Hyperbolic and asymptotically constrained systems in the Ashtekar formulation of general relativity *

Gen Yoneda† and Hisa-aki Shinkai‡

† *Dept. of Mathematical Science, Waseda University, Shinjuku, Tokyo 169-8555, Japan*
‡ *Center for Gravitational Physics and Geometry, 104 Davey Lab., Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802-6300, USA*

(December 22, 1999)

Abstract

Hyperbolic formulations of the equations of motion are essential technique for proving the well-posedness of the Cauchy problem of a system, and are also helpful for implementing stable long time evolution in numerical applications. We, here, present three kinds of hyperbolic systems in the Ashtekar formulation of general relativity for Lorentzian vacuum spacetime. We exhibit several (I) weakly hyperbolic, (II) diagonalizable hyperbolic, and (III) symmetric hyperbolic systems, with each their eigenvalues, and discuss how gauge conditions and reality conditions are constrained toward constructing a symmetric hyperbolic system. As an application of this formulation, we present a set of dynamical equations which forces the space-time to evolve to the manifold that satisfies the constraint equations or the reality conditions or both as the attractor against perturbative errors. The obtained systems may be useful for future numerical studies using Ashtekar's variables. Several current works are also described.

1 Introduction

1.1 Hyperbolic formulations in General Relativity

Developing hyperbolic formulations of the Einstein equation is growing into an important research areas in general relativity (see e.g. [1]). These formulations are used in the analytic proof of the existence, uniqueness and stability (well-posedness) of the solutions of the Einstein equation [2]. So far, several first order hyperbolic formulations have been proposed; some of them are flux conservative [3, 4], some of them are symmetrizable or symmetric hyperbolic systems [5, 6, 7, 8]. The recent interest in hyperbolic formulations arises from their application to numerical relativity. One of the most useful features is the existence of characteristic speeds in hyperbolic systems. We expect more stable evolutions and expect improved boundary conditions in their numerical simulation. Some numerical tests have been reported along this direction [9, 10]. It might be worth remarking that the standard Arnowitt-Deser-Misner (ADM) formulation cannot be a first-order hyperbolic form, since there are curvature terms in the r.h.s. of dynamical equations.

For the reader's convenience, we provide our definitions of hyperbolic systems here.

Definition 1 *We assume a certain set of (complex) variables u_α $(\alpha = 1, \cdots, n)$ forms a first-order (quasi-linear) partial differential equation system,*

$$\partial_t u_\alpha = J^{l\beta}{}_\alpha(u)\,\partial_l u_\beta + K_\alpha(u), \tag{1}$$

where J (the characteristic matrix) and K are functions of u but do not include any derivatives of u. We say that the system (1) is:

(I). **weakly hyperbolic**, *if all the eigenvalues of the characteristic matrix are real.*

(II). **diagonalizable hyperbolic**, *if the characteristic matrix is diagonalizable and has all real eigenvalues.*

---

*Proceedings of *The 9th Workshop on General Relativity and Gravitation* at Hiroshima, Japan, October 1999.

†Email: yoneda@mn.waseda.ac.jp
‡Email: shinkai@gravity.phys.psu.edu

(III). **symmetric hyperbolic**, *if the characteristic matrix is a Hermitian matrix.*

We treat $J^{l\beta}{}_{\alpha}$ as a $n \times n$ matrix when the l-index is fixed. We say λ^l is an eigenvalue of $J^{l\beta}{}_{\alpha}$ when the characteristic equation, $\det(J^{l\beta}{}_{\alpha} - \lambda^l \delta^{\beta}{}_{\alpha}) = 0$, is satisfied. The eigenvectors, p^{α}, are given by solving $J^{l\alpha}{}_{\beta} p^{l\beta} = \lambda^l p^{l\alpha}$.

These three classes have the relation (III) $\in$ (II) $\in$ (I). The symmetric hyperbolic system gives us the energy integral inequalities, which are the primary tools for studying well-posedness of the system. As was discussed by Geroch [11], most physical systems can be expressed as symmetric hyperbolic systems. In order to define the symmetric hyperbolic system, (III), we need to declare an inner product $\langle u|u\rangle$ to judge whether $J^{l\beta}{}_{\alpha}$ is Hermitian. In other words, we are required to define the way of lowering the index α of u^{α}. We say $J^{l\beta}{}_{\alpha}$ is Hermitian with respect to this index rule, when $J^{l}{}_{\beta\alpha} = \bar{J}^{l}{}_{\alpha\beta}$ for every l, where the overhead bar denotes complex conjugate.

For more concrete descriptions for each systems, please refer our paper [12].

1.2 Ashtekar's formulations of General Relativity

Ashtekar's formulation of general relativity [13] has many advantages. By using his special pair of variables, the constraint equations which appear in the theory become low-order polynomials, and the theory has the correct form for gauge theoretical interpretation. These features suggest the possibility for developing a nonperturbative quantum description of gravity. The detail equations are reviewed in Sec.2.

From the point of hyperbolicity of the system, Ashtekar's original set of equations of motion already forms a weakly hyperbolic system, as we will show in §3. A symmetric hyperbolic formulation of Ashtekar's variables was first developed by Iriondo, Leguizamón and Reula (ILR) [14]. Unfortunately, in their first short paper [14], they did not discuss the consistency of their system with the reality conditions, which are crucial in the study of the Lorentzian dynamics using the Ashtekar variables. We considered this point in [15], and found that there are strict reality constraints (alternatively they can be interpreted as gauge conditions). There are also another debates between ILR's system and ours: on the definition of the symmetric hyperbolic systems and on the treatment of the triad lapse freedom. These are described in [16] and [12].

As can been seen above, there is always a problem of reality conditions in applying Ashtekar formulation to classical dynamics. Fortunately, it was shown that the secondary condition of the reality of the metric will be automatically preserved during the evolution, if the initial data satisfies both primary and secondary metric reality conditions [17]. If we impose the reality condition on the triad (triad reality condition), then we have additional conditions that can be controlled by a part of a gauge variable, triad lapse $\mathcal{A}^a_0$ (defined later) [18]. Therefore the reality conditions are controllable, and we think that applying the Ashtekar formulation to dynamics is quite attractive, and broadens our possibilities to attack dynamical issues. It is, however, also the case that preliminary numerical simulations of the spacetime using Ashtekar's variables show that the system will not normally recover real-valued spacetime if we relax the metric reality condition locally during the evolution [19]. Therefore, we desire a system that is robust for controlling both the reality conditions and the constraint equations for stable long-term integration. We present a new dynamical system to solve these problems in §4.

1.3 Short description of this report

In this *report*, we summarize our recent works [15, 12, 20], and comments on our current directions.

The first main results obtained are the several levels of hyperbolic formulations of Ashtekar's system. There are two reasons to consider (I) weakly and (II) diagonalizable hyperbolic systems, rather than (III) symmetric hyperbolic system. First, as we found in [15], the symmetric hyperbolic system we obtained has strict restrictions on the gauge conditions, while the original Ashtekar equations constitute a weakly hyperbolic system. We are interested in these differences, and show how additional constraints appear during the steps toward constructing a symmetric hyperbolic system. Second, many numerical experiments show that there are several advantages if we apply a certain form of hyperbolic formulation. Therefore, we think that presenting these three hyperbolic systems is valuable to stimulate the studies in this field. To aid in possibly applying these systems in numerical applications, we present characteristic speeds of each system we construct.

The second result is a proposal of a new dynamical systems, which is robust for the perturbative error for violations of the constraint equations and the reality conditions. This is an extension of the idea by Brodbeck, Frittelli, Hübner and Reula (BFHR) [21] who constructed an asymptotically *stable* system (i.e., it approaches to the constraint surface) for the Einstein equation. (A similar effort can be found also in [22].) BFHR introduced additional dynamical variables, λs, which obey dissipative dynamical equations and which evolve the spacetime to the constraint surface of general relativity as the attractor in the extended spacetime.

The layout of this paper is as follows: In §2, we briefly review Ashtekar's formulation and the way of handling reality conditions. In §3, we show several classes of hyperbolic system, including the construction of symmetric hyperbolic system. In §4, we show a dynamical system which asymptotically evolves to constrained manifold. Summary and discussion are in §5.

In this report, we discuss only the case of a vacuum spacetime, but including matter is straightforward.

2 Ashtekar formulation

We start by giving a brief review of the Ashtekar formulation and the way of handling reality conditions.

2.1 Variables and Equations

The key feature of Ashtekar's formulation of general relativity [13] is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write the metric $g_{\mu\nu}$ using the tetrad E^I_μ as $g_{\mu\nu} = E^I_\mu E^J_\nu \eta_{IJ}$ 1). Define its inverse, E^μ_I, by $E^\mu_I := E^J_\nu g^{\mu\nu}\eta_{IJ}$ and we impose $E^0_a = 0$ as the gauge condition. We define SO(3,C) self-dual and anti self-dual connections ${}^{\pm}\mathcal{A}^a_\mu := \omega^{0a}_\mu \mp (i/2)\epsilon^a{}_{bc}\,\omega^{bc}_\mu$, where ω^{IJ}_μ is a spin connection 1-form (Ricci connection), $\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu E^J_\nu$. Ashtekar's plan is to use only the self-dual part of the connection ${}^{+}\mathcal{A}^a_\mu$ and to use its spatial part ${}^{+}\mathcal{A}^a_i$ as a dynamical variable. Hereafter, we simply denote ${}^{+}\mathcal{A}^a_\mu$ as $\mathcal{A}^a_\mu$.

The lapse function, N, and shift vector, N^i, both of which we treat as real-valued functions, are expressed as $E^\mu_0 = (1/N, -N^i/N)$. This allows us to think of E^μ_0 as a normal vector field to Σ spanned by the condition $t = x^0$ =const., which plays the same role as that of ADM formulation. Ashtekar treated the set $(\tilde{E}^i_a,\, \mathcal{A}^a_i)$ as basic dynamical variables, where $\tilde{E}^i_a$ is an inverse of the densitized triad defined by $\tilde{E}^i_a := eE^i_a$, where $e := \det E^a_i$ is a density. This pair forms the canonical set.

In the case of pure gravitational spacetime, the Hilbert action takes the form

$$S \;=\; \int \mathrm{d}^4x[(\partial_t\mathcal{A}^a_i)\tilde{E}^i_a + (i/2)\underset{\sim}{N}\tilde{E}^i_a\tilde{E}^j_b F^c_{ij}\epsilon^{ab}{}_c - e^2\Lambda\underset{\sim}{N} - N^i F^a_{ij}\tilde{E}^j_a + \mathcal{A}^a_0\,\mathcal{D}_i\tilde{E}^i_a], \qquad (2)$$

where $\underset{\sim}{N} := e^{-1}N$, $F^a_{\mu\nu} := 2\partial_{[\mu}\mathcal{A}^a_{\nu]} - i\epsilon^a{}_{bc}\,\mathcal{A}^b_\mu\mathcal{A}^c_\nu$ is the curvature 2-form, Λ is the cosmological constant, $\mathcal{D}_i\tilde{E}^j_a := \partial_i\tilde{E}^j_a - i\epsilon_{ab}{}^c\,\mathcal{A}^b_i\tilde{E}^j_c$, and $e^2 = \det\tilde{E}^i_a = (\det E^a_i)^2$ is defined to be $\det\tilde{E}^i_a = (1/6)\epsilon^{abc}\underset{\sim}{\epsilon}_{ijk}\tilde{E}^i_a\tilde{E}^j_b\tilde{E}^k_c$, where $\epsilon_{ijk} := \epsilon_{abc}E^a_iE^b_jE^c_k$ and $\underset{\sim}{\epsilon}_{ijk} := e^{-1}\epsilon_{ijk}$ 2).

Varying the action with respect to the non-dynamical variables $\underset{\sim}{N}$, N^i and $\mathcal{A}^a_0$ yields the constraint equations,

$$\mathcal{C}_H \;:=\; (i/2)\epsilon^{ab}{}_c\,\tilde{E}^i_a\tilde{E}^j_b F^c_{ij} - \Lambda\det\tilde{E} \approx 0, \qquad (3)$$

$$\mathcal{C}_{Mi} \;:=\; -F^a_{ij}\tilde{E}^j_a \approx 0, \qquad (4)$$

$$\mathcal{C}_{Ga} \;:=\; \mathcal{D}_i\tilde{E}^i_a \approx 0. \qquad (5)$$

1) We use $\mu,\nu = 0,\cdots,3$ and $i,j = 1,\cdots,3$ as spacetime indices, while $I, J = (0),\cdots,(3)$ and $a, b = (1),\cdots,(3)$ are $SO(1,3)$, $SO(3)$ indices respectively. We raise and lower $\mu,\nu,\cdots$ by $g^{\mu\nu}$ and $g_{\mu\nu}$ (the Lorentzian metric); $I, J,\cdots$ by $\eta^{IJ} = \mathrm{diag}(-1,1,1,1)$ and η_{IJ}; $i,j,\cdots$ by γ^{ij} and γ_{ij} (the 3-metric); $a,b,\cdots$ by δ^{ab} and δ_{ab}. We also use volume forms ϵ_{abc}: $\epsilon_{abc}\epsilon^{abc} = 3!$.

2) When $(i,j,k) = (1,2,3)$, we have $\epsilon_{ijk} = e$, $\underset{\sim}{\epsilon}_{ijk} = 1$, $\epsilon^{ijk} = e^{-1}$, and $\tilde{\epsilon}^{ijk} = 1$.

The equations of motion for the dynamical variables ($\tilde{E}^i_a$ and $\mathcal{A}^a_i$) are

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c, \quad (6)$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i, \quad (7)$$

where $\mathcal{D}_j X^{ji}_a := \partial_j X^{ji}_a - i\epsilon_{ab}{}^c \mathcal{A}^b_j X^{ji}_c$, for $X^{ij}_a + X^{ji}_a = 0$.

2.2 Reality conditions

In order to construct the metric from the variables $(\tilde{E}^i_a, \mathcal{A}^a_i, \underset{\sim}{N}, N^i)$, we first prepare the tetrad E^μ_I as $E^\mu_0 = (1/e\underset{\sim}{N}, -N^i/e\underset{\sim}{N})$ and $E^\mu_a = (0, \tilde{E}^i_a/e)$. Using them, we obtain the metric $g^{\mu\nu}$ such that $g^{\mu\nu} := E^\mu_I E^\nu_J \eta^{IJ}$.

This metric, in general, is not real-valued in the Ashtekar formulation. To ensure that the metric is real-valued, we need to impose real lapse and shift vectors together with two *metric reality* conditions;

$$\mathrm{Im}(\tilde{E}^i_a \tilde{E}^{ja}) = 0, \quad (8)$$

$$W^{ij} := \mathrm{Re}(\epsilon^{abc} \tilde{E}^k_a \tilde{E}^{(i}_b \mathcal{D}_k \tilde{E}^{j)}_c) = 0, \quad (9)$$

where the latter comes from the secondary condition of reality of the metric $\mathrm{Im}\{\partial_t(\tilde{E}^i_a \tilde{E}^{ja})\} = 0$ [17], and we assume $\det \tilde{E} > 0$ (see [18]).

For later convenience, we also prepare stronger reality conditions, *triad reality* conditions. The primary and secondary conditions are written respectively as

$$U^i_a := \mathrm{Im}(\tilde{E}^i_a) = 0, \quad (10)$$

$$\text{and} \ \ \mathrm{Im}(\partial_t \tilde{E}^i_a) = 0. \quad (11)$$

Using the equations of motion of $\tilde{E}^i_a$, the gauge constraint (5), the metric reality conditions (8), (9) and the primary condition (10), we see that (11) is equivalent to [18]

$$\mathrm{Re}(\mathcal{A}^a_0) = \partial_i(\underset{\sim}{N})\tilde{E}^{ia} + (1/2e)E^b_i \underset{\sim}{N} \tilde{E}^{ja} \partial_j \tilde{E}^i_b + N^i \mathrm{Re}(\mathcal{A}^a_i), \quad (12)$$

or with un-densitized variables,

$$\mathrm{Re}(\mathcal{A}^a_0) = \partial_i(N)E^{ia} + N^i\,\mathrm{Re}(\mathcal{A}^a_i). \quad (13)$$

From this expression we see that the secondary triad reality condition restricts the three components of the "triad lapse" vector $\mathcal{A}^a_0$. Therefore (12) is not a restriction on the dynamical variables ($\tilde{E}^i_a$ and $\mathcal{A}^a_i$) but on the slicing, which we should impose on each hypersurface.

Throughout the discussion in this report, we assume that the initial data of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ for evolution are solved so as to satisfy all three constraint equations and the metric reality condition (8) and (9). Practically, this is obtained, for example, by solving ADM constraints and by transforming a set of initial data to Ashtekar's notation.

3 Constructing hyperbolic systems

We have constructed several hyperbolic systems based on the Ashtekar formulation of general relativity, together with discussions of the required gauge conditions and reality conditions. We summarize their features in Table 1. For details, see [12].

For the later convenience, we show the explicit form of the system (*IIIa*). For a pair of $u^{(D)}_\alpha = (\tilde{E}^i_a, \mathcal{A}^a_i)$, a symmetric hyperbolic system is obtained by modifying the right-hand-side of the dynamical

system	Eqs of motion	reality condition	gauge conditions required	first order	all real eigenvals	diagonal-izable	sym. matrix
Ia	original	metric	-	yes	yes	no	no
Ib	original	triad	$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$, $\partial_i N = 0$	yes	yes	no	no
IIa	original	metric	$N^l \neq 0, \pm N\sqrt{\gamma^{ll}}$ $(\gamma^{ll} \neq 0)$	yes	yes	yes	no
IIb	modified	metric	$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$	yes	yes	yes	no
IIIa	modified	triad	$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$, $\partial_i N = 0$	yes	yes	yes	yes

Table 1: List of obtained hyperbolic systems. The system *I, II* and *III* denote weakly hyperbolic, diagonalizable hyperbolic and symmetric hyperbolic systems, respectively.

equations using appropriate combinations of the constraint equations. The modification we made is to add terms to the original evolution equation (6) and (7) as:

$$\begin{aligned} \text{adding term to } \partial_t \tilde{E}^i_a &= (N^i \delta_{ab} + i \underset{\sim}{N} \epsilon_{ab}{}^c \tilde{E}^i_c) \mathcal{C}^b_G, \\ \text{adding term to } \partial_t \mathcal{A}^a_i &= e^{-2} \underset{\sim}{N} \tilde{E}^a_i \mathcal{C}_H - i e^{-2} \underset{\sim}{N} \epsilon^{abc} \tilde{E}_{bi} \tilde{E}^j_c \mathcal{C}_{Mj}. \end{aligned}$$

Then the principal part of evolution equation becomes

$$\partial_t \begin{pmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \end{pmatrix} \cong \begin{pmatrix} A^l{}_a{}^{bi}{}_j & 0 \\ 0 & D^l{}^a{}_{bi}{}^j \end{pmatrix} \partial_l \begin{pmatrix} \tilde{E}^j_b \\ \mathcal{A}^b_j \end{pmatrix},$$

where $\cong$ means that we have extracted only the terms which appear in the principal part of the system, and the actual block diagonal components are

$$\begin{aligned} A^{labij} &= i \epsilon^{abc} \underset{\sim}{N} \tilde{E}^l_c \gamma^{ij} + N^l \gamma^{ij} \delta^{ab}, && (14) \\ D^{labij} &= i \underset{\sim}{N} (\epsilon^{abc} \tilde{E}^j_c \gamma^{li} - \epsilon^{abc} \tilde{E}^l_c \gamma^{ji} \\ &\quad - e^{-2} \tilde{E}^{ia} \epsilon^{bcd} \tilde{E}^j_c \tilde{E}^l_d - e^{-2} \epsilon^{acd} \tilde{E}^i_d \tilde{E}^l_c \tilde{E}^{jb} + e^{-2} \epsilon^{acd} \tilde{E}^i_d \tilde{E}^j_c \tilde{E}^{lb}) + N^l \delta^{ab} \gamma^{ij}. && (15) \end{aligned}$$

The inner product of a set of the variables is

$$\langle (\tilde{E}^i_a, \mathcal{A}^a_i) | (\tilde{E}^i_a, \mathcal{A}^a_i) \rangle = \gamma_{ij} \delta^{ab} \tilde{E}^i_a \bar{\tilde{E}}^j_b + \gamma^{ij} \delta_{ab} \mathcal{A}^a_i \bar{\mathcal{A}}^b_j. \quad (16)$$

We note that this symmetric hyperbolic system is obtained under the assumption of the triad reality condition, together with gauge conditions, $\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$ and $\partial_i N = 0$.

4 Asymptotically constrained system

4.1 Propagations of constraints

Frittelli [23] showed that the propagation of the constraint equations in the standard ADM system of the Einstein equation forms a symmetric hyperbolic system. This fact suggests that a small violation of the constraint equations such as a truncation error in numerical simulation does not behave in a fatal way immediately.

Similarly, we can show the set of constraint equations (3), (4) and (5), forms a symmetric hyperbolic system in its evolution equations. The principal part of the time derivatives of $\mathcal{C}_H$, $\tilde{\mathcal{C}}_{Mi} := e\,\mathcal{C}_{Mi}$ and $\mathcal{C}_{Ga}$ become

$$\partial_t \begin{pmatrix} \mathcal{C}_H \\ \tilde{\mathcal{C}}_{Mi} \\ \mathcal{C}_{Ga} \end{pmatrix} \cong \begin{pmatrix} N^l & -e\underset{\sim}{N}\gamma^{li} & 0 \\ -e\underset{\sim}{N}\delta_i{}^l & N^l \delta_i{}^j + i\underset{\sim}{N}\tilde{\epsilon}^{lj}{}_i & 0 \\ 0 & 0 & i\underset{\sim}{N}\epsilon_a{}^{bc}\tilde{E}^l_c + N^l \delta_a{}^b \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{C}_H \\ \tilde{\mathcal{C}}_{Mj} \\ \mathcal{C}_{Gb} \end{pmatrix}, \quad (17)$$

which forms a Hermitian matrix under the inner product rule of

$$\langle (\mathcal{C}_H, \mathcal{C}_{Mi}, \mathcal{C}_{Ga}) | (\mathcal{C}_H, \mathcal{C}_{Mi}, \mathcal{C}_{Ga}) \rangle := \mathcal{C}_H \bar{\mathcal{C}}_H + \gamma^{ij} \mathcal{C}_{Mi} \bar{\mathcal{C}}_{Mj} + \delta^{ab} \mathcal{C}_{Ga} \bar{\mathcal{C}}_{Gb}. \tag{18}$$

We note that non-principal parts of the dynamical equations of a set of $u_\alpha^{(C)} = (\mathcal{C}_H, \tilde{\mathcal{C}}_M, \mathcal{C}_G)$ include the terms of $u_\alpha^{(D)}$ and $u_\alpha^{(C)}$. These facts suggest that all constraint equations have a well-posed feature. Iriondo *et al* [16] present a similar result, but our definition of the inner product does not include any coefficients. We also remark that all other occurrences of the inner products throughout this report also do not include any coefficients (i.e., obey the normal index notation). Thus we omit to express the inner product hereafter.

4.2 Asymptotically constrained system

Following the BFHR procedure [21], we next construct a dynamical system which evolves the spacetime to the constrained surface, $\mathcal{C}_H \approx \mathcal{C}_{Mi} \approx \mathcal{C}_{Ga} \approx 0$ as the attractor. We introduce new variables $(\lambda, \lambda_i, \lambda_a)$, as they obey the dissipative evolution equations

$$\begin{aligned} \partial_t \lambda &= \alpha_1 \, \mathcal{C}_H - \beta_1 \, \lambda, & (19)\\ \partial_t \lambda_i &= \alpha_2 \, \tilde{\mathcal{C}}_{Mi} - \beta_2 \, \lambda_i, & (20)\\ \partial_t \lambda_a &= \alpha_3 \, \mathcal{C}_{Ga} - \beta_3 \, \lambda_a, & (21) \end{aligned}$$

where $\alpha_i \neq 0$ (allowed to be complex numbers) and $\beta_i > 0$ (real numbers) are constants.

If we take $u_\alpha^{(DL)} = (\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a)$ as a set of dynamical variables, then the principal part of (19)-(21) can be written as

$$\begin{aligned} \partial_t \lambda &\cong -i\alpha_1 \epsilon^{bcd} \tilde{E}^j_c \tilde{E}^l_d (\partial_l \mathcal{A}^b_j), & (22)\\ \partial_t \lambda_i &\cong \alpha_2 [-e \delta^l_i \tilde{E}^j_b + e \delta^j_i \tilde{E}^l_b] (\partial_l \mathcal{A}^b_j), & (23)\\ \partial_t \lambda_a &\cong \alpha_3 \partial_l \tilde{E}^l_a. & (24) \end{aligned}$$

The characteristic matrix of the system $u_\alpha^{(DL)}$ does not form a Hermitian matrix. However, if we modify the right-hand-side of the evolution equation of $(\tilde{E}^i_a, \mathcal{A}^a_i)$, then the set becomes a symmetric hyperbolic system. This is done by adding $\bar{\alpha}_3 \gamma^{il} (\partial_l \lambda_a)$ to the equation of $\partial_t \tilde{E}^i_a$, and by adding $i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d (\partial_l \lambda) + \bar{\alpha}_2 (-e \gamma^{lm} \tilde{E}^a_i + e \delta^m_i \tilde{E}^{la}) (\partial_l \lambda_m)$ to the equation of $\partial_t \mathcal{A}^a_i$. The final principal part, then, is written as

$$\partial_t \begin{pmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \\ \lambda \\ \lambda_i \\ \lambda_a \end{pmatrix} \cong \begin{pmatrix} A^l{}_a{}^{bi}{}_m & 0 & 0 & 0 & \bar{\alpha}_3 \gamma^{il} \delta_a{}^b \\ 0 & D^{l\,a}{}_{ib}{}^m & i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d & \bar{\alpha}_2 e (\delta^m_i \tilde{E}^{la} - \gamma^{lm} \tilde{E}^a_i) & 0 \\ 0 & -i\alpha_1 \epsilon_b{}^{cd} \tilde{E}^m_c \tilde{E}^l_d & 0 & 0 & 0 \\ 0 & \alpha_2 e (\delta^m_i \tilde{E}^l_b - \delta^l_i \tilde{E}^m_b) & 0 & 0 & 0 \\ \alpha_3 \delta_a{}^b \delta^l{}_m & 0 & 0 & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} \tilde{E}^m_b \\ \mathcal{A}^b_m \\ \lambda \\ \lambda_m \\ \lambda_b \end{pmatrix}. \tag{25}$$

Clearly, the solution $(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a) = (\tilde{E}^i_a, \mathcal{A}^a_i, 0, 0, 0)$ represents the original solution of the Ashtekar system. If the λs decay to zero after the evolution, then the solution also describes the original solution of the Ashtekar system in that stage. Since the dynamical system of $u_\alpha^{(DL)}$, (25), constitutes a symmetric hyperbolic form, the solutions to the λ-system are unique. BFHR showed analytically that such a decay of λs can be seen for λs sufficiently close to zero with a choice of appropriate combination of αs and βs, and that statement can be also applied to our system. Therefore, the dynamical system, (25), is useful for stabilizing numerical simulations from the point that it recovers the constraint surface automatically.

4.3 Experiments of λ-system using linearized equations

In order to show the system (25) evolves into asymptotically constrained manifold, we here demonstrate the eigenvalues in the case of linearized system. This process is in accordance with the process of section 3 of [21].

The linearization, we mean, is obtained by truncating higher order deviations from the background (0th order) spacetime, $(\tilde{E}^i_a, \mathcal{A}^a_i) = (\delta^i_a, 0)$. That is, we expand

$$\begin{aligned} \tilde{E}^i_a &= \delta^i_a + {}^{(1)}\tilde{E}^i_a + {}^{(2)}\tilde{E}^i_a + \cdots \\ \mathcal{A}^a_i &= 0 + {}^{(1)}\mathcal{A}^a_i + {}^{(2)}\mathcal{A}^a_i + \cdots \end{aligned}$$

and take only up to the terms which has 1st order contributions. For example, 2-form curvature F^a_{ij} becomes $F^a_{ij} = \partial_i \mathcal{A}^a_j - \partial_j \mathcal{A}^a_i - i\epsilon^a{}_{bc}\mathcal{A}^b_i \mathcal{A}^c_j \simeq \partial_i({}^{(1)}\mathcal{A}^a_j) - \partial_j({}^{(1)}\mathcal{A}^a_i)$.

Applying this linearization procedure to the evolution equations of $\mathcal{C}_H, \tilde{\mathcal{C}}_{Mi}, \mathcal{C}_{Ga}$ (and λs), we obtain

$$\begin{aligned} \partial_t \mathcal{C}_H &= -\partial_k \tilde{\mathcal{C}}_{Mk} + 2\bar{\alpha}_1(\partial_k \partial_k \lambda), && (26) \\ \partial_t \tilde{\mathcal{C}}_{Mi} &= -\partial_i \mathcal{C}_H - i\epsilon^a{}_i{}^j(\partial_a \tilde{\mathcal{C}}_{Mj}) + \bar{\alpha}_2(\partial_i \partial_l \lambda_l) + \bar{\alpha}_2(\partial_l \partial_l \lambda_i), && (27) \\ \partial_t \mathcal{C}_{Ga} &= -i\epsilon^{jb}{}_a \partial_j \tilde{\mathcal{C}}_{Gb} - 2\tilde{\mathcal{C}}_{Ma} + 2\bar{\alpha}_1(\partial_a \lambda) - i\bar{\alpha}_2 \epsilon^{alm}(\partial_l \lambda_m) + \bar{\alpha}_3(\partial_i \partial_i \lambda_a). && (28) \end{aligned}$$

Fourier transformation yields the equations for transformed variables $(\hat{\mathcal{C}}, \hat{\tilde{\mathcal{C}}}_{Mi}, \hat{\mathcal{C}}_{Ga}, \hat{\lambda}, \hat{\lambda}_i, \hat{\lambda}_a)$,

$$\begin{aligned} \partial_t \hat{\mathcal{C}}_H &= -ik_m \hat{\tilde{\mathcal{C}}}_{Mm} - 2\bar{\alpha}_1 k_m k^m \hat{\lambda}, && (29) \\ \partial_t \hat{\tilde{\mathcal{C}}}_{Mi} &= -ik_i \hat{\mathcal{C}}_H + k_l \epsilon^l{}_i{}^m \hat{\tilde{\mathcal{C}}}_{Mm} - k_i k^m \bar{\alpha}_2 \hat{\tilde{\mathcal{C}}}_{Mm} - \bar{\alpha}_2 k_p k^p \delta^{mi} \hat{\lambda}_i, && (30) \\ \partial_t \mathcal{C}_{Ga} &= +\epsilon^{jb}{}_a k_j \hat{\mathcal{C}}_{Gb} - 2\hat{\tilde{\mathcal{C}}}_{Mi}\delta^i_a + 2i\bar{\alpha}_1 k_a \hat{\lambda} + \bar{\alpha}_2 \epsilon^{alm} k_l \hat{\lambda}_m - \bar{\alpha}_3 k_m k^m \hat{\lambda}_a, && (31) \\ \partial_t \hat{\lambda} &= \alpha_1 \hat{\mathcal{C}}_H - \beta_1 \hat{\lambda}, && (32) \\ \partial_t \hat{\lambda}_i &= \alpha_2 \hat{\tilde{\mathcal{C}}}_{Mi} - \beta_2 \hat{\lambda}_i, && (33) \\ \partial_t \hat{\lambda}_a &= \alpha_3 \hat{\mathcal{C}}_{Ga} - \beta_3 \hat{\lambda}_a. && (34) \end{aligned}$$

We can show that the eigenvalues of this coefficient matrix are all negative (real part), for certain pairs of αs and βs. Such example can be obtained from a combination of $\alpha_1 = \alpha_2 = \alpha_3 = 1$, and $\beta_1 = \beta_2 = \beta_3 = 1$ in one dimensional situation $k_1 = 1, k_2 = k_3 = 0$. This suggests that λ-system, (25), works for decreasing the errors of constraints.

5 Future issues

Up to the previous sections, we summarized our recent works. These are, (1) efforts for constructing several levels of hyperbolic formulations of Ashtekar's dynamical equations [15, 12], and (2) a proposal of a new set of dynamical variables which force the spacetime evolution onto its constrained manifold asymptotically [20]. Here as the concluding remarks of this report, we mention some current status of our future works together with these backgrounds.

5.1 Relation between hyperbolic formulations and stabilities in numerics

First issue to concern is the relation between hyperbolic formulations and stabilities in numerical simulations. Up to a couple of years ago, ADM decomposition was taken as the standard formulation for numerical relativists. Difficulties in stable long-term evolutions were supposed to be solved by choosing proper gauge conditions and boundary conditions.

As we mentioned in the introduction, one of the alternative approaches to ADM is to formulate Einstein equations as they reveal hyperbolicity. Several numerical tests show nice properties than those by original ADM (e.g. tests [4] of Bona-Massó's flux conservative form [3], tests [10] of Choquet-Bruhat and York (95)'s symmetrizable form [7]). It is worth remarking the following two points here. (i) These numerical tests were performed using spherical symmetric spacetime. Therefore these advantages in full three-dimensional simulations are not yet confirmed. (ii) Both two hyperbolic formulations are not of first-order symmetric hyperbolic form. Therefore a question arises: Is a symmetric hyperbolic form necessary for numerical relativity?

The recent studies can be checked from another direction. There are also several papers which mention that the original ADM formulation is not the best one to be used. For example, a modified ADM formulation that was applied by Shibata and Nakamura (SN) [24] was shown to give us stable time evolutions than those of the original ADM [25]. The propagation of the original ADM constraint equations obeys well-posed behavior [23] (and this fact can be applied also to SN's modified version), but dynamical equations of both ADM and ADM-SN are not a first-order hyperbolic formulation. Therefore another question arises: Will hyperbolic formulations really help for numerical relativity?

Alcubierre *et al* [26] analyzed eigenvalues of both ADM and ADM-SN equations of their linearized version, and found that ADM-SN does not have zero-eigenvalues in its characteristic matrix. They conjectured that non-zero eigenvalues in the evolution system contribute stable long-term numerical evolution. This, we think, implies a certain kind of hyperbolicity as a threshold for stable simulations.

In order to answer these questions, we are now preparing numerical tests using our three-levels of hyperbolic equations based on the Ashtekar formulation. Such comparisons are suitable when the fundamental equations cast on the similar interface, and that is available only in our formulations in any gravity theories at this moment.

5.2 Towards an initial boundary value problem

The second issue is a direction to understand the system as an initial boundary value problem (IBVP). All the numerical simulations are of IBVP, that is we have to impose proper boundary conditions at the edges of numerical region during the time evolution, together with data on initial hypersurface. In order not to pick up unphysical reflected information from the boundaries, several techniques (such as out-going wave conditions) are so far developed. However, we have not yet obtained a "well-posed" boundary condition based on the mathematical consideration.

The actual researches of IBVP in general relativity began only recently. A satisfactory theory of IBVP is only available currently for linearized problems, and from this reason the base system is preferred to be a symmetric hyperbolic form. Therefore the discussion is available only limited cases. As far as we know, there are only two references on this topic: Stewart [27] studied IBVP for Frittelli and Reula's symmetric hyperbolic formulation [8], and Friedrich and Nagy [28] for Friedrich's system [6]. Along to this IBVP, we are now trying to find out a proper additional conditions for our symmetric hyperbolic formulation based on Ashtekar's variables.

5.3 Asymptotically constrained systems

In §4, we showed a set of dynamical equations, which has the constraint surface as its attractor, by introducing new additional variables that obey dissipative equations of motion. Based on BFHR's analytical proof [21], we expect that this set of equations is robust against a perturbative error of the constraint equations. Thus, the system may be useful for future numerical studies with its stability property.

In [20], we also showed an advanced set of equations that has its attractor also in the real-valued surface. Since our symmetric hyperbolic system of the original Ashtekar's variables requires the reality condition on the triad, the new system is designed as such a way. The same above discussion can be applied to this advanced set, and we expect the asymptotically real-valued feature in its evolution.

The problem of these systems might be that they require many additional variables. From a view point of numerical applications, this claim would not be so serious a problem, as we see a success of a dissipative maximal slicing condition [29] ('K-driver' in the literature). Actually, the λ-system of the Einstein equations was already tested and confirmed to work appropriately in numerical applications at least in one dimensional space evolution models [30]. Therefore we expect our system also shows the desired asymptotic behaviors. We are in preparation of presenting such a numerical result. We are also trying to reduce the number of the variables in order to find out clear geometrical meanings of our λ-system. We will report these efforts elsewhere.

Acknowledgments

HS would like to thank Abhay Ashtekar and CGPG group members for their hospitality. HS is supported by the Japan Society for the Promotion of Science (JSPS) as a research fellow abroad.

References

[1] Recent review is given by O. A. Reula, Livng Rev. Relativ. **1998-3** at http://www.livingreviews.org/.

[2] Y. Choquet-Bruhat and J.W. York. Jr., in *General Relativity and Gravitation* , vol. 1, ed. by A. Held, (Plenum, New York, 1980).

[3] C. Bona, J. Massó, Phys. Rev. Lett. **68**, 1097 (1992).

[4] C. Bona, J. Massó, E. Seidel and J. Stela, Phys. Rev. Lett. **75**, 600 (1995); Phys. Rev. **D56**, 3405 (1997).

[5] A. E. Fischer and J. E. Marsden, Commun. Math. Phys. **28**, 1 (1972).

[6] H. Friedrich, Proc. Roy. Soc. **A375**, 169 (1981); Proc. Roy. Soc. **A378**, 401 (1981); Comm. Math. Phys **100**, 525 (1985); J. Diff. Geom. **34**, 275 (1991); Class. Quantum Grav. **13**, 1451 (1996).

[7] Y. Choquet-Bruhat and J.W. York, Jr., C. R. Acad. Sci. Paris, **t. 321**, Série I, 1089 (1995) [gr-qc/9506071]; A. Abrahams, A. Anderson, Y. Choquet-Bruhat and J.W. York, Jr., Phys. Rev. Lett. **75**, 3377 (1995); C. R. Acad. Sci. Paris, **t. 323**, Série II b, 835 (1996); Class. Quantum Grav. **14**, A9 (1997); A. Anderson and J.W. York, Jr. Phys. Rev. Lett. **82**, 4384 (1999).

[8] S. Frittelli and O.A. Reula, Phys. Rev. Lett. **76**, 4667 (1996).

[9] M. Alcubierre, Phys. Rev. **D55**, 5981 (1997); M. Alcubierre and J. Massó, Phys. Rev. **D57**, 4511 (1998).

[10] M.A. Scheel, T.W. Baumgarte, G.B. Cook, S.L. Shapiro, and S.A. Teukolsky, Phys. Rev. **D56** 6320 (1997); *ibid.* **D58**, 044020 (1998).

[11] R. Geroch, *Partial Differential Equations in Physics*, gr-qc/9602055.

[12] G. Yoneda and H. Shinkai, to be published in Int. J. Mod. Phys. D. (gr-qc/9901053).

[13] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. **D36**, 1587 (1987); *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).

[14] M.S. Iriondo, E.O. Leguizamón and O.A. Reula, Phys. Rev. Lett. **79**, 4732 (1997).

[15] G. Yoneda and H. Shinkai, Phys. Rev. Lett. **82**, 263 (1999).

[16] M.S. Iriondo, E.O. Leguizamón and O.A. Reula, Adv. Theor. Math. Phys. **2**, 1075 (1998).

[17] A. Ashtekar, J. D. Romano and R. S. Tate, Phys. Rev. **D40**, 2572 (1989).

[18] G. Yoneda and H. Shinkai, Class. Quantum Grav. **13**, 783 (1996).

[19] G. Yoneda, H. Shinkai and A. Nakamichi, Phys. Rev. **D56**, 2086 (1997).

[20] H. Shinkai and G. Yoneda, Phys. Rev. **D60**, 101502 (1999).

[21] O. Brodbeck, S. Frittelli, P. Hübner and O.A. Reula, J. Math. Phys. **40**, 909 (1999).

[22] S. Detweiler, Phys. Rev. **D35**, 1095 (1987).

[23] S. Frittelli, Phys. Rev. **D55**, 5992 (1997).

[24] M. Shibata and T. Nakamura, Phys. Rev. **D52**, 5428 (1995).

[25] T. W. Baumgarte and S. L. Shapiro, Phys. Rev. **D59**, 024007 (1999).

[26] M. Alcubierre, G. Allen, B. Brügmann, E. Seidel and W-M. Suen, gr-qc/9908079.

[27] J.M. Stewart, Class. Quant. Grav. **15**, 2865 (1998).

[28] H. Friedrich and G. Nagy, Comm. Math. Phys. **201**, 619 (1999).

[29] J. Balakrishna *et al*, Class. Quant. Grav. **96**, L135 (1997).

[30] In private commucations with P. Hübner and with G. Cook.