Hyperbolic Formulations and Numerical Relativity [*]

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We present our numerical comparisons between weakly hyperbolic, strongly hyperbolic, and symmetric hyperbolic systems based on Ashtekar's connection variables. Our numerical code demonstrates gravitational wave propagation in plane symmetric spacetime, and we compare the stability and/or accuracy by monitoring the violation of the constraints. We also study alternative approaches to obtain stable evolutions, that can be called "asymptotically constrained systems". We demonstrate"λ-system" and "adjusted-system" for the Ashtekar system, and show they work as desired. We propose a new mechanism to control the stability by evaluating the eigenvalues of constraint propagation equations, and this mechanism affects more than the system's symmetric hyperbolicity.

This report is for formulations to obtain long-term stable numerical evolution of the Einstein equations. According to the recent studies, the standard Arnowitt-Deser-Misner (ADM) decomposition of the Einstein equations may not be the best formulation to use in numerics. One direction in the community is to apply conformally decoupled and tracefree re-formulation of the ADM system which were first used by Nakamura *et al.* [1]. Although there is an effort to show why this re-formulation is better than ADM, we do not yet know this method is robust for all situations.

Another alternative approach to ADM is to formulate the Einstein equations to reveal hyperbolicity (see the references in [2]). A certain kind of hyperbolicity of the dynamical equations is essential to analyze their propagation features mathematically, and are known to be useful in numerical simulations.

Our purpose here is to report our numerical comparisons between several mathematical levels of hyperbolicity: (I) weakly, (II) strongly and (III) symmetric hyperbolic systems. These three classes have the relation (III) $\in$ (II) $\in$ (I). The symmetric hyperbolic system gives us the energy integral inequalities, which are the primary tools for studying well-posedness of the system.

Such comparisons are appropriate when the fundamental equations are cast in the same interface, and that is possible using Ashtekar's connection variables[3]. More precisely, (a) the original set of dynamical equations proposed by Ashtekar already forms a weakly hyperbolic system [2], (b) by requiring additional gauge conditions *and/or* adding constraints to the dynamical equations, we can obtain a strongly or symmetric hyperbolic system [2], and (c) based on the above symmetric hyperbolic system, we can construct a system which is robust against perturbative errors for constraints and reality conditions [4] (*aka.* λ-system [5]).

We developed a numerical code which handles gravitational wave propagation in the plane symmetric spacetime. Due to the limited space, we have to omit details. More complete discussion can be found in our recent articles [6,7].

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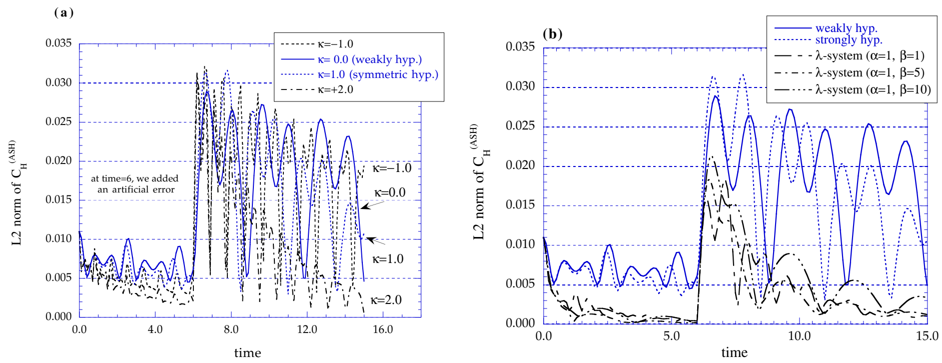

Figure 1: Demonstration of the adjusted-system [Fig.(a)] and λ-system [Fig.(b)] in the Ashtekar equation. We plot the violation of the constraint (L2 norm of the Hamiltonian constraint equation), for the cases of plane wave propagation under the periodic boundary. To see the effect more clearly, we added artificial error at $t = 6$.

Fig.1 shows a part of our results. Here "adjusted-system" [Fig.(a)] means that we modified RHS of dynamical equations using constraint terms. The parameter κ indicates the multiplier of the adjust terms, and it changes the system's hyperbolicity. We see $\kappa = 2$ (strongly hyp.) is more effective than $\kappa = 1$ (sym. hyp.), this indicates that the symmetric hyperbolicity does not always the best for numerics. Fig.(b) is of λ-system, which introduce another variables λ and external flow so as to evolve the system to the constraint surface. α and β are parameters in λ-system. We see this idea actually works as desired.

We explain why such better performance appears by evaluating the eigen values of propagataion equations of the constraint equations. We conjectured that the stability can be obtained when the system has negative or non-zero (pure-imaginary) eigenvalues[7].

In the end, we would like to announce that this is the first numerical simulations that uses Ashtekar's connection variables.

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