P31

Singularity Avoidance in Gauss-Bonnet gravity

Takashi Torii & Hisa-aki Shinkai
(Osaka Inst. Technology, Japan)
鳥居隆，真貝寿明（大阪工業大）

1. Motivation

Dynamics in Gauss-Bonnet gravity?

- Action

$$S=\int_{\mathcal{M}} d^{N+1}x\sqrt{-g}\Big[\frac{1}{2\kappa^2}\{\alpha_1\mathcal{R}+\alpha_2\mathcal{L}_{\rm GB}\}+\mathcal{L}_{\rm matter}\Big]$$

where $\mathcal{L}_{GB}=\mathcal{R}^2-4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}+\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$

- Field equation

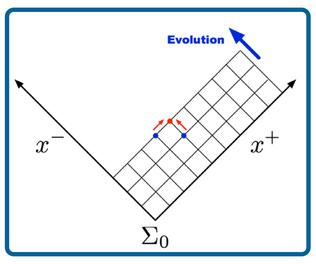
$$\alpha_1 G_{\mu\nu}+\alpha_2 H_{\mu\nu}+g_{\mu\nu}\Lambda=\kappa^2 T_{\mu\nu}$$

where $H_{\mu\nu}=2[\mathcal{R}\mathcal{R}_{\mu\nu}-2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu}-2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta}+\mathcal{R}_{\mu}{}^{\alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}]-\frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB}$

- **has GR correction terms from String Theory**
- **has two solution branches (GR/non-GR).**
- **is expected to have singularity avoidance feature. (but has never been demonstrated.)**

- **new topic in numerical relativity.**
  S Golod & T Piran, PRD 85 (2012) 104015
  N Deppe+, PRD 86 (2012) 104011
  F Izaurieta & E Rodriguez, 1207.1496
- **much attentions in WH community**
  H Maeda & M Nozawa, PRD 78 (2008) 024005
  P Kanti, B Kleihaus & J Kunz, PRL 107 (2011) 271101
  P Kanti, B Kleihaus & J Kunz, PRD 85 (2012) 044007

Field Eqs.



Field Equations (1)

Formulation for evolution [dual null]

Metric n-dimensional, dual-null coordinate, $2+(n-2)$ decomposition

$$ds^2=-2e^{-f(x^+,x^-)}dx^+dx^-+r^2(x^+,x^-)\gamma_{ij}dx^idx^j \quad (1)$$

Variables

$\Omega=\frac{1}{r}$	Conformal factor
$\vartheta_\pm=(n-2)\partial_\pm r$	expansion
f	lapse function
$\nu_\pm=\partial_\pm f$	inaffinity (shift)
ψ	scalar field (normal)
$\pi_\pm=r\partial_\pm\psi$	scalar momentum
ϕ	scalar field (ghost)
$p_\pm=r\partial_\pm\phi$	scalar momentum

Parameters

n	dimension
k	curvature
Λ	cosmological constant

For simplicity, we define

$$\tilde{\alpha}=(n-3)(n-4)\alpha_2, \quad (2)$$
$$A=\alpha_1+2\tilde{\alpha}\Omega^2 Z, \quad (3)$$
$$W=\frac{2e^f}{(n-2)^2}\vartheta_+\vartheta_-, \quad (4)$$
$$Z=k+W, \quad (5)$$
$$\eta=\Omega^2\frac{(n-2)(n-3)}{2}e^{-f}Z, \quad (6)$$

Field Equations (2)

matter variables

normal field $\psi(u,v)$ and/or ghost field $\phi(u,v)$

$$T_{\mu\nu}=T_{\mu\nu}(\psi)+T_{\mu\nu}(\phi)=\Big[\psi_{,\mu}\psi_{,\nu}-g_{\mu\nu}\Big(\frac{1}{2}(\nabla\psi)^2+V_1(\psi)\Big)\Big]+\Big[-\phi_{,\mu}\phi_{,\nu}-g_{\mu\nu}\Big(-\frac{1}{2}(\nabla\phi)^2+V_2(\phi)\Big)\Big]$$

this derives Klein-Gordon equations

$$\Box\psi=\frac{dV_1}{d\psi}, \qquad \Box\phi=\frac{dV_2}{d\phi}.$$

Scalar field variables

$$\pi_\pm\equiv r\partial_\pm\psi=\frac{1}{\Omega}\partial_\pm\psi$$
$$p_\pm\equiv r\partial_\pm\phi=\frac{1}{\Omega}\partial_\pm\phi$$

Klein-Gordon eqs.

$$\Box\phi=-\frac{e^f}{r}(2r\phi_{uv}+(n-2)r_u\phi_v+(n-2)r_v\phi_u)=-2e^f\phi_{uv}-e^f\Omega^2(\vartheta_-p_++\vartheta_+p_-)$$

Energy-momentum tensor

$$T_{++}=\Omega^2(\pi_+^2-p_+^2)$$
$$T_{--}=\Omega^2(\pi_-^2-p_-^2)$$
$$T_{+-}=-e^{-f}(V_1(\psi)+V_2(\phi))$$
$$T_{zz}=e^f(\pi_+\pi_--p_+p_-)-\frac{1}{\Omega^2}(V_1(\psi)-V_2(\phi))$$

Field Equations (3)

evolution equations (1)

Equations for x^+ direction

$$\partial_+\Omega=-\frac{1}{n-2}\vartheta_+\Omega^2 \quad (7)$$
$$\partial_+\vartheta_+=-\vartheta_+\nu_+-\frac{1}{\Omega A}\kappa^2T_{++}=-\vartheta_+\nu_+-\frac{1}{A}\kappa^2\Omega(\pi_+^2-p_+^2) \quad (8)$$
$$\partial_+\vartheta_-=\frac{1}{A}\frac{e^{-f}}{\Omega}\Big[-\alpha_1\Omega^2\frac{(n-2)(n-3)}{2}Z+\Lambda+\kappa^2(V_1+V_2)\Big]-\frac{\tilde{\alpha}}{A}\Omega^2e^{-f}\frac{(n-2)(n-5)}{2}\left[Z^2+W\right] \quad (9)$$
$$\partial_+f=\nu_+ \quad (10)$$
$$\partial_+\nu_+=\text{no evolution eq. exists}$$
$$\partial_+\nu_-=\frac{\alpha_1}{A}Ze^{-f}\Omega^2\frac{(n-3)}{2}\Big\{-\frac{\alpha_1}{A}2(n-3)+n-4\Big\}+\frac{1}{A}\Omega^2e^{-f}\kappa^2(\pi_+\pi_--p_+p_-)+\frac{1}{A}e^{-f}\Big\{\frac{\alpha_1}{A}\frac{2(n-3)}{(n-2)}-1\Big\}\{\Lambda+\kappa^2(V_1+V_2)\}$$
$$-\frac{\tilde{\alpha}}{A}e^{-f}\Omega^2(n-5)\times\Big[\frac{\alpha_1}{A}\Omega^2(n-3)\left\{k^2+2WZ+2Z^2\right\}+\frac{\tilde{\alpha}}{A}\Omega^2 2(n-5)\left\{k^2+2WZ\right\}Z\Big]$$
$$+\frac{\tilde{\alpha}}{A}e^{-f}\Omega^2(n-5)\times\Big[\frac{1}{2}\Omega^2\left\{(n-2)k^2+2WZ-4Z^2\right\}+\frac{1}{A}\frac{4}{n-2}Z\left\{\Lambda+\kappa^2(V_1+V_2)\right\}\Big]$$
$$-\frac{\tilde{\alpha}}{A}e^{-f}\Omega^2\frac{4}{(n-2)^2}\left\{\nu_+\vartheta_+(\partial_-\vartheta_-)+\nu_-\vartheta_-(\partial_+\vartheta_+)+(\partial_+\vartheta_+)(\partial_-\vartheta_-)+\nu_+\nu_-\vartheta_+\vartheta_--(\partial_-\vartheta_+)^2\right\} \quad (11)$$
$$\partial_+\psi=\Omega\pi_+ \quad (12)$$
$$\partial_+\phi=\Omega p_+ \quad (13)$$
$$\partial_+\pi_+=\text{no evolution eq. exists}$$
$$\partial_+\pi_-=\Big(\frac{1}{n-2}-\frac{1}{2}\Big)\Omega\vartheta_+\pi_--\frac{1}{2}\Omega\vartheta_-\pi_+-\frac{1}{2e^f\Omega}\frac{dV_1}{d\psi} \quad (14)$$
$$\partial_+p_+=\text{no evolution eq. exists}$$
$$\partial_+p_-=\Big(\frac{1}{n-2}-\frac{1}{2}\Big)\Omega\vartheta_+p_--\frac{1}{2}\Omega\vartheta_-p_+-\frac{1}{2e^f\Omega}\frac{dV_2}{d\phi} \quad (15)$$

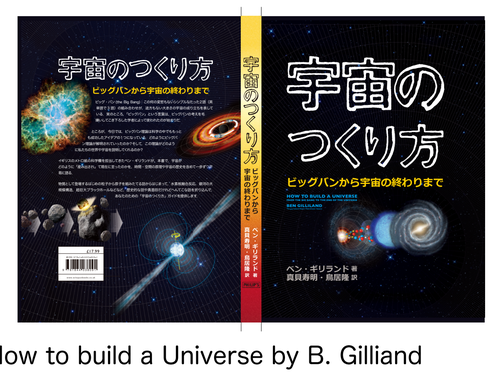
Field Equations (4)

evolution equations (2)

Equations for x^- direction

$$\partial_-\Omega=-\frac{1}{n-2}\vartheta_-\Omega^2 \quad (16)$$
$$\partial_-\vartheta_+=(9) \quad (17)$$
$$\partial_-\vartheta_-=-\vartheta_-\nu_--\frac{1}{\Omega A}\kappa^2T_{--}=-\vartheta_-\nu_--\frac{1}{A}\Omega\kappa^2(\pi_-^2-p_-^2) \quad (18)$$
$$\partial_-f=\nu_- \quad (19)$$
$$\partial_-\nu_+=(11) \quad (20)$$
$$\partial_-\nu_-=\text{no evolution eq. exists}$$
$$\partial_-\psi=\Omega\pi_- \quad (21)$$
$$\partial_-\phi=\Omega p_- \quad (22)$$
$$\partial_-\pi_+=-\frac{1}{2}\Omega\vartheta_+\pi_-+\Big(\frac{1}{n-2}-\frac{1}{2}\Big)\Omega\vartheta_-\pi_+-\frac{1}{2e^f\Omega}\frac{dV_1}{d\psi} \quad (23)$$
$$\partial_-\pi_-=\text{no evolution eq. exists}$$
$$\partial_-p_+=-\frac{1}{2}\Omega\vartheta_+p_-+\Big(\frac{1}{n-2}-\frac{1}{2}\Big)\Omega\vartheta_-p_+-\frac{1}{2e^f\Omega}\frac{dV_2}{d\phi} \quad (24)$$
$$\partial_-p_-=\text{no evolution eq. exists}$$

This constitutes the first-order dual-null form, suitable for numerical coding.



How to build a Universe by B. Gilliand
Japanese version translated by H.Shinkai, T. Torii
(Maruzen Pub.) will be released on Dec, 2016

Outline & Summary

We numerically investigated how the dynamics depends on the dimensionality and how the higher-order curvature terms affect to singularity formation in two models: (i) perturbed wormhole in spherically symmetric space-time, and (ii) colliding scalar pulses in planar space-time. Our numerical code uses dual-null formulation, and we compare the dynamics in 5, 6 and 7-dimensional General Relativity and Gauss-Bonnet (GB) gravity.

Both results suggest that **GB correction works for avoiding singularity formation in their dynamics**. We also found that **the existence of the trapped surface in GB gravity does not directly indicates formation of BH.**

Wormhole Evolutions

For wormhole dynamics, we observe that the perturbed throat will be easily enhance in the presence of GB term. If we inject large positive energy, then the throat turns to a blackhole, but that threshold of energy becomes larger for larger coupling constant, alpha, and for larger dimensions.

1. Motivation

Why Wormhole?

They increase our understanding of gravity when the usual energy conditions are not satisfied, due to quantum effects (Casimir effect, Hawking radiation) or alternative gravity theories, brane-world models etc.

They are very similar to black holes --both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH).

Wormhole = Hypersurface foliated by marginally trapped surfaces

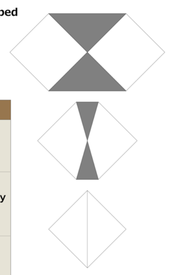
BH & WH are interconvertible?

S.A. Hayward, Int. J. Mod. Phys. D 8 (1999) 373

They are very similar -- both contain (marginally) trapped surfaces and can be defined by trapping horizons (TH)

Only the causal nature of the THs differs, whether THs evolve in plus / minus density which is given locally.

	Black Hole	Wormhole
Locally defined by	Achronal (spatial/null) outer TH → 1-way traversable	Temporal (timelike) outer THs → 2-way traversable
Einstein eqs.	Positive energy density normal matter (or vacuum)	Negative energy density "exotic" matter
Appear-ance	occur naturally	Unlikely to occur naturally, but constructible??



initial data

- Static condition

$$(\partial_++\partial_-)\Omega=0 \implies \vartheta_++\vartheta_-=0$$
$$(\partial_++\partial_-)\psi=0 \implies \pi_++\pi_-=0$$
$$(\partial_++\partial_-)\phi=0 \implies p_++p_-=0$$
$$\left.\begin{array}{l}(\partial_++\partial_-)\vartheta_+=0\\(\partial_++\partial_-)\vartheta_-=0\end{array}\right\} \implies \vartheta_+\nu_++\frac{1}{A}\Omega\kappa^2(\pi_+^2-p_+^2)=\vartheta_-\nu_-+\frac{1}{A}\Omega\kappa^2(\pi_-^2-p_-^2)$$

- Solve x^+ and x^- equations with the starting condition at the throat

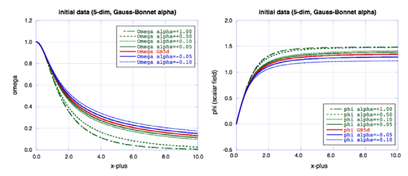
$$\vartheta_+=\vartheta_-(=0)$$
$$\nu_+=\nu_-(=0)$$
$$-\kappa^2\Omega(\pi_+^2-p_+^2)e^f=-\frac{1}{\Omega}\Big[-\alpha_1\Omega^2\frac{(n-2)(n-3)}{2}k+\Lambda+\kappa^2(V_1+V_2)\Big]+\tilde{\alpha}\Omega^3\frac{(n-2)(n-5)}{2}k^2$$

If we assume only ghost field ϕ, then

$$p_+=-p_-=\sqrt{\frac{1}{\kappa^2e^f}\Big[\alpha_1\frac{(n-2)(n-3)}{2}k-\frac{1}{\Omega^2}(\Lambda+\kappa^2V_2)+\tilde{\alpha}\Omega^2\frac{(n-2)(n-5)}{2}k^2\Big]}$$

- add perturbation

$$p_+(x^+=x,x^-=0)=p_+(\text{solution})+a\exp[-100(x-0.5)^2]$$

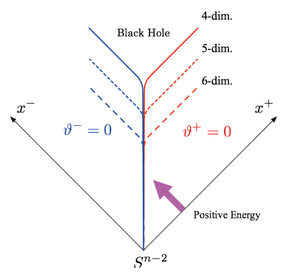


conformal factor　　scalar field

in GR

n-dim GR $\alpha_{\rm GB}=0$ Torii-HS PRD 88 (2013) 064027

Wormhole evolution in n-dim.



PHYSICAL REVIEW D 88, 064027 (2013)
TABLE I. The negative eigenvalues ω^2.

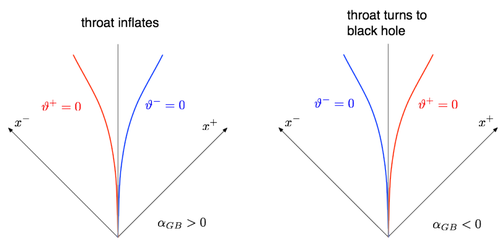
n	ω^2
4	−1.39705243371511
5	−2.98495893027790
6	−4.68662054299460
7	−6.46258414126318
8	−8.28975936306259
9	−10.1535530451867
10	−12.0442650147438
11	−13.9552091676647
20	−31.5751101285105
50	−91.3457759137153
100	−191.283017729717

$$f(t,r)=f_0(r)+\varepsilon f_1(r)e^{i\omega t}, \quad (3.1)$$
$$\delta(t,r)=\delta_0(r)+\varepsilon\delta_1(r)e^{i\omega t}, \quad (3.2)$$
$$R(t,r)=R_0(r)+\varepsilon R_1(r)e^{i\omega t}, \quad (3.3)$$
$$\phi(t,r)=\phi_0(r)+\varepsilon\phi_1(r)e^{i\omega t}. \quad (3.4)$$

Higher dim -> catch instability earlier

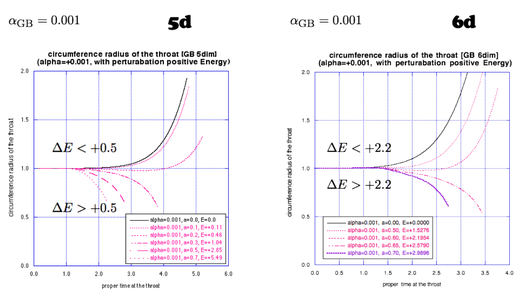
in GB

5d GR vs Gauss-Bonnet WH : instability appears



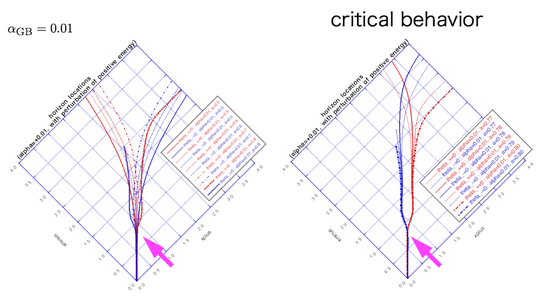
GB terms make wormhole unstable. Final structure depends on the signature of coupling constant, but normal GB prefers not to form a blackhole.

5d, 6d Gauss-Bonnet WH



need more positive energy for transition to BH in 6dim

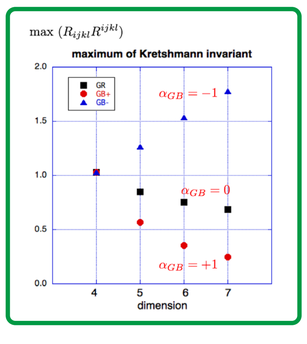
5d Gauss-Bonnet WH : trapped surface



existence of trapped surface
—> not necessary to form a BH

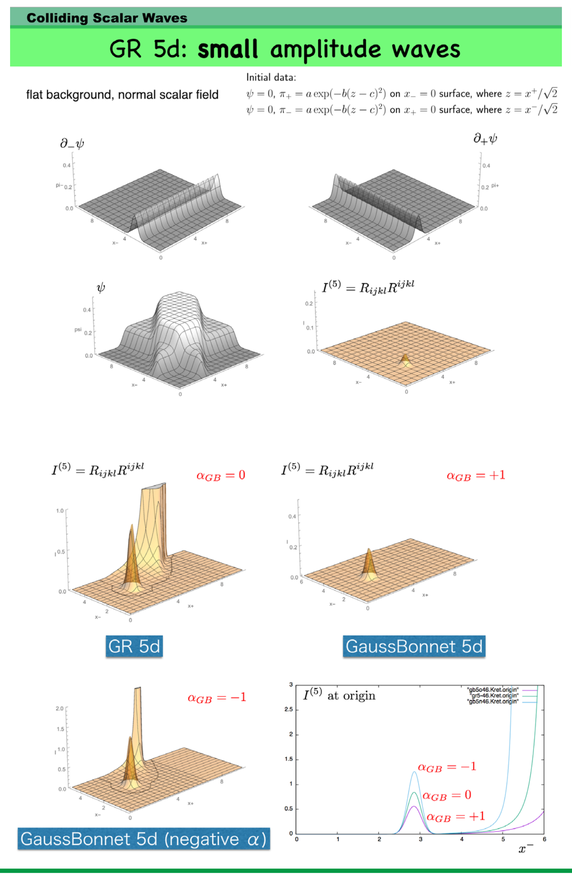
Colliding Scalar Waves

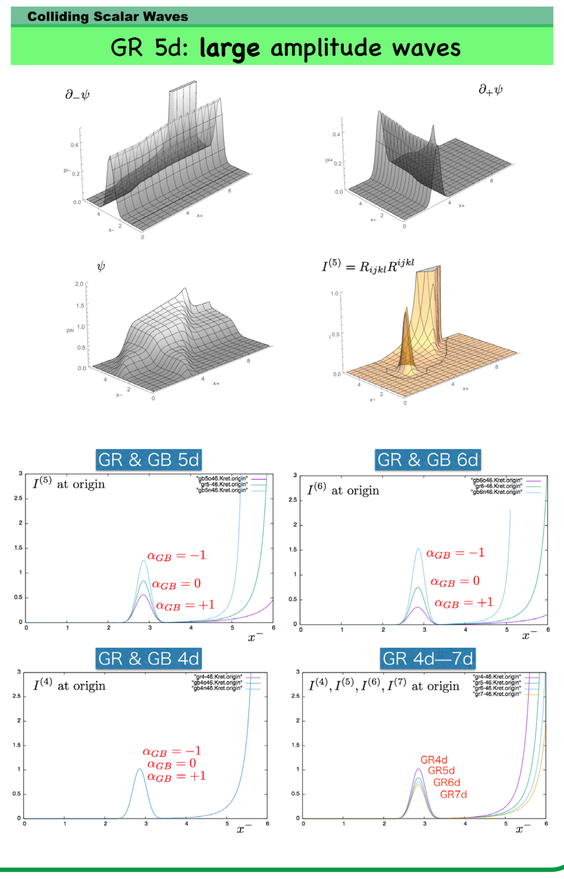
For scalar wave collisions, we observe that curvature (Kretschmann invariant) evolves milder in the presence of GB term and/or in higher-dimensional space-time, while the singularity formation is inevitable.



Colliding Scalar Waves

GR 5d: **small** amplitude waves



Colliding Scalar Waves

GR 5d: **large** amplitude waves