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Fate of the Kaluza-Klein bubble

Hisa-aki Shinkai*
Center for Gravitational Physics and Geometry, 104 Davey Lab., Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802-6300

Tetsuya Shiromizu†
DAMTP, University of Cambridge, Silver Street, Cambridge CB3 9EW, United Kingdom, Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan, and Research Centre for the Early Universe (RESCEU), The University of Tokyo, Tokyo 113-0033, Japan
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We numerically study classical time evolutions of Kaluza-Klein bubble space-time which has negative energy after a decay of vacuum. As the zero-energy Witten bubble space-time, where the bubble expands infinitely, the subsequent evolutions of the Brill-Horowitz momentarily static initial data show that the bubble will expand in terms of the area. At first glance, this result may support the Corley-Jacobson conjecture that the bubble will expand forever as well as Witten's bubble. The irregular signatures, however, can be seen in the behavior of the lapse function in the maximal slicing gauge and the divergence of the Kretchman invariant. Since there is no appearance of the apparent horizon, we suspect the appearance of a naked singularity as the final fate of this space-time.

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I. INTRODUCTION

It is likely that superstring or M theory governs the physics of gravity or space-time in higher energy stages [1]. Such theories are naturally formulated in higher dimensions than four. We expect a plausible scenario that such a higher dimensional space-time somehow evolves to the stable four-dimensional space-time according to the history of the Universe [2–4]. The so-called brane world scenario [2] is the most actively being investigated along this line. This scenario is motivated by the Horava Witten theory [5] which shows that an eleven-dimensional supergravity theory on the orbihold $\mathbf{R}^{10}\times S^1/Z_2$ is related to the ten-dimensional $E_8\times E_8$ heterotic string theory. Therein the matters are confined to the ten-dimensional space-time (three-brane) and gravitons are propagating in the full eleven dimensions. The brane world space-time should be stable.

Although the brane world scenario may be plausible at the reduction from eleven to ten dimensions, the space-time will be still compactified to four dimensions in the normal Calabi-Yau's way. Regarding the full scenario of the compactification, the stability of the space-time becomes the important issue to be investigated. The positive energy theorem guarantees the stability of the four-dimensional asymptotically flat space-time in the framework of general relativity [6]. Surprisingly, the existence of the extra dimensions can drastically change the situation. Witten showed that the five-dimensional Minkowski space-time decays into the so-called Kaluza-Klein (KK) bubble space-time unless we assume the existence of the elementary fermion related to supersymmetry [7,8], of which existence we cannot expect generally. This also may indicate that the ''bubble'' appears somewhere at the bulk or on the brane in the brane world scenario and disturbed the three-brane where we are living.

The metric of the KK bubble space-time given by Witten is written as [7]

$$ds_5^2=-r^2\,dt^2+\left(1-\frac{r_0^2}{r^2}\right)d\chi^2+\left(1-\frac{r_0^2}{r^2}\right)^{-1}dr^2 + r^2\cosh^2 t\,d\Omega^2, \tag{1.1}$$

where the χ direction will be compactified and $r\geqslant r_0$, and $d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2$. In the general case, the metric has a conical singularity at r_0. However, if we carefully take a periodicity along the χ direction, the metric can be regularized. More precisely to see this, we write the metric near $r=r_0$ as

$$\begin{aligned} ds_5^2 &\simeq -r_0^2\,dt^2+\frac{2(r-r_0)}{r_0}d\chi^2+\frac{r_0}{2(r-r_0)}dr^2 \\ &\quad + r_0^2\cosh^2 t\,d\Omega^2 \\ &= -r_0^2\,dt^2+2r_0\left[R^2 d\left(\frac{\chi}{r_0}\right)^2+dR^2\right]+r_0^2\cosh^2 t\,d\Omega^2, \end{aligned} \tag{1.2}$$

where $R=\sqrt{r-r_0}$. Then we realize that the period should be set to be $\chi_p=2\pi r_0$. As one can see, the ''boundary of bubble'' located at $r=r_0$ expands rapidly like $\cosh t$ and the space-time does not have naked singularities. Here, we remind you that the total energy is zero. We are imaging the boundary of the space-time, $r=r_0$, as the surface of the ''bubble.''

Interestingly Brill and Pfister [9] gave initial data which has the negative total energy related to the size of the compactified dimension. The space-time with the negative en-

*Electronic address: shinkai@gravity.phys.psu.edu
†Electronic address: T.Shiromizu@damtp.cam.ac.uk

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ergy may be in favor in the aspect of energetics. One year later, Brill and Horowitz [10] gave initial data in a simple way. (We will briefly review their construction in Sec. II.) Contrasted to the ''Witten bubble,'' their solution has arbitrary negative energy regardless of the size of the compactified space. This is so far from our intuition that the negative energy is proportional to the Casimir energy due to the boundary effect of the compactified space. Therefore, it is difficult to imagine the classical evolution after the vacuum decay.

Corley and Jacobson [11] discussed the subsequent evolution of the Brill-Horowitz initial data. They found the positive acceleration of the bubble's surface area for the negative mass bubbles, and they conjectured that the KK bubble with negative energy cannot collapse. However, their study is not sufficient to conclude the final fate of the bubble, as they already mentioned, because they considered only the initial behavior of the time-symmetric data and they did not do any dynamical studies.

In this paper, we report our numerical analysis on this final fate problem of the KK bubble, especially of the negative mass bubble. We start our numerical simulation from the Brill-Horowitz initial data, and evolve the space-time using the standard Arnowitt-Deser-Misner formulation [but (4+1)-dimensional decomposition]. We will show that the space-time initially behaves as the Corley-Jacobson analysis, and expands forever, although the acceleration will be negative. Despite the expanding, we will observe the irregular behavior of the curvature invariant.

This paper is organized as follows. In Sec. II, we give a brief review of the Brill-Horowitz construction of their initial data. In Sec. III, we describe numerical method and equations. The results of our simulations are shown in in Sec. IV. Finally, we summarize our results in Sec. V.

II. BRILL-HOROWITZ'S INITIAL DATA

In this section we briefly review the Brill-Horowitz argument [10]. Let us consider an initial slice with $K_{ij}=0$ in five-dimensional vacuum space-times, where K_{ij} is the extrinsic curvature of a four-dimensional spacelike hypersurface. In this slice the Hamiltonian constraint equation becomes ${}^{(4)}R=0$, where ${}^{(4)}R$ is the four-dimensional Ricci scalar. Here one can easily see that the Euclidean Reissner-Nordstrom metric with imaginary ''charge'' iq satisfies the Hamiltonian constraint equation, because the ''energy-momentum'' tensor of the four-dimensional Maxwell field is traceless. The metric of the hypersurface is given by

$$ {}^{(4)}g = U(r)d\chi^2 + \frac{dr^2}{U(r)} + r^2 d\Omega^2, \tag{2.1} $$

where $U(r)=1-2m/r-q^2/r^2$ and $r \geqslant r_+ := m+\sqrt{m^2+q^2}$. In the same way as the previous Witten's example, the metric is approximately written as

$$ {}^{(4)}g \simeq \frac{4}{U'(r_+)}\left[R^2 d\left(\frac{U'(r_+)\chi}{2} \right)^2 + dR^2 \right] + r_+^2 d\Omega^2, \tag{2.2} $$

near $r=r_+$, where $R=\sqrt{r-r_+}$. To avoid a conical singularity at $r=r_+$, we assume the period $\chi_p = 4\pi/U'(r_+) = 2\pi r_+^2/(r_+-m)$ along the χ direction.

The total energy is evaluated as $E=m/2$. The m is arbitrary parameter and q determines the size of the compactified space. So the total energy can be arbitrary negative.

III. FIELD EQUATIONS AND OUR NUMERICAL METHOD

To know the final fate of the KK bubbles with the negative energy, we study the subsequent time evolution for a long time numerically. We apply the (4+1) decomposition of the space-time to the Arnowitt-Deser-Misner formulation for the actual time integrations. We describe equations and basic numerical techniques in this section.

The metric of the full space-time is assumed to be

$$ ds_5^2 = -N(r,t)^2 dt^2 + e^{2a(r,t)}U(r)d\chi^2 + e^{2b(r,t)}U(r)^{-1}dr^2 + r^2 e^{2c(r,t)}d\Omega^2, \tag{3.1} $$

where $U(r)=1-2m/r-q^2/r^2$, N is the lapse function, and the metric components a, b, and c are now time dependent. The evolution equations of the four-metric γ_{ij} and the extrinsic curvature K_{ij} become[1]

$$ \dot{K}^i_j = N({}^{(4)}R^i_j + KK^i_j) - {}^{(4)}D^{i(4)}D_j N, \tag{3.2} $$

$$ \dot{\gamma}_{ij} = -2NK_{ij}, \tag{3.3} $$

a dot denotes the time derivative, and ${}^{(4)}R^i_j$ and ${}^{(4)}D^i$ denote the four-dimensional Ricci curvature and the covariant derivative, respectively. For the reader's convenience, we write down several terms in Eq. (3.2) for the metric (3.1) as

$$ {}^{(4)}R^\chi_\chi = e^{-2b}\left[\left(-a'^2 - a'' - 2a'c' - \frac{2a'}{r} + a'b' \right) U + \left(-\frac{3}{2}a' + \frac{1}{2}b' - c' - \frac{1}{r} \right) U' - \frac{1}{2}U'' \right], \tag{3.4} $$

$$ {}^{(4)}R^r_r = e^{-2b}\left[\left(-a'^2 - a'' - 2c'^2 - 2c'' - \frac{4c'}{r} + a'b' + 2b'c' + \frac{2b'}{r} \right) U + \left(-\frac{3}{2}a' + \frac{1}{2}b' - c' - \frac{1}{r} \right) \times U' - \frac{1}{2}U'' \right], \tag{3.5} $$

[1]Here, for simplicity, we tacitly supposed the boundary condition so that the location of the bubble is ''fixed'' under the variation of the action. As a result we obtain the five-dimensional vacuum Einstein equation and can show the consistent result given in the Appendix. Since the Cauchy development of the initial data cannot cover all regions outside the bubble, one may be able to consider another boundary conditions, which might be artificial.

$$
\begin{aligned}
{}^{(4)}R^{\theta}_{\theta} &= {}^{(4)}R^{\varphi}_{\varphi} \\
&= e^{-2b}\left[\left(-2c'^{2}-c''-\frac{4c'}{r}-\frac{1}{r^{2}}+c'b'\right.\right. \\
&\quad \left.\left.+\frac{b'}{r}-a'c'-\frac{a'}{r}\right)U+\left(-c'-\frac{1}{r}\right)U'+\frac{e^{2b-2c}}{r^{2}}\right],
\end{aligned}
\tag{3.6}
$$

and

$$
{}^{(4)}D^{(4)}_{\chi}D^{\chi}N = e^{-2b}\left(a'+\frac{1}{2}\frac{U'}{U}\right)UN', \tag{3.7}
$$

$$
{}^{(4)}D^{(4)}_{r}D^{r}N = e^{-2b}\left[(N''-b'N')U+\frac{1}{2}N'U'\right], \tag{3.8}
$$

$$
{}^{(4)}D^{(4)}_{\theta}D^{\theta}N = e^{-2b}\left(c'+\frac{1}{r}\right)N'U, \tag{3.9}
$$

where a dash denotes the derivative on r.

We start our simulation from the initial data of the Brill-Horowitz momentarily static solution, such as

$$
a(r,0)=b(r,0)=c(r,0)=0, \tag{3.10}
$$

$$
K^{\chi}_{\chi}(r,0)=K^{r}_{r}(r,0)=K^{\theta}_{\theta}(r,0)=0. \tag{3.11}
$$

The numerical region is taken as $r_{+}\leqslant r\leqslant r_{e}$, where $r_{+} := m+\sqrt{m^{2}+q^{2}}$ is the location of the bubble at the initial data and r_{e} is the numerical outer boundary. We stress from the construction that the Kaluza-Klein bubble space-time is restricted in $r_{+}\leqslant r\leqslant\infty$ [10]. We apply the Robin boundary condition at $r=r_{e}$ such that all the components fall off as they form an asymptotically flat spacetime. At the inner boundary $r=r_{+}$, we use the fact that both a and b evolve synchronously as we describe in the Appendix, and use both evolution equations for $\mathrm{tr}\,K$,

$$
\dot{K}=NK_{ij}K^{ij}-{}^{(4)}D^{i(4)}D_{i}N, \tag{3.12}
$$

where we used the Hamiltonian constraint equation, and the momentum constraint equation,

$$
{}^{(4)}D_{j}K^{j}_{r}-{}^{(4)}D_{r}K=0, \tag{3.13}
$$

so the system evolves properly.

In order to specify the lapse function, N, we apply both the geodesic slicing condition, $N=1$, and the maximal slicing condition, $K=0$, which becomes [directly from Eq. (3.12)]

$$
{}^{(4)}\Delta N = NK_{ij}K^{ij}. \tag{3.14}
$$

This elliptic equation is solved using the incomplete Cholsky conjugate gradient method. The outer boundary for N is set again as asymptotically flat, and the inner boundary at r_{+} for solving Eq. (3.14) we lineally extrapolate four-metric components.

We apply the Brailovskaya integration scheme (a second-order predictor-corrector method) [12] for the time evolution. The accuracy of the calculation is checked by monitoring the violation of the Hamiltonian constraint equation. The numerical code passed convergence tests, and the results shown in this paper are all obtained with acceptable accuracy.

IV. RESULTS

A. Acceleration of the bubble surface

We first check whether our code reveals the initial behavior discussed by Corley and Jacobson [11]. We calculate the area of the bubble,

$$
A(t)=4\pi g_{\theta\theta}(r_{+},t), \tag{4.1}
$$

together with its time derivative $\dot{A}$, and its acceleration,

$$
\ddot{A}=4\pi\ddot{g}_{\theta\theta}=4\pi r^{2}[-2N\dot{K}^{\theta}_{\theta}-2\dot{N}K^{\theta}_{\theta}+4N^{2}(K^{\theta}_{\theta})^{2}]e^{2c}. \tag{4.2}
$$

We first show this acceleration in Figs. 1(a) and 1(b), since this was the quantity discussed by Corley and Jacobson. Figures 1(a) and 1(b) are of the geodesic slicing condition and of the maximal slicing condition, respectively. We fix the charge q and varied m from negative to positive values. Except for the transition at $m=0$, as one can see later, our result is not qualitatively sensitive under changes of m/q. Under both slicing conditions, we see that the negative mass bubble start expanding (positive $\ddot{A}$) initially, yet will soon be in the deaccelerating phase (negative $\ddot{A}$), while the positive mass bubble keep accelerating all the way in Figs. 1(a), and in the region in Figs. 1(b). More precisely, for the positive mass cases in Figs. 1(b), we observe from the numerical results that the acceleration will reach and stay at a positive value in the final stage, even if it goes negative for a short time, which happens to quite small positive mass cases.

Such an initial behavior (for both positive and negative mass bubbles) does agree with the Corley and Jacobson analysis (we remark that their analysis was under the geodesic slicing condition). However, the turning behavior into the deaccelarating phase could not be found in their analysis. The deaccelarating does not mean collapsing feature directly. Actually, up to the time we stop our time evolution, the numerical data of the area Eq. (4.1) monotonically increases [Figs. 1(c)], while its velocity goes down for negative mass bubbles [Figs. 1(d)]. However, from those facts, we cannot say that negative mass bubbles will expand forever, because we can see the blowups of the Riemann invariant and collapsing lapse behavior as we show next. (We had to stop time evolution for negative mass bubble case when we face the blowups of the Riemann invariant.)

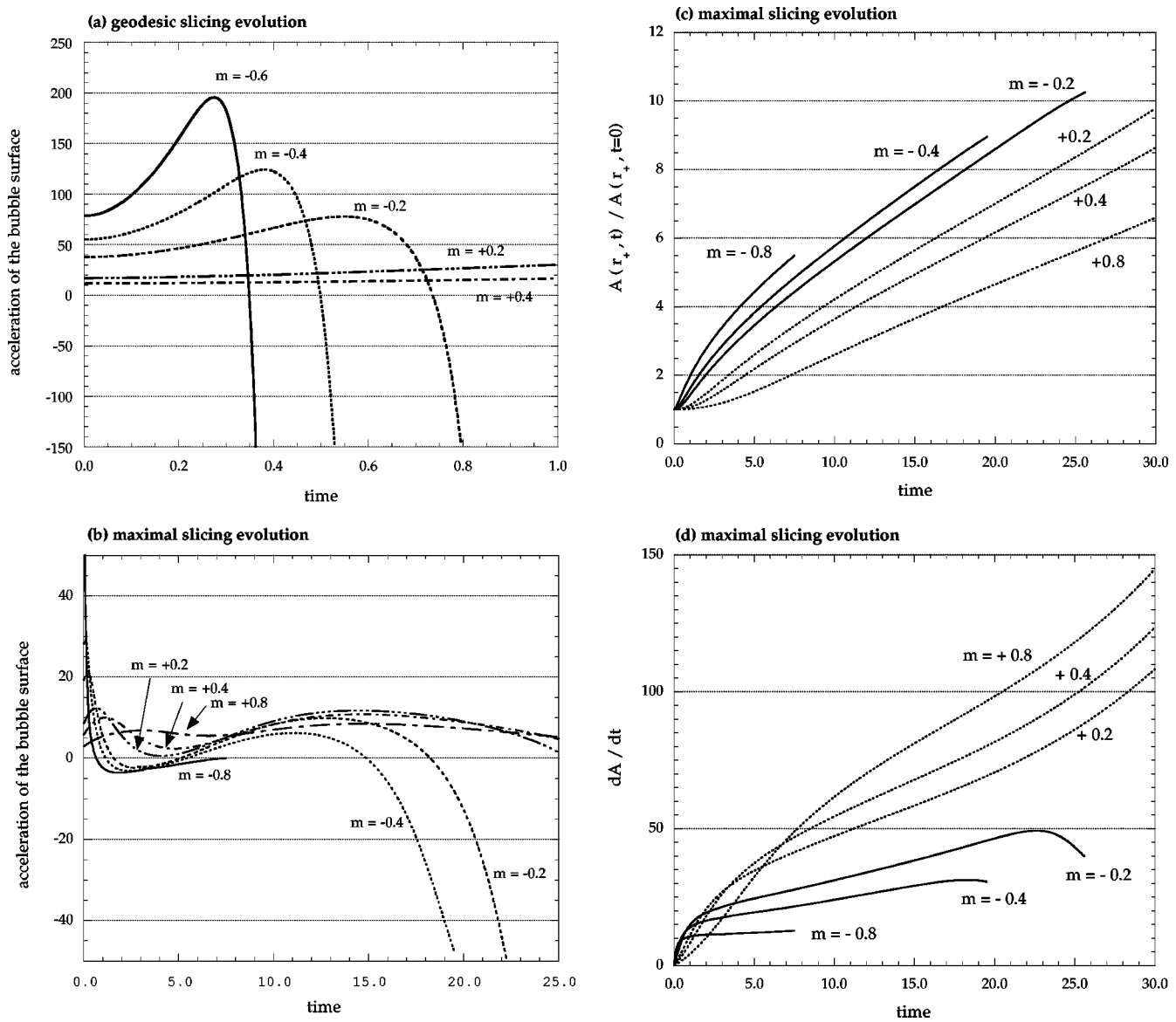

FIG. 1. Acceleration of the bubble surface, Eq. (4.2), versus time. (a) and (b) are of the geodesic slicing condition and of the maximal slicing condition, respectively. In both figures, we see that the negative mass bubble will soon be in collapse phase, although they start expanding initially. We set $q=1$ (hereafter for all figures). For the case of evolutions with the maximum slicing condition, we plot (c) area of the bubble surface A, Eq. (4.1), versus time, and (d) the velocity of the bubble surface, dA/dt.

B. Collapse of lapse

Since we found that the time integration using the maximal slicing condition survives a longer term time evolution than that of the geodesic slicing condition, we will show only the results of the maximal slicing condition hereafter.

The maximal slicing condition is known as a robust gauge condition for singularity avoidance (or, exactly speaking, avoiding the vanishing of the volume elements of the associated Eulerian observers) [13]. This is because the lapse will become quite small value in the strong gravitational field. In contrast, we may guess whether the space-time will collapse or not by monitoring the lapse function.

We plot the lapse function N in Fig. 2. Figure 2(a) is the lapse function at the bubble surface, $r=r_+$, versus time. We see the lapse evolves as a small value for the case of negative mass bubble space-time. The lines end at the time when the violation of the constraint equation begin growing. From above standard behavior of the maximal sliced lapse functions, we may say that the negative mass bubble space-time is ''collapsing'' in some senses. Figure 2(b) is snapshots of N at several times for the case of the negative mass bubble space-time.

C. Riemann invariant

In order to confirm our guess of the ''collapsing'' behavior of the negative mass bubble, we calculated the Kretchman invariant (Riemann invariant) $R_{ijkl}R^{ijkl}$ of both the four and five-dimensional Riemann tensor. We see both blowups in the cases of negative mass bubbles. We plotted a typical behavior of the invariant as a function of r and t, in Fig. 3.

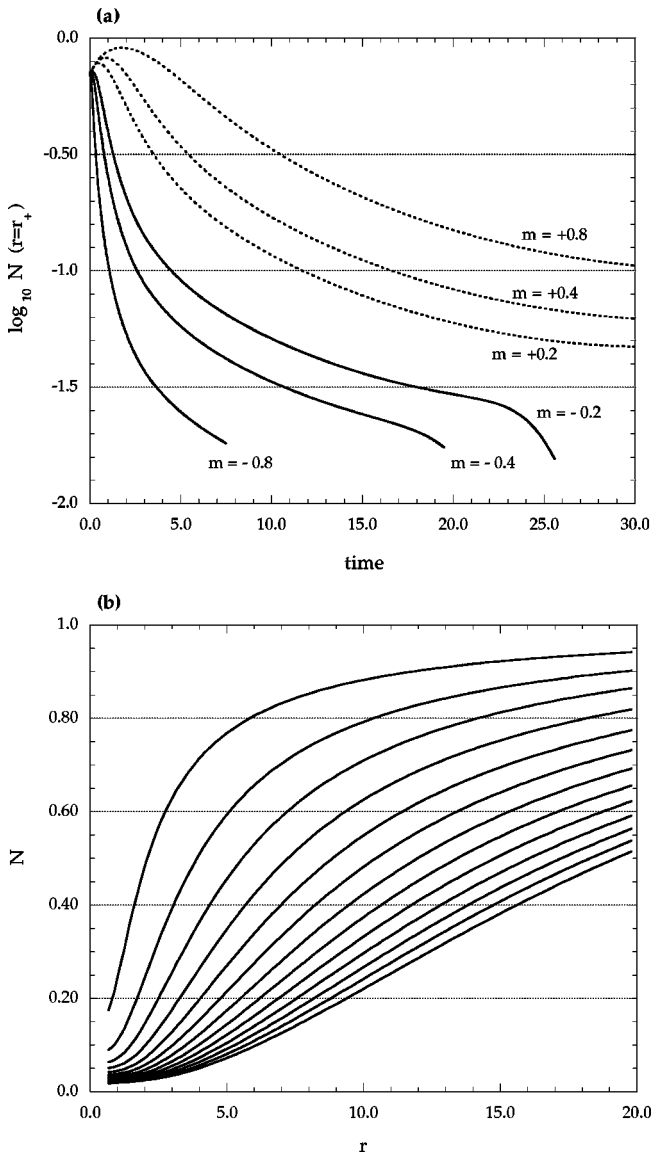

FIG. 2. The lapse function N is plotted (the solutions of maximal slicing condition). (a) is N at $r=r_+$ versus time. We see the lapse evolves at a small value for the case of negative mass bubble space-time. (b) is snapshots of N at several times for a case of negative mass bubble ($m=-0.4$) space-time.

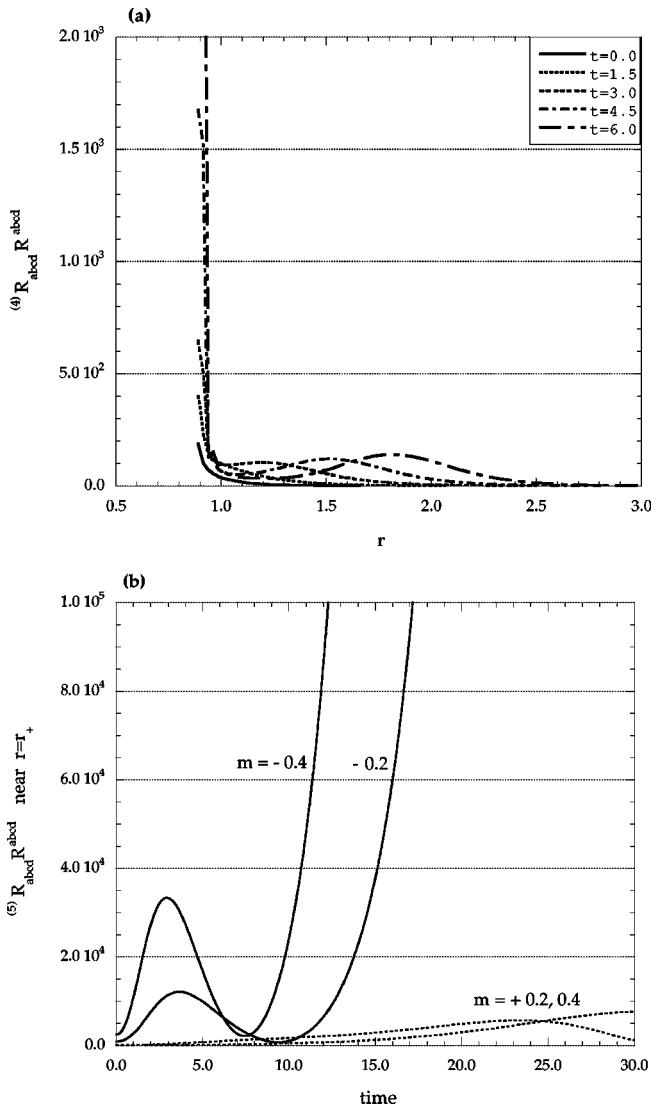

FIG. 3. (a) Typical snapshots of the Riemann invariant $R_{ijkl}R^{ijkl}$ of four-dimensional Riemann curvature for the case of negative mass bubble ($m=-0.4$) are plotted. Only the region near the bubble surface is drawn. (b) The Riemann invariant $R_{ijkl}R^{ijkl}$ of five-dimensional Riemann curvature are plotted as a function of time. We see blowups in the cases of negative mass bubbles (we cut the display range at 10^5). The values are evaluated at a point right from the bubble surface (that is, at $r_+ + \Delta r$).

These lines suggest the possibility of the formation of a singularity in the final phase of evolution.

D. Apparent horizons

In order to confirm whether a black hole is formed or not in such a case, we check the appearance of the apparent horizon. The location of the apparent horizon is given by the position of the expansion rate, and where the outgoing null geodesic congruence turns into the negative. The appearance of the apparent horizon indicates the existence of the event horizon.

The definition of the apparent horizon might not be unique in our five-dimensional spacetime because it depends on the dimension of the space-time which the null geodesic congruence runs.[2] If the null congruence is propagating in the full five-dimensional space-time, the expansion rate, ${}^{(4)}\theta_+$, is given by

$$
\begin{aligned}
{}^{(4)}\theta_+ &= {}^{(4)}\nabla_a s^a - {}^{(4)}K + s^a s^b \ {}^{(4)}K_{ab} \\
&= \left(a' + 2c' + \frac{2}{r} + \frac{U'}{2U} \right) \sqrt{U} e^{-b} \\
&\quad - (K^\chi_\chi + K^\theta_\theta + K^\varphi_\varphi),
\end{aligned} \tag{4.3}
$$

[2]In the usual Kaluza-Klein picture, it is natural that the null geodesic congruence runs in the four-dimensional part.

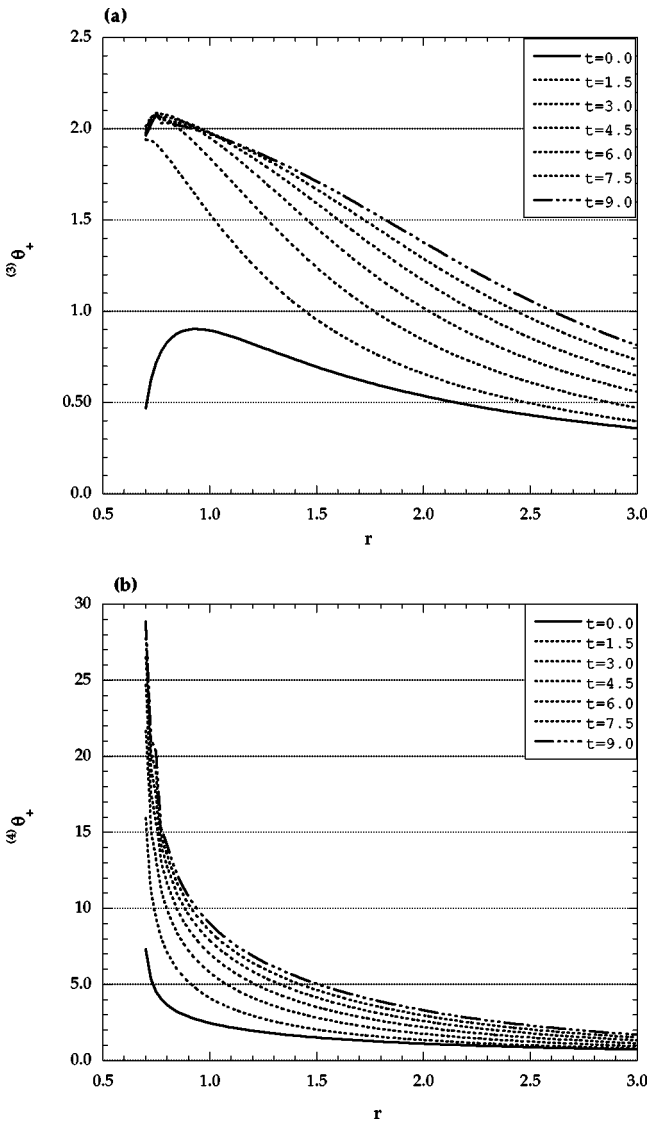

FIG. 4. A typical sample of the outgoing null expansion rate ${}^{(3)}\theta_+$ and ${}^{(4)}\theta_+$ are plotted for the case of negative mass ($m=-0.4$) bubbles.

where $s^a=(0,1/\sqrt{g_{rr}},0,0)$ is a outer pointing vector in our spatial four metric. On the other hand, if the null is confined to noncompactified four dimensions, the expansion rate, ${}^{(3)}\theta_+$, is given similarly by

$$
\begin{aligned}
{}^{(3)}\theta_+ &= {}^{(3)}\nabla_a s^a - {}^{(3)}K + s^a s^b \ {}^{(3)}K_{ab} \\
&= 2\left(c' + \frac{1}{r}\right)\sqrt{U}e^{-b} - (K^\theta_\theta + K^\varphi_\varphi). \qquad (4.4)
\end{aligned}
$$

We analyzed both ${}^{(3)}\theta_+$ and ${}^{(4)}\theta_+$ in our evolving spacetime. Surprisingly, in all cases (positive and negative mass bubbles), both expansions remain positive definite everywhere as we show in Fig. 4. These suggest us no-appearance of apparent horizons.

V. DISCUSSION

We numerically studied the dynamical evolution of the Brill- and Horowitz initial data which can have negative energy. Similar to the zero-energy Witten's bubble space-time, we show that the ''bubbles'' with negative energy will expand by means of area up to the time we stop the simulations. At first glance this result supports the Corley-Jacobson conjecture. However, from the facts that the curvature invariant blows up, and with no appearance of the apparent horizon, we suspect a formation of a naked singularity as the final fate of Kaluza-Klein negative energy bubble.[3] Hence, we may have to consider seriously the decay problem from the Kaluza-Klein vacuum to the Witten-type bubble spacetime. A possible resolution to this may be given by assuming the supersymmetry which may forbid the decay [7,8], or by constructing quantum gravity theory which may smooth out singularities as has normally been expected.

Although the negative mass bubbles are expanding, we obtained the result that the bubble spacetime terminates at the singularity. At first glance, they are incompatible, because of the naive picture, in which the expanding keeps regularity. However, the picture may be based on the Raychaudhri-type equation and this equation does not hold in the present case. Moreover, the area cannot properly describe whether the system will collapse or not. We need the proper radius from the center which is absent in the present case.

Finally, we would like to comment on the so-called brane world scenario [2–4]. The brane world is motivated by the reduction from the M theory to the $E_8\times E_8$ heterotic superstring theory. This reduction drastically changes the picture of the reduction [2] because ''matters'' are confined to the ten dimensions and gravitons are propagating in 11 dimensions. Here we call the timelike hypersurface, where matters are confined, a brane. A plausible history of the compactifications is still quite actively discussed and we have not reached a consensus yet. Apart from this compactification scenario, the reduction from ten to four dimensions follows the well-known Kaluza-Klein type or Calabi-Yau compactification. Thus, our present analysis is basically applicable to the space-time on the brane, because we supposed the well-known Kaluza-Klein compactification. More precisely, we may say that the Witten-type Kaluza-Klein bubble space-time on the four brane will be reduced from at least six-dimensional space-time. As was recently reported [3], the effective Einstein equations on the brane are different from the normal Einstein equations. Therefore, it might be worth reasking what is the final fate of Kaluza-Klein bubble if we describe the space-time by such a modified Einstein equation when we take a brane world scenario.

Note added in proof. After acceptance of our manuscript, we found a related work by Fabinger and Horava [14]. They also discussed the decay into ''Witten bubble'' spacetime with the zero energy in the brane world.

[3]We note the similarity of the positive and negative mass bubble results. However, from the results we obtained, we believe that there are qualitative differences between positive and negative mass cases in their dynamical behaviors. (We remark that our simulations are only up to a finite time in order to keep the resolution against the expansion of the spacetime.)

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APPENDIX: THE DYNAMICAL EQUATIONS AT THE SURFACE OF THE BUBBLE

In this appendix, we show that $\dot{a}=\dot{b}$ at the location of the bubble, $r=r_+$, during the time evolution.

The explicit expression of Eqs. (3.2) and (3.3) are given by

$$\left(\frac{\dot{a}}{N}\right)^{\cdot} = -Ne^{-2b}\left[\left(-a'^2-a''-2a'c'-\frac{2a'}{r}+a'b'\right)U + \left(-\frac{3}{2}a'+\frac{1}{2}b'-c'-\frac{1}{r}\right)U'-\frac{1}{2}U''\right] - \frac{1}{N}(\dot{a}+\dot{b}+2\dot{c})\dot{a}+e^{-2b}\left(a'U+\frac{1}{2}U'\right)N', \quad \text{(A1)}$$

$$\left(\frac{\dot{b}}{N}\right)^{\cdot} = -Ne^{-2b}\left[\left(-a'^2-a''-2c'^2-2c''-\frac{4c'}{r}+a'b' + 2b'c'+\frac{2b'}{r}\right)U + \left(-\frac{3}{2}a'+\frac{1}{2}b'-c'-\frac{1}{r}\right)U'-\frac{1}{2}U''\right] - \frac{1}{N}(\dot{a}+\dot{b}+2\dot{c})\dot{b}+e^{-2b} \times\left[(N''-b'N')U+\frac{1}{2}N'U'\right], \quad \text{(A2)}$$

$$\left(\frac{\dot{c}}{N}\right)^{\cdot} = -Ne^{-2b}\left[\left(-2c'^2-c''-\frac{4c'}{r}-\frac{1}{r^2}+c'b' + \frac{b'}{r}-a'c'-\frac{a'}{r}\right)U+\left(-c'-\frac{1}{r}\right)U'+\frac{e^{2b-2c}}{r^2}\right] - \frac{1}{N}(\dot{a}+\dot{b}+2\dot{c})\dot{c}+e^{-2b}\left(c'+\frac{1}{r}\right)N'U. \quad \text{(A3)}$$

At $r=r_+$, we can truncate U, since $U=0$. By adding a suffix $+$ for the variables which is evaluated at $r=r_+$, the above equations become

$$\left(\frac{\dot{a}_+}{N_+}\right)^{\cdot} = -N_+e^{-2b_+}\left[\left(-\frac{3}{2}a'_+ +\frac{1}{2}b'_+ -c'_+ -\frac{1}{r_+}\right) \times U'_+ -\frac{1}{2}U''_+\right] -\frac{1}{N_+}(\dot{a}_+ +\dot{b}_+ +2\dot{c}_+)\dot{a}_+ + \frac{1}{2}e^{-2b_+}U'_+N'_+, \quad \text{(A4)}$$

$$\left(\frac{\dot{b}_+}{N_+}\right)^{\cdot} = -N_+e^{-2b_+}\left[\left(-\frac{3}{2}a'_+ +\frac{1}{2}b'_+ -c'_+ -\frac{1}{r_+}\right) \times U'_+ -\frac{1}{2}U''_+\right] -\frac{1}{N_+}(\dot{a}_+ +\dot{b}_+ +2\dot{c}_+)\dot{b}_+ + \frac{1}{2}e^{-2b_+}U'_+N'_+. \quad \text{(A5)}$$

Subtracting Eq. (A5) from Eq. (A4), we obtain

$$\left(\frac{\dot{a}_+-\dot{b}_+}{N_+}\right)^{\cdot} = -(\dot{a}_+ +\dot{b}_+ +2\dot{c}_+)\frac{\dot{a}_+-\dot{b}_+}{N_+}. \quad \text{(A6)}$$

Therefore, we get

$$\frac{\dot{a}_+-\dot{b}_+}{N_+} = Ae^{-(a_++b_++2c_+)}, \quad \text{(A7)}$$

where A is a constant. Since the initial conditions, Eqs. (3.10) and (3.11), imply $a_+=b_+$, which implies $A=0$. Therefore, at the boundary, $r=r_+$, we can set $a_+=b_+$ even after the long time integration.

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