0} = -\frac{\ell}{2}\frac{Q^2}{r_+^4} \leq 0, \tag{3.12}$$

and

$$\ddot{R}(\chi)|_{\chi=0} = \frac{3Q^2 + \beta}{r_+^4} - \frac{\ell^2 Q^4}{2r_+^8}. \tag{3.13}$$

For model (i) ($Q = 0$), $\dot{R}(\chi)|_{\chi=0} = 0$, but $\ddot{R}(\chi)|_{\chi=0} = \beta/r_+^4$. This gives $\ddot{R}(\chi)|_{\chi=0} < 0$ for the case with $\beta < 0$, which indicates that the ratio decreases, while $\ddot{R}(\chi)|_{\chi=0} > 0$ for the case with $\beta > 0$, which indicates that the ratio increases. We have plotted the numerical results for this ratio in Fig.1 (a) and (b) (henceforth we shall set $\ell = G_4 = 1$ and assume $M \gg 1$, as appropriate for an astrophysical black hole.). Fig.1 (a) and (b) suggests that a negative value for β is the natural choice since the apparent horizon grows (relative to the black string) in the fifth dimension when β is positive.

For model (ii) ($\beta = 0$), $\dot{R}(\chi)|_{\chi=0} < 0$ and the ratio always decreases [see Fig.1 (c)]. Model (iii) ($Q \neq 0$ and $\beta \neq 0$) is non-trivial. We present numerical results in Fig.2. The plot is for $M = 5$, $Q = 3$ and $\beta = 0, \pm 5, \pm 10, \pm 15$, where $\beta = 15$ is close to the extreme[**] case for this choice of Q. The qualitative features are combinations of the plots in Fig.1. Note that β seems to have the greatest effect on the bulk evolution. Again, the case with negative β appears to be the natural choice since positive β gives a growing horizon.

C. Bulk Geometry: extent of the horizon

In this section, we shall estimate how far the horizon extends into the fifth dimension by combining analytical and numerical work. Following the conjugate points theorem [26], we shall show that for a charged black hole, the trace of the extrinsic curvature diverges at a finite distance from the brane. The trace of the evolutional equation is given by

$$\dot{K} = {}^{(4)}R - K^2 + \frac{16}{\ell^2} = -K_{\mu\nu}K^{\mu\nu} + \frac{4}{\ell^2}, \tag{3.14}$$

where we used the Hamiltonian constraint in the second line. Now define $k_{\mu\nu}$ as

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[§]The reason for measuring the size of the horizon relative to that of the black string is because we want to distinguish the closing-off of the horizon from the exponential collapse of hypersurfaces of constant χ arising from the AdS nature of the geometry.

[**]By extreme, we mean that $U(r)$ has a double root at $r = r_+$.

$$K^\mu_\nu =: -\frac{1}{\ell}h^\mu_\nu + k^\mu_\nu. \tag{3.15}$$

The trace part of $k_{\mu\nu}$, $k = k^\mu_\mu$, is expected to measure the volume expansion relative to the AdS "background" geometry. In term of $k_{\mu\nu}$, (3.14) can be written as

$$\dot{k} - \frac{2}{\ell}k + \frac{1}{4}k^2 = -\tilde{k}_{\mu\nu}\tilde{k}^{\mu\nu} \leq 0, \tag{3.16}$$

where $\tilde{k}_{\mu\nu}$ is the traceless part of $k_{\mu\nu}$. On the brane the "initial" condition is

$$k_{\mu\nu}|_{\rm brane} = \tilde{k}_{\mu\nu}|_{\rm brane} = -4\pi G_5 T_{\mu\nu}, \tag{3.17}$$

which implies

$$k|_{\rm brane} = 0 \tag{3.18}$$

For the case with $Q \neq 0$,

$$\tilde{k}_{\mu\nu}\tilde{k}^{\mu\nu} > 0, \tag{3.19}$$

so

$$\dot{k}|_{\rm brane} < 0 \tag{3.20}$$

This implies that there is a $\chi = \chi_0$ such that

$$k = k_0 < 0 \tag{3.21}$$

From (3.16), one obtains

$$1 + \frac{8}{\ell|k|} \leq \left(1 + \frac{8}{\ell|k_0|}\right) e^{2(\chi_0 - \chi)/\ell}, \tag{3.22}$$

from which it follows that k diverges before $\chi = \chi_{crit}$, where

$$\chi_{crit} = \chi_0 + \frac{\ell}{2}\log\Bigl(1 + \frac{8}{\ell|k_0|}\Bigr). \tag{3.23}$$

The divergence in k implies that K also diverges. Near $\chi = \chi_{crit}$, $|k|$ behaves like

$$k \leq -\frac{4}{\chi_{crit} - \chi}. \tag{3.24}$$

The case with $Q = 0$ is more difficult to analyze because $\dot{k}|_{\rm brane} = 0$. We can use equation (2.15) (with $N = 1$) to give

$$k|_{r=r_+} = 2\partial_\chi \left(a + c + \frac{2\chi}{\ell}\right), \tag{3.25}$$

where we have used the synchronous evolution boundary condition $a = b$ at $r = r_+$. In Fig.3, we have plotted $a + c + 2\chi/\ell$ at $r = r_+$. It is clear from this plot that k becomes negative in the bulk when $\beta < 0$. In fact k also becomes negative when $\beta > 0$. Thus, even in the $Q = 0$ case, there exists a $\chi = \chi_0$ such that $k = k_0 < 0$. The above argument can then be used to show that when $Q = 0$ and $\beta \neq 0$, K must diverge before $\chi = \chi_{crit}$, where χ_{crit} is given by equation (3.23). We have therefore proved that if $Q \neq 0$ or $\beta \neq 0$ then K diverges before $\chi = \chi_{crit}$.

It follows from equations (3.24) and (3.25) that

$$(a + c)|_{r=r_+} \leq 2\log\left(\chi_{crit} - \chi\right), \tag{3.26}$$

which implies that $\sqrt{-g}$ tends to zero at least as fast as $(\chi_{crit} - \chi)^4$ as $\chi \to \chi_{crit}$.

Conservatively, the divergence of K indicates that the geodesic slicing has broken down (when $N = 1$, ∂_χ is the tangent vector of spacelike geodesics), in other words a caustic has occurred. The numerical study therefore cannot be extended further using this slicing. This has, however, a physical meaning because the apparent horizon is located at constant $r = r_+$ in the bulk. The horizon will encounter the caustic before reaching the AdS Cauchy horizon. The caustic can therefore be viewed as the endpoint of the horizon, i.e., the tip of the black cigar. Our analysis has only shown that the geodesic slicing must break down at the caustic so, in principle, this point may be regular, i.e., there may exist a coordinate chart that covers a neighbourhood of this point††. However, we do not regard this as very likely. Our guess for the induced metric on the domain wall is unlikely to be exactly correct, so in general we would expect some pathology such as a naked curvature singularity to appear in the bulk. We cannot check whether curvature invariants diverge at $\chi = \chi_{crit}$ since our numerical evolution cannot be extended as far as $\chi = \chi_{crit}$.

Whether the bulk solution is regular or not, equation (3.23) gives us an upper bound on the extent of the horizon in the direction transverse to the domain wall, i.e., the length of the black cigar. We have plotted this upper bound in Fig.4 taking the values for χ_0 and k_0 at the endpoint of our numerical evolution. The first graph shows how χ_{crit} depends on Q and β when M is fixed. Note that when $Q = \beta = 0$, the numerical solution is simply the black string‡‡, which has $\chi_{crit} = \infty$. Increasing Q clearly has the effect of decreasing χ_{crit}, which is not surprising since increasing Q also shrinks the horizon radius on the domain wall r_+. Perhaps more surprising is that making β more negative also appears to decrease χ_{crit} even though this *increases* the horizon radius on the wall [see equation (3.9)]. The solid curve on this diagram has both M and r_+ fixed. It is clear that χ_{crit} decreases along this curve as Q or β increases.

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††It is not even clear from our analysis whether the caustic occurs at a single point or is spread over a region of spacetime.

‡‡The reader may find this surprising since the black string is unstable [6], and small numerical errors might be expected to act as perturbations. However, the string is unstable to *long* wavelength perturbations, and the numerical errors will only be relevant at short wavelengths.

The second graph of Fig.4 plots the same curve (fixed M and fixed r_+) for different values of M. The trend seems to be the same in each case.

The final graph of Fig.4 is for fixed r_+ (rather than fixed M). Increasing β appears to decrease χ_{crit} when Q is small but has no significant effect when Q is large. When β is non-zero, increasing Q has the effect of initially slightly increasing χ_{crit}, but ultimately decreases it substantially. The gross trend appears to be that increasing either type of charge leads to a decrease in the length of the horizon.

In most of these graphs, $\chi_{crit} < r_+$, so the extent of the horizon in the fifth dimension is smaller than the horizon radius on the domain wall, just as for the uncharged black cigar.

IV. SUMMARY AND DISCUSSION

In this paper we have studied charged black holes in the second RS model. We have seen that two types of charge can arise on the brane, one coming from the bulk Weyl tensor [10] and one from a Maxwell field *trapped on the brane*. Starting from an ansatz for the induced metric on the brane, we have solved the constraint equations of 4+1 dimensional gravity to find metrics describing charged brane-world black holes. In the absence of Maxwell charge, one can obtain a Reissner-Nordstrom solution [10]. If Maxwell charge is included then one can obtain a geometry that is Reissner-Nordstrom with small corrections.

Using these induced metrics as "initial" data, we have solved the bulk field equations numerically. We have found that the RN solution of [10] has an apparent horizon that grows (relative to the black string apparent horizon) in the dimension transverse to the brane unless the "charge squared" parameter β is taken to be negative[§§]. It therefore seems unlikely that this solution really corresponds to a charged brane-world black hole. Of course, if a bulk gauge field is included then the work of [10] (with $\beta > 0$) has a natural interpretation as the induced metric on the brane arising from the charged black string solution of [14,15].

If $\beta < 0$ and/or $Q \neq 0$ then the horizon shrinks relative to the black string horizon. For all cases (including $\beta > 0$), we have found that the trace of the extrinsic curvature diverges at a finite distance from the brane, with the volume element of the spacetime tending to zero. For $\beta \leq 0$, we have interpreted this as the end point of the horizon of the black hole. Our results suggest that increasing the charges of a brane-world black hole will decrease the length of its horizon in the fifth dimension, even when the horizon radius on the brane is kept fixed. This implies that, by adjusting Q, one can change the five dimensional horizon area while keeping the four dimensional horizon area fixed. One might think that this would lead to a difference between the entropies calculated from these horizon areas, which would be bad news for hopes of recovering General Relativity as the effective four dimensional theory of gravity on the brane. However, the exponential decrease in the volume element as one moves away from the brane implies that the dominant contribution to the five dimensional area comes from the region of the horizon that is closest to the brane [9]. Changes near the other end of the horizon give only subleading corrections to the five dimensional area, allowing the four and five dimensional entropies to agree at leading order.

We suspect that our solutions will generically have a curvature singularity at the point where the trace of the extrinsic curvature diverges. This is because it seems rather improbable that our ansatz for the induced metric on the brane should turn out to be exactly right. However, we expect that for each value of Q there will be some value of β for which a small change in our initial data would smooth out this singularity, leading to a regular geometry describing a brane-world black hole carrying Maxwell charge Q. This smoothing would probably not significantly affect the position of the "tip" of the horizon, for which we have obtained an upper bound on the distance from the brane. This is to be contrasted with the uncharged case in which one takes the induced metric on the brane to be Schwarzschild. Evolving this into the bulk gives the black string metric, for which the singularity occurs at the AdS horizon, which is at *infinite* proper distance from the brane along spacelike geodesics. A small perturbation of the metric on the brane takes one from the black string to the black cigar, which has a regular AdS horizon and a black hole horizon with a tip at finite distance from the brane.

For the black string, the stability analysis of [6] shows that the horizon extends a distance of order $d = \ell \log(G_4 M/\ell)$ into the fifth dimension, so $d \ll r_+$. Our results give only an upper bound for d in the charged case. It would be nice if the stability analysis could be extended to the charged case. However, the instability only sets in when the proper radius of the horizon becomes smaller than the anti-de Sitter length scale and we were not able to extend our numerical evolution this far. Our upper bound seems rather on the large side, since it appears to give $d \sim r_+$ for small Q and β. However, for large Q and/or β, figure 4(c) shows that $d \ll r_+$, so our upper bound is probably tighter in this case.

The main outstanding problem remains to find the exact bulk metric that describes a brane-world black hole. This was solved for uncharged black holes in the 3 di-

[§§]In [24], the evolution of Kaluza-Klein bubbles was studied numerically and it was found that even though negative mass bubbles start off with accelerating expansion [23], the acceleration ultimately becomes negative. It is conceivable that something analogous could happen here but we have found no evidence for such behaviour.

mensional RS model by using the 4 dimensional AdS C-metric in the bulk [9]. Unfortunately, the higher dimensional generalization of this metric is not known. It would be interesting to see whether charged black holes in the 3 dimensional RS model could be constructed by using the same bulk as in [9] but simply slicing along a different hypersurface. It would also be interesting to use the methods of [18–20] to find linearized solutions describing spherical distributions of matter charged with respect to a brane-world gauge field.

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APPENDIX A: BRANE BENDING AND THE BLACK STRING

One candidate for a black hole formed by gravitational collapse of charged brane-world matter on a domain wall in AdS is the black string solution in AdS, which has the metric

$$ds^2 = \frac{\ell^2}{z^2}(-U(r)dt^2 + U(r)^{-1}dr^2 + r^2 d\Omega_2^2 + dz^2) \quad \text{(A1)}$$

where $U(r) = 1 - 2G_4M/r$. As discussed in [6], surfaces of constant z trivially satisfy the Israel matching conditions provided that the tension satisfies $\lambda = \pm 6/{\kappa_5}^2\ell$. Thus, we may slice the spacetime along such a surface of constant z, and match to a mirror image, in order to obtain the Schwarzschild solution on the domain wall.

We now want to consider what happens when we allow the black hole to be electrically charged with respect to some $U(1)$ gauge field living on the brane. Thus, we must add in an extra term to the brane-world stress energy tensor of the form

$$T_{\mu\nu} = \frac{1}{4\pi G_4}(F_{\mu\rho}F_\nu^\rho - \frac{1}{4}q_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \quad \text{(A2)}$$

where the electric gauge potential has the form

$$A = -\Phi(r)dt \quad \text{(A3)}$$

so that

$$F = \Phi'(r)dt \wedge dr \quad \text{(A4)}$$

where $'$ denotes differentiation with respect to r.

Now, as a first guess we might try to support this stress-energy on the brane by allowing the brane to bend in the black string background in such a way that the extrinsic curvatures would still satisfy the Israel equations.

In other words, we allow the position z of the brane to depend on the radial direction r. Solving the Maxwell equations yields

$$\Phi'(r) = -\frac{Q}{r^2}\left(1 + z'^2 U\right)^{1/2} \quad \text{(A5)}$$

To compute the extrinsic curvature of the timelike hypersurface swept out by $z = z(r)$, we introduce an orthonormal basis which consists of a unit normal vector

$$n = \frac{\epsilon\ell}{z\sqrt{1+Uz'^2}}\left(dz - z'dr\right), \quad \text{(A6)}$$

where $\epsilon = \pm 1$, a unit timelike tangent

$$u = \frac{z}{\ell}U^{-1/2}\frac{\partial}{\partial t}, \quad \text{(A7)}$$

and the spacelike tangents

$$t = \frac{z}{\ell}\sqrt{\frac{U}{1+Uz'^2}}\left(z'\frac{\partial}{\partial z} + \frac{\partial}{\partial r}\right), \quad \text{(A8)}$$

$$e_\phi = \frac{z}{\ell r\sin\theta}\frac{\partial}{\partial\phi}, \quad \text{(A9)}$$

$$e_\theta = \frac{z}{\ell r}\frac{\partial}{\partial\theta} \quad \text{(A10)}$$

It follows that the non-vanishing components of the extrinsic curvature in this basis are

$$K_{uu} = \frac{\epsilon}{\ell\sqrt{1+Uz'^2}}(1 + \frac{1}{2}U'zz'), \quad \text{(A11)}$$

$$K_{\theta\theta} = K_{\phi\phi} = \frac{-\epsilon}{\ell\sqrt{1+Uz'^2}}\left(1 + \frac{U}{r}zz'\right), \quad \text{(A12)}$$

$$K_{tt} = -\frac{\epsilon U}{\ell\left(1+Uz'^2\right)^{3/2}}\left(zz'' + z'^2 + U^{-1} + \frac{U'zz'}{2U}\right). \quad \text{(A13)}$$

Under the assumption of Z_2 symmetry, the Israel equations reduce to (2.12). The three independent components of $K_{\mu\nu}$ give three independent equations:

$$K_{tt} = \frac{1}{\ell} - \frac{z^4Q^2}{2\ell^3 r^4}$$
$$K_{uu} = -\frac{1}{\ell} + \frac{z^4Q^2}{2\ell^3 r^4} \qquad \text{(A14)}$$
$$K_{\theta\theta} = \frac{1}{\ell} + \frac{z^4Q^2}{2\ell^3 r^4}.$$

It is straightforward to show that it is impossible to solve these three equations simultaneously unless one takes $Q = 0$ and z =constant, which is the uncharged solution of [6]. It is therefore not possible to support the stress-energy of a point charge by simply allowing the brane to bend in the black string background. It follows that the *bulk* has to change once the brane-world charge is introduced. In other words, brane-world charge will induce changes in the bulk Weyl tensor, and this is exactly what we have found in our numerical analysis.

APPENDIX B: KALUZA-KLEIN BUBBLE

The double Wick rotation($\chi \to it, t \to i\tau$) of the metric of Eq. (2.13) gives us the Euclidean induced metric:

$$ds^2 = U(r)d\tau^2 + \frac{dr^2}{U(r)} + r^2 d\Omega_2^2. \qquad \text{(B1)}$$

The largest $r = r_+$ such that $U(r_+) = 0$ is interpreted as the position of the bubble surface. Around $r = r_+$, the metric can be expanded

$$ds^2 \simeq U'(r_+)(r - r_+)d\tau^2 + \frac{dr^2}{U'(r_+)(r - r_+)} + r_+^2 d\Omega_2^2. \quad \text{(B2)}$$

In term of the new coordinate $R := \sqrt{r - r_+}$,

$$ds^2 \simeq \frac{4}{U'(r_+)}\Big[R^2 d\Big(\frac{U'(r_+)\tau}{2}\Big)^2 + dR^2\Big] + r_+^2 d\Omega_2^2. \quad \text{(B3)}$$

We can see easily that the metric will be regular if we assume that the τ direction is periodic with period $4\pi/U'(r_+)$.

In the case of $U(r) = 1 - r_0^2/r^2$ with $\lambda = \Lambda = 0$, the exact five dimensional solution for time-symmetric initial data ($K_{\mu\nu} = 0$) is

$$ds_5^2 = -r^2dt^2 + U(r)d\tau^2 + U^{-1}(r)dr^2 + r^2\cosh^2 t\, d\Omega_2^2. \quad \text{(B4)}$$

This is the Witten-bubble spacetime [21]. Another example of initial data for a Kaluza-Klein bubble spacetime was given in Ref. [22] and its classical time evolution has been investigated in Ref. [23,24].

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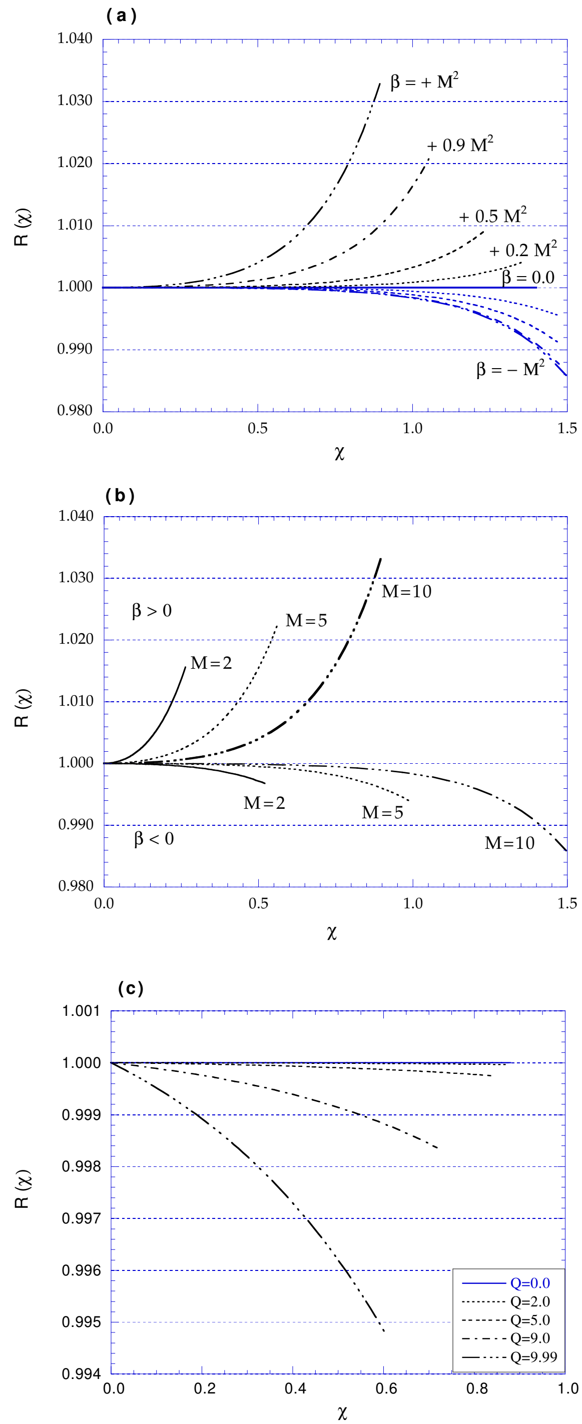

FIG. 1. Ratio of the physical size of apparent horizon to size of black string apparent horizon, $R(\chi)$ [cf. eq. (3.10)], plotted as a function of χ. Fig.(a) is for model (i). Lines are of $\beta = 0, \pm 0.2M^2, \pm 0.5M^2, \pm 0.9M^2$ and $\pm M^2$, where $M = 10.0$. We see that the qualitative behaviour of $R(\chi)$ depends on the sign of β. Fig.(b) is for the extremal case of model (i), $\beta = M^2$ with different values of M. Results are also plotted for $\beta = -M^2$. Fig.(c) is for model (ii) for which $R(\chi)$ is monotonically decreasing.

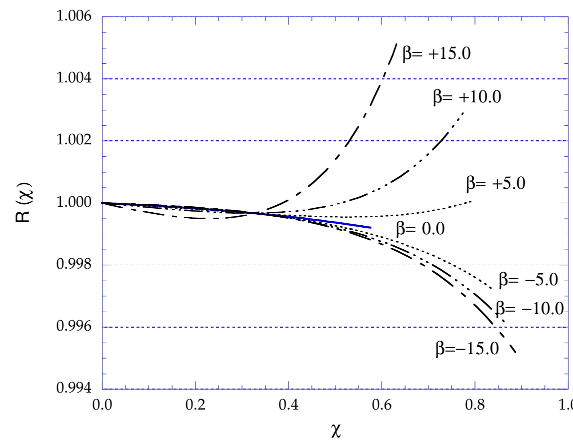

FIG. 2. Ratio of physical size of apparent horizon to size of black string apparent horizon, $R(\chi)$ [cf. eq. (3.10)], for nonzero Q and β. We have set $M = 5$, $Q = 3$ and $\beta = 0, \pm 5, \pm 10, \pm 15$ for this plot. The main features are a combination of plots in Fig.1.

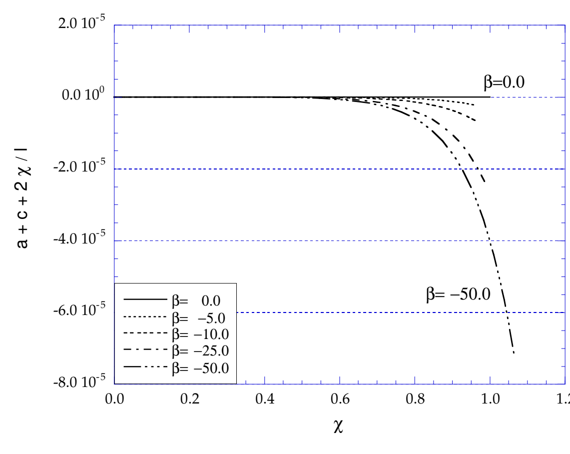

FIG. 3. The quantity $a + c + 2\chi/\ell$ at $r = r_+$ plotted for $M = 5, Q = 0$ and $\beta = 0, -5, -10, -25$ and -50.

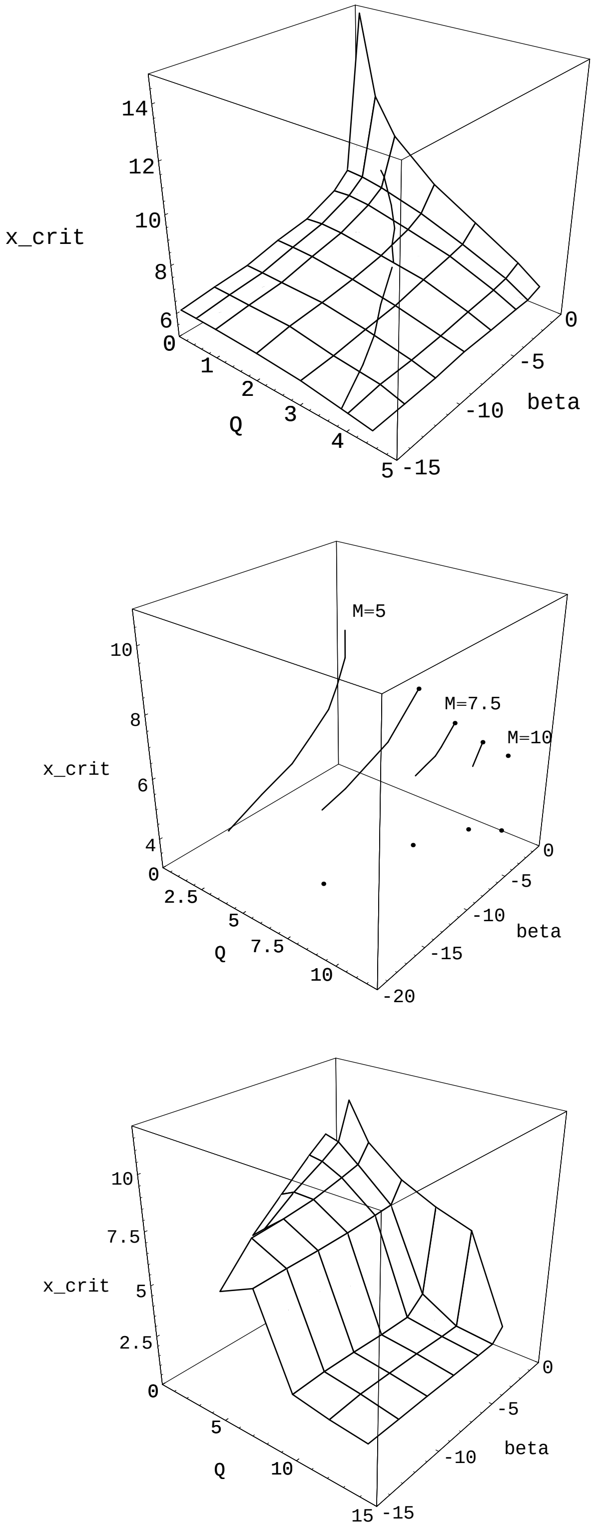

FIG. 4. Critical value χ_{crit} [eq.(3.23)] plotted for non-zero Q and $\beta (\leq 0)$ black holes. (a) [above left] We have set $M = 5$ for this plot. Note that for the uncharged case, $\chi_{crit} = \infty$. The solid curve is for the special cases with $r_+ = 10.0$. (b) [above right] Critical value χ_{crit} for combinations of parameters (Q, β) which produce a black hole with $r_+ = 10.0$. $M = 5.0, 6.25, 7.5, 8.75$ and 10.0 are chosen for these plots. The black dots denote the ends of the lines at $\beta = 0$ and the other ends projected onto the $\chi_{crit} = 3$ plane. (c) [below] The same plot as for (b), but M is specified so as to fix $r_+ = 10$ for given (Q, β).