一般相対論の数値計算手法

近畿大セミナー
2011/12/9-10

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Contents

Procedure of the Standard Numerical Relativity

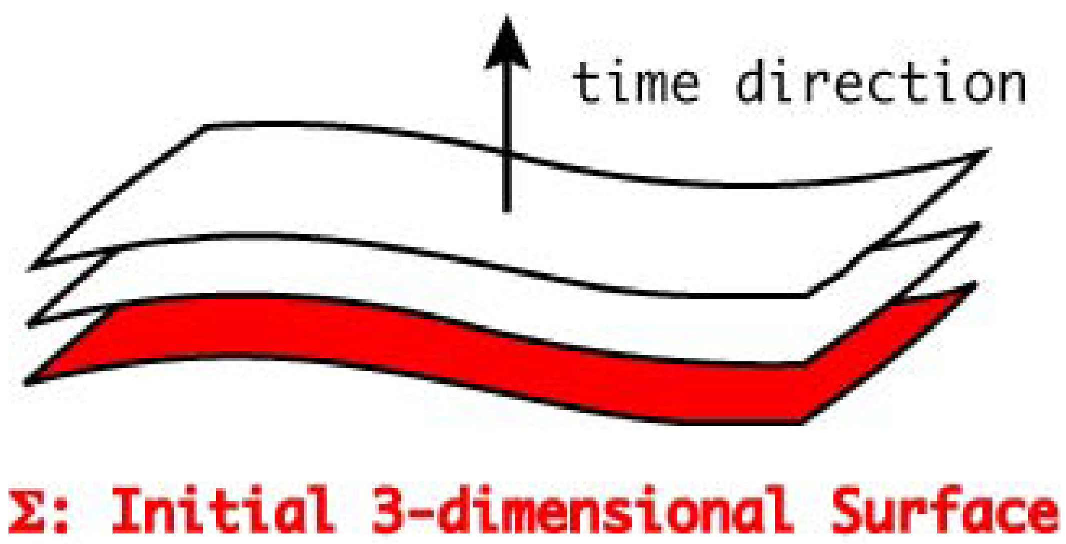
- 3+1 (ADM) formulation

- Preparation of the Initial Data
  - Assume the background metric
  - Solve the constraint equations

- Time Evolution

  do time=1, time_end
  - Specify the slicing condition
  - Evolve the variables
  - Check the accuracy
  - Extract physical quantities

  end do



The 3+1 decomposition of space-time, The ADM formulation

[1] R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: An Introduction to Current Research*, ed. by L.Witten, (Wiley, New York, 1962).
[2] J.W. York, Jr. in *Sources of Gravitational Radiation*, (Cambridge, 1979)

Dynamics of Space-time = Foliation of Hypersurface

- Evolution of t =const. hypersurface $\Sigma(t)$.

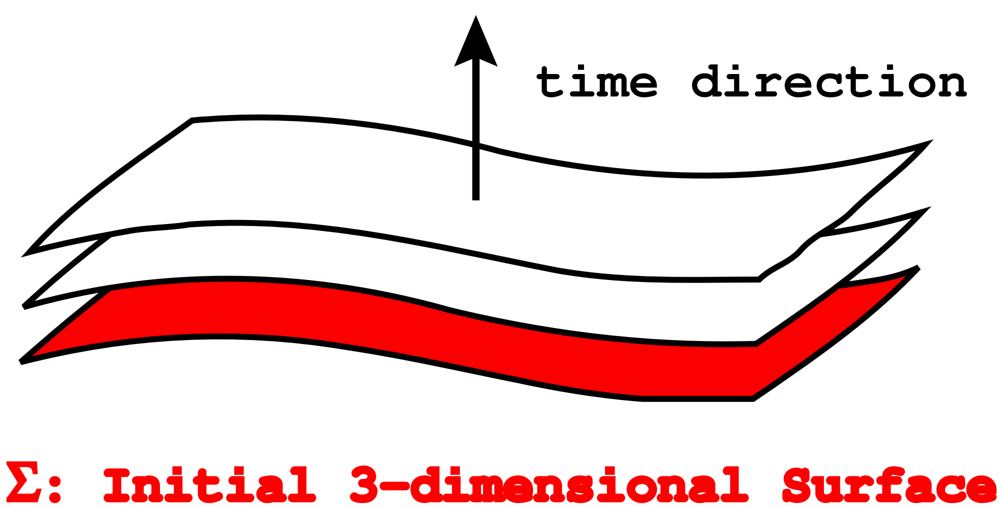
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3)$$

on $\Sigma(t)$... $$d\ell^2 = \gamma_{ij} dx^i dx^j, \qquad (i, j = 1, 2, 3)$$

- The unit normal vector of the slices, n^μ.

$$n_\mu = (-\alpha, 0, 0, 0)$$
$$n^\mu = g^{\mu\nu} n_\nu = (1/\alpha, -\beta^i/\alpha)$$

- The lapse function, α. The shift vector, β^i.



$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

The decomposed metric:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$
$$= (-\alpha^2 + \beta_l \beta^l) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j$$

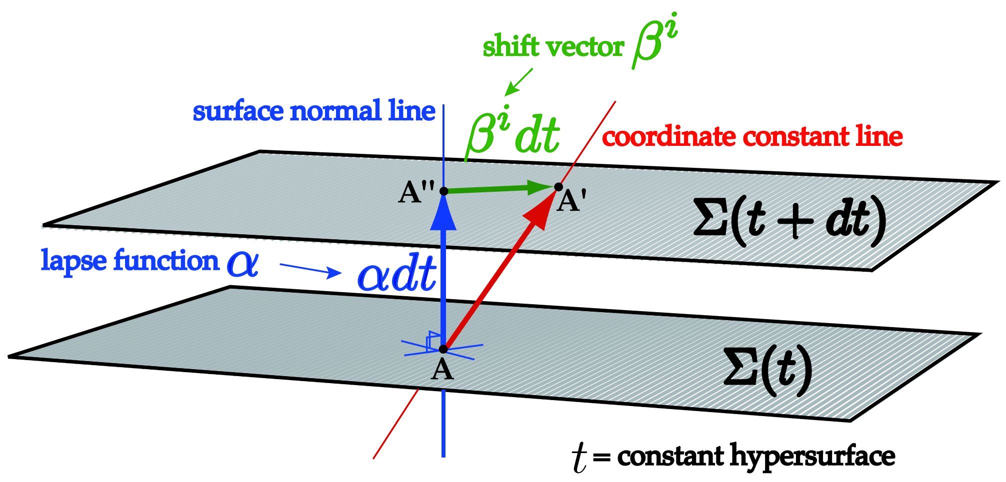
$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_l \beta^l & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}$$

where α and β_j are defined as $\alpha \equiv 1/\sqrt{-g^{00}}, \quad \beta_j \equiv g_{0j}$.

- The unit normal vector of the slices, n^μ.

$$n_\mu = (-\alpha, 0, 0, 0)$$
$$n^\mu = g^{\mu\nu} n_\nu = (1/\alpha, -\beta^i/\alpha)$$

- The lapse function, α.
- The shift vector, β^i.



Projections to Hypersurface Σ (spacelike or timelike) (1)

- Projection operator (or intrinsic 3-metric) to $\Sigma(t)$,

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$
$$\gamma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu \equiv \perp^\mu_\nu$$

where n_μ is the unit-normal timelike vector to Σ.

Remark: the projection operator can be defined also to the timelike hypersurface.

$$\perp_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu, \quad n_\mu n^\mu = \varepsilon \tag{1}$$

where n_μ is the unit-normal vector to Σ with n_μ is timelike (if $\varepsilon = -1$) or spacelike (if $\varepsilon = 1$). Σ is spacelike (timelike) if n_μ is timelike (spacelike).

- Projection of the Einstein equation:

$$G_{\mu\nu} n^\mu n^\nu = \kappa^2 T_{\mu\nu} n^\mu n^\nu \equiv \kappa^2 \rho_H \quad \Rightarrow \text{the Hamiltonian constraint eq.} \tag{2}$$
$$G_{\mu\nu} n^\mu \perp^\nu_i = \kappa^2 T_{\mu\nu} n^\mu \perp^\nu_i \equiv -\kappa^2 J_i \quad \Rightarrow \text{the momentum constraint eqs.} \tag{3}$$
$$G_{\mu\nu} \perp^\mu_i \perp^\nu_j = \kappa^2 T_{\mu\nu} \perp^\mu_i \perp^\nu_j \equiv \kappa^2 S_{ij} \quad \Rightarrow \text{the evolution eqs.} \tag{4}$$

where we defined

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}, \;\; T = -\rho_H + S^\ell{}_\ell$$

Projections to Hypersurface Σ (spacelike or timelike) (2)

- The projections of the Eisntein equation:

$$G_{\mu\nu}\, n^\mu\, n^\nu = \kappa^2\, T_{\mu\nu}\, n^\mu\, n^\nu =: \kappa^2 \rho_H,$$
$$G_{\mu\nu}\, n^\mu \perp^\nu_\rho = \kappa^2\, T_{\mu\nu}\, n^\mu \perp^\nu_\rho =: -\kappa^2 J_\rho,$$
$$G_{\mu\nu} \perp^\mu_\rho \perp^\nu_\sigma = \kappa^2\, T_{\mu\nu} \perp^\mu_\rho \perp^\nu_\sigma =: \kappa^2 S_{\rho\sigma},$$

where we defined

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}, \;\; T = -\rho_H + S^\ell{}_\ell$$

- Introduce the extrinsic curvature K_{ij},

$$K_{ij} := -\frac{1}{2}\pounds_n h_{ij} = -\perp^\mu_i \perp^\nu_j \nabla_\nu n_\mu = -(\delta^\mu_i + n^\mu n_i)(\delta^\nu_j + n^\nu n_j)\nabla_\nu n_\mu$$
$$= -\nabla_j n_i = \Gamma^\alpha_{ij} n_\alpha = \cdots = \frac{1}{2\alpha}\left(-\partial_t \gamma_{ij} + D_j \beta_i + D_i \beta_j\right). \quad (5)$$

where $\pounds_n$ denotes the Lie derivative in the 3-dimension and ∇ and D_i is the covariant differentiation with respect to $g_{\mu\nu}$ and γ_{ij}, respectively.

Projections to Hypersurface Σ (spacelike or timelike) (3)

- Projection of the $(3+1)$-dimensional Riemann tensor onto Σ_N

Gauss eq. $$\mathcal{R}_{\alpha\beta\gamma\delta} \perp^{\alpha}_{\ i} \perp^{\beta}_{\ j} \perp^{\gamma}_{\ k} \perp^{\delta}_{\ l} = R_{ijkl} - \varepsilon K_{ik}K_{jl} + \varepsilon K_{il}K_{jk}, \tag{6}$$

Codacci eq. $$\mathcal{R}_{\alpha\beta\gamma\delta} \perp^{\alpha}_{\ i} \perp^{\beta}_{\ j} \perp^{\gamma}_{\ k} n^{\delta} = -2D_{[i}K_{j]k}, \tag{7}$$

$$\mathcal{R}_{\alpha\beta\gamma\delta} \perp^{\alpha}_{\ i} \perp^{\gamma}_{\ k} n^{\beta} n^{\delta} = \pounds_n K_{ik} + K_{i\ell}\, K^{\ell}_{\ k}\,, \tag{8}$$

- Curvature relations

$$\begin{aligned}\mathcal{R}_{\mu\nu\rho\sigma} = {} & R_{\mu\nu\rho\sigma} - \varepsilon(K_{\mu\rho}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\rho} - n_\mu D_\rho K_{\nu\sigma} + n_\mu D_\sigma K_{\rho\nu} + n_\nu D_\rho K_{\sigma\mu} - n_\nu D_\sigma K_{\rho\mu} \\ & - n_\rho D_\mu K_{\nu\sigma} + n_\rho D_\nu K_{\mu\sigma} + n_\sigma D_\mu K_{\nu\rho} - n_\sigma D_\nu K_{\mu\rho}) \\ & + n_\mu n_\rho K_{\nu\alpha}K^{\alpha}_{\ \sigma} - n_\mu n_\sigma K_{\nu\alpha}K^{\alpha}_{\ \rho} - n_\nu n_\rho K_{\mu\alpha}K^{\alpha}_{\ \sigma} + n_\nu n_\sigma K_{\mu\alpha}K^{\alpha}_{\ \rho} \\ & + n_\mu n_\rho \pounds_n K_{\nu\sigma} - n_\mu n_\sigma \pounds_n K_{\nu\rho} - n_\nu n_\rho \pounds_n K_{\mu\sigma} + n_\nu n_\sigma \pounds_n K_{\mu\rho},\end{aligned} \tag{9}$$

$$\begin{aligned}\mathcal{R}_{\mu\nu} = {} & R_{\mu\nu} - \varepsilon[K K_{\mu\nu} - 2K_{\mu\alpha}K^{\alpha}_{\ \nu} + n_\mu\left(D_\alpha K^{\alpha}_{\ \nu} - D_\nu K\right) + n_\nu\left(D_\alpha K^{\alpha}_{\ \mu} - D_\mu K\right)] \\ & + n_\mu n_\nu K_{\alpha\beta}K^{\alpha\beta} + \varepsilon \pounds_n K_{\mu\nu} + n_\mu n_\nu \gamma^{\alpha\beta}\pounds_n K_{\alpha\beta},\end{aligned} \tag{10}$$

$$\mathcal{R} = R - \varepsilon(K^2 - 3K_{\alpha\beta}K^{\alpha\beta} - 2\gamma^{\alpha\beta}\pounds_n K_{\alpha\beta}). \tag{11}$$

The Standard ADM formulation (aka York 1978):

The fundamental dynamical variables are (γ_{ij}, K_{ij}), the three-metric and extrinsic curvature. The three-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i, \\
\partial_t K_{ij} &= \alpha\ {}^{(3)}R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - D_i D_j \alpha \\
&\quad + (D_i \beta^k) K_{kj} + (D_j \beta^k) K_{ki} + \beta^k D_k K_{ij} \\
&\quad - 8\pi G \alpha \{ S_{ij} + (1/2) \gamma_{ij} (\rho_H - \mathrm{tr} S) \},
\end{aligned}$$

  where $K = K^i{}_i$, and ${}^{(3)}R_{ij}$ and D_i denote three-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$\begin{aligned}
&\text{Hamiltonian constr.} \qquad \mathcal{H}^{ADM} := {}^{(3)}R + K^2 - K_{ij} K^{ij} \approx 0, \\
&\text{momentum constr.} \qquad \mathcal{M}_i^{ADM} := D_j K^j{}_i - D_i K \approx 0,
\end{aligned}$$

  where ${}^{(3)}R = {}^{(3)}R^i{}_i$.

Original ADM The original construction by ADM uses the pair of (h_{ij}, π^{ij}).

$$\mathcal{L} = \sqrt{-g}R = \sqrt{h}N[^{(3)}R - K^2 + K_{ij}K^{ij}], \quad \text{where } K_{ij} = \frac{1}{2}\pounds_n h_{ij}$$

$$\text{then} \quad \pi^{ij} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}),$$

The Hamiltonian density gives us constraints and evolution eqs.

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} = \sqrt{h}\left\{N\mathcal{H}(h,\pi) - 2N_j\mathcal{M}^j(h,\pi) + 2D_i(h^{-1/2}N_j\pi^{ij})\right\},$$

$$\begin{cases} \partial_t h_{ij} = \dfrac{\delta \mathcal{H}}{\delta \pi^{ij}} = 2\dfrac{N}{\sqrt{h}}(\pi_{ij} - \dfrac{1}{2}h_{ij}\pi) + 2D_{(i}N_{j)}, \\ \partial_t \pi^{ij} = -\dfrac{\delta \mathcal{H}}{\delta h_{ij}} = -\sqrt{h}N(^{(3)}R^{ij} - \dfrac{1}{2}{}^{(3)}Rh^{ij}) + \dfrac{1}{2}\dfrac{N}{\sqrt{h}}h^{ij}(\pi_{mn}\pi^{mn} - \dfrac{1}{2}\pi^2) - 2\dfrac{N}{\sqrt{h}}(\pi^{in}\pi_n{}^j - \dfrac{1}{2}\pi\pi^{ij}) \\ \qquad + \sqrt{h}(D^iD^jN - h^{ij}D^mD_mN) + \sqrt{h}D_m(h^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)} \end{cases}$$

Standard ADM (by York) NRists refer ADM as the one by York with a pair of (h_{ij}, K_{ij}).

$$\begin{cases} \partial_t h_{ij} = -2NK_{ij} + D_jN_i + D_iN_j, \\ \partial_t K_{ij} = N(\,^{(3)}R_{ij} + KK_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN + (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} \end{cases}$$

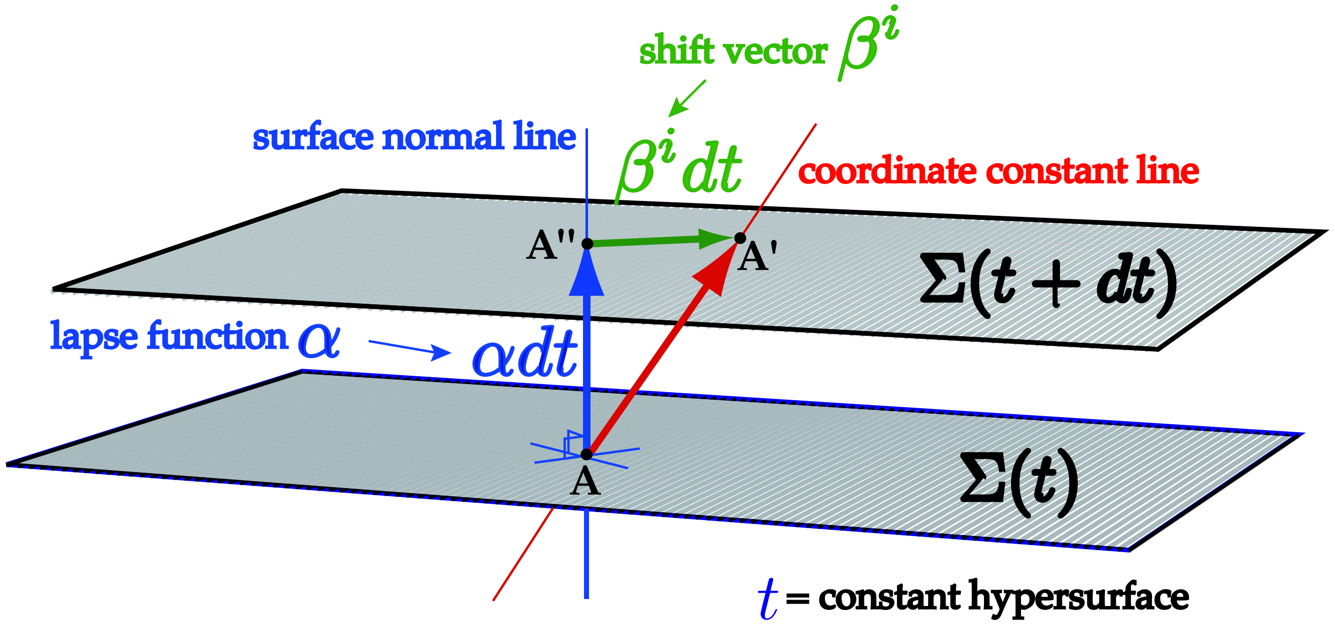
In the process of converting, $\mathcal{H}$ was used, i.e. the standard ADM has already adjusted.

strategy 0 The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

3+1 decomposition of the spacetime.

Evolve 12 variables (γ_{ij}, K_{ij})

with a choice of gauge condition.



	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$ div $\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} = \text{rot } \mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -\text{rot } \mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

S. Frittelli, Phys. Rev. D55, 5992 (1997)

HS and G. Yoneda, Class. Quant. Grav. 19, 1027 (2002)

The Constraint Propagations of the Standard ADM:

$$\begin{aligned}\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha\gamma^{ij}(\partial_i \mathcal{M}_j) \\ &\quad +\alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \\ \partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\ &\quad +\alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j.\end{aligned}$$

That is, the constraints are the first class.

From these equations, we know that

if the constraints are satisfied on the initial slice Σ,
then the constraints are satisfied throughout evolution (in principle).

But this is not true in numerics....

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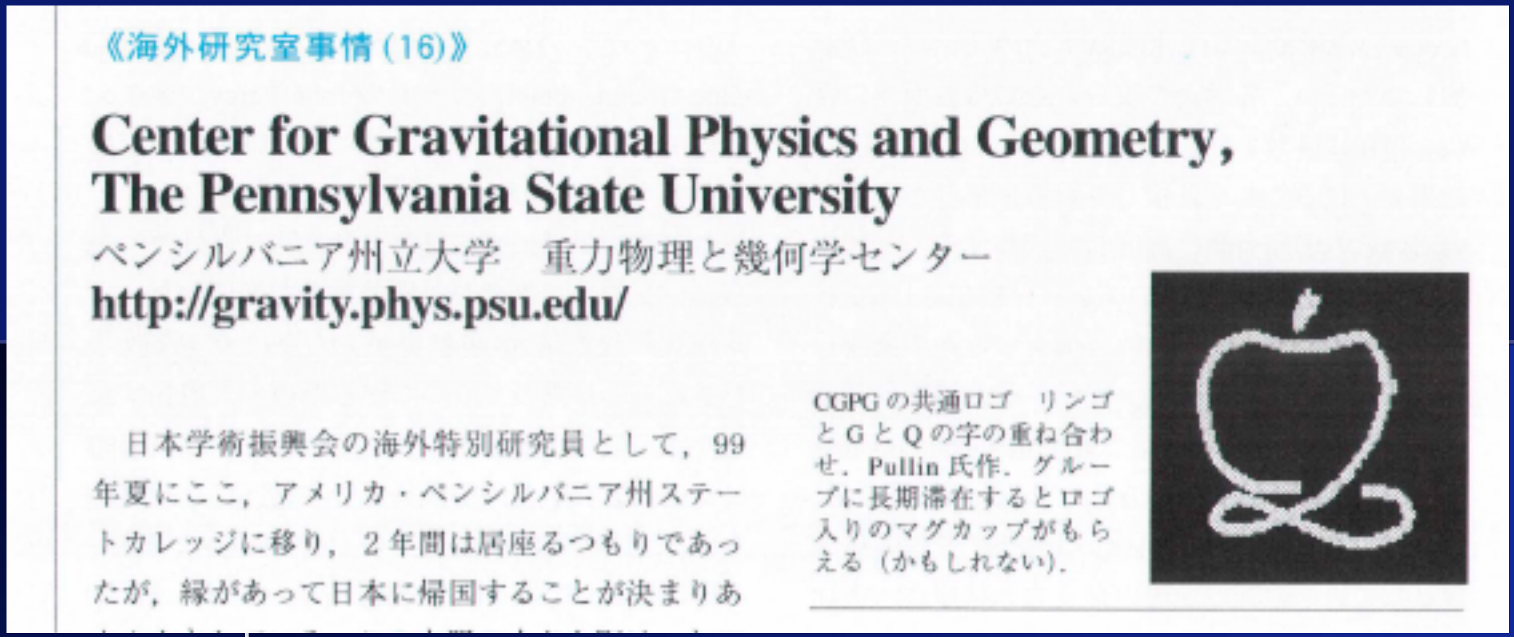


《海外研究室事情 (16)》

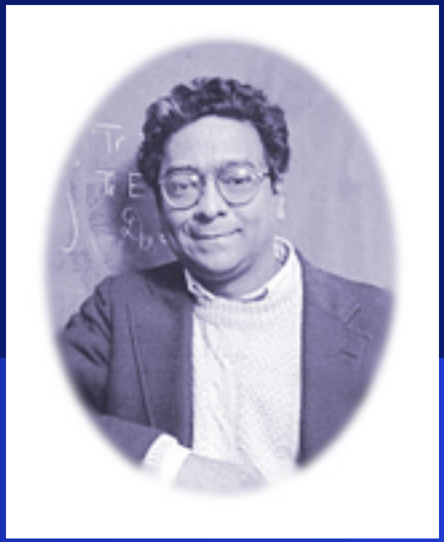
Center for Gravitational Physics and Geometry, The Pennsylvania State University

ペンシルバニア州立大学　重力物理と幾何学センター

http://gravity.phys.psu.edu/

日本学術振興会の海外特別研究員として，99年夏にここ，アメリカ・ペンシルバニア州ステートカレッジに移り，2年間は居座るつもりであったが，縁があって日本に帰国することが決まりあ

CGPG の共通ロゴ　リンゴとGとQの字の重ね合わせ．Pullin 氏作．グループに長期滞在するとロゴ入りのマグカップがもらえる（かもしれない）．



「天文月報」2001年4月号

http://www.gravity.psu.edu/

重力場と電磁気場の比較

- 一般相対論は，時空の各点で局所座標系を選択できる（等価原理）とする．
- ゲージ理論は，理論が局所対称性（不変性）をもつ，とする．

$\Psi(x) \longrightarrow e^{i\alpha}\Psi(x)$ ：大域対称性（大局的ゲージ不変性）$\Longrightarrow$ 保存則

$\Psi(x) \longrightarrow e^{i\alpha(x)}\Psi(x)$ ：局所対称性（局所的ゲージ不変性）$\Longrightarrow$ 相互作用

	重力場	電磁気場
自由場の方程式	$\frac{d^2}{dt^2}\boldsymbol{x} = 0$	$(i\gamma^\mu\partial_\mu - m)\Psi = 0$
対称性	一般座標変換 $(x^\mu \longrightarrow x^{\mu'})$	局所ゲージ変換 $(\Psi \longrightarrow e^{i\alpha(x)}\Psi)$
共変微分	$\nabla_\mu = \partial_\mu + \Gamma$	$D_\mu \equiv \partial_\mu + iqA_\mu$
接続係数	${\Gamma^\mu}_{\alpha\beta}$ (局所的に 0 とできる)	A_μ (直接観測できない，ゲージ依存)
相互作用	$\frac{d^2x^\alpha}{d\tau^2} + {\Gamma^\alpha}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$	$(i\gamma^\mu(\partial_\mu + iqA_\mu) - m)\,\Psi = 0$
共変微分の非可換性	曲率テンソル　${R^\mu}_{\alpha\beta\gamma}$ (観測可能量)	電磁場テンソル　$F_{\mu\nu}$ (ゲージ不変量)

From Einstein to Ashtekar: via transformation of Lagragian (1)

Step 1: Einstein-Hilbert action (metric $g_{\mu\nu}$)

$$S_E[g] = \int d^4x \sqrt{-g}\, R(g) \quad \sim \quad g\, \partial^2 g + (\partial g)^2 \tag{1}$$

- Construct a canonical theory by means of the ADM method;

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2\, dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$
$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & q_{ij} \end{pmatrix}.$$

- The canonical action, then, is given by

$$S_E[q,p] = \int d^4x\, [\dot{q}_{ij}\, p^{ij} - N\, \mathcal{C}_{\mathcal{H}} - N_i\, \mathcal{C}_{\mathcal{M}}{}^i] \tag{2}$$
$$\text{where} \quad \mathcal{C}_{\mathcal{H}} := G_{ijkl} p^{ij} p^{kl} - \sqrt{q}{}^{(3)}R$$
$$\mathcal{C}_{\mathcal{M}}{}^i := -2\nabla_j p^{ij}$$

where $G_{ijkl} = \dfrac{1}{2\sqrt{q}}(q_{ik}q_{jl} + q_{il}q_{jk} - q_{ij}q_{kl})$.

From Einstein to Ashtekar: via transformation of Lagragian (2)

Step 2: Palatini action (metric $g_{\mu\nu}$, Affine connection $\Gamma^{\alpha}_{\mu\nu}$)

- Einstein-Hilbert action consists of the terms with the second-order derivative or the square of the first order derivative of metric $g_{\mu\nu}$.
- Palatini's idea is to introduce the Affine connection $\Gamma^{\alpha}_{\mu\nu} (= \Gamma^{\alpha}_{\nu\mu})$ to be independent to the metric $g_{\mu\nu}$. The Palatini action is

$$S_P[g, \Gamma] = \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma) \quad \sim \quad g(\partial\Gamma + \Gamma\,\Gamma) \tag{3}$$

- $S_P = S_E$ when $\Gamma^{\lambda}_{\mu\nu}$ satisfies the definition of the Christoffel symbol, $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}(g) \sim \partial g$. This condition is derived from the variation with respect to $\Gamma^{\alpha}_{\mu\nu}$,

$$\frac{\delta}{\delta \Gamma^{\alpha}_{\mu\nu}} S_P[g, \Gamma] = 0.$$

- The action (3) contains up to the first-order derivatives.

From Einstein to Ashtekar: via transformation of Lagragian (Summary)

Theory	action		order of ∂_μ	independent variables
Einstein	Einstein-Hilbert action	S_E	2nd order	metric $(g_{\mu\nu})$
	Palatini action	S_P	1st order	metric $(g_{\mu\nu})$ & Affine connection $(\Gamma^\lambda_{\mu\nu})$
	Tetrad Palatini action	S_T	1st order	tetrad $(e^a{}_\mu)$ & spin connection (ω^{ab}_μ)
Ashtekar	Jacobson-Smolin action	${}^+S_T$	1st order	tetrad $(e^a{}_\mu)$ & self-dual connection $({}^+\omega^{ab}_\mu)$

S_P with the Christoffel condition for Γ	$\Longrightarrow$	S_E
S_T with the Levi-Civita condition for ω^{ab} (torsion free condition)	$\Longrightarrow$	S_P
${}^+S_T$ with the Bianchi identity for R_{ab} $(R_{\mu[\nu\alpha\beta]} = 0)$	$\Longrightarrow$	S_T

From Einstein to Ashtekar: via transformation of Lagragian (3)

Step 3: Tetrad Palatini action (tetrad e^a_μ, spin connection ω^{ab}_μ)

- The next step is the introduction of the internal symmetry, that is, to introduce the local Lorentz transformation as a gauge symmetry.
- We employ the orthonormal tetrad e^a_μ in stead of the metric $g_{\mu\nu}$, which acts as a basis of the local Lorentz frame.
- We also employ the spin connection $\omega^{ab}_\mu (= -\omega^{ba}_\mu)$ instead of the Affine connection $\Gamma^\alpha_{\mu\nu}$, which acts as a gauge field of the local Lorentz albebra so(3,1).
- The internal indices $a, b, \cdots$ are lowering and raising by the metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. The tetrad plays a role of a square root of the metric,

$$g_{\mu\nu} = \eta_{ab}\, e^a_\mu\, e^b_\nu. \tag{4}$$

- The Palatini action in the tetrad form

$$S_P[g, \Gamma] = \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma)$$
$$S_T[e, \omega] = \int d^4x\, e\, E^\mu_a E^\nu_b R^{ab}{}_{\mu\nu}(\omega) \tag{5}$$

where e is the determinant of e^a_μ, and the E^μ_a is the inverse tetrad,

$$e := \det e^a_\mu = \sqrt{-g}, \qquad E^\mu_a = e^b_\nu\, g^{\mu\nu}\, \eta_{ab}.$$

- Now that the internal symmetry is taken into account, the Riemann curvature $R^\alpha{}_{\beta\mu\nu}$ will be replaced by the curvature $R^{ab}_{\mu\nu}(\omega)$ of the spin connection ω^{ab}_μ defined by

$$R^{ab}_{\mu\nu}(\omega) := \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^a_{\mu c}\omega^{cb}_\nu - \omega^a_{\nu c}\omega^{cb}_\mu,$$
$$\text{i.e.} \quad R^{ab}(\omega) := d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$$

- The action (5), then, can be expressed also as

$$S_T[e, \omega] = \int (1/2)\, \varepsilon_{abcd} R^{ab}(\omega) \wedge e^c \wedge e^d. \tag{6}$$

- $S_P = S_T$ only when $\omega^{ab}(e)$ satisfies the torsion free condition (Levi-Civita connection)

$$De^a := de^a + \omega^a{}_b \wedge e^b = 0$$

which is derived from $\dfrac{\delta}{\delta\omega^{ab}} S_T[e, \omega] = 0.$

自己双対性 (self-duality), anti-self-duality

- For an anti-symmetric tensor F_{ab}, the duality transformation is defined as
$$ {}^{*}F_{ab} := \frac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd}. $$
- The dual of dual is equal to the minus of the original,
$$ {}^{*}({}^{*}F_{ab}) = -F_{ab} $$
when we choose the Lorentzian signature and use the metric η_{ab}.
- If we suppose the complex combinations
$$ {}^{\pm}F_{ab} = \frac{1}{2}(F_{ab} \mp i\,{}^{*}F_{ab}), $$
then this satisfies the eigen-equations
$$ {}^{*}({}^{\pm}F_{ab}) = \pm i\,{}^{\pm}F_{ab}. $$

${}^{+}F_{ab}$: self-dual part of F_{ab}
${}^{-}F_{ab}$: anti-self-dual part of F_{ab}.

From Einstein to Ashtekar: via transformation of Lagragian (4)

Step 4: Self-dual action (tetrad e^a_μ, self-dual connection ${}^+\omega^{ab}_\mu$)

- Introduction of the self-dual connection ${}^+\omega^{ab}_\mu$.

$$\omega^{ab} = {}^+\omega^{ab} + {}^-\omega^{ab}.$$

- The curvature 2-form R^{ab} results in

$$R^{ab}(\omega) = R^{ab}({}^+\omega^{ab} + {}^-\omega^{ab}) = R^{ab}({}^+\omega^{ab}) + R^{ab}({}^-\omega^{ab}) := {}^+R^{ab} + {}^-R^{ab}$$

- The previously mentioned tetrad-Palatini action (6)

$$S_T[e,\omega] = \int \frac{1}{2}\varepsilon_{abcd}R^{ab}(\omega) \wedge e^c \wedge e^d = \int {}^*R_{cd}(\omega) \wedge e^c \wedge e^d$$

is decomposed as

$$\begin{aligned} S_T[e,\omega] &= \int {}^*R_{ab}({}^+\omega) \wedge e^a \wedge e^b + \int {}^*R_{ab}({}^-\omega) \wedge e^a \wedge e^b \\ &:= {}^+S_T[e, {}^+\omega] + {}^-S_T[e, {}^-\omega] \end{aligned} \tag{7}$$

with regard to the contributions of self-dual and anti-self-dual connections.

Ashtekar's idea is to consider just a self-dual part of the action.

- When the self-dual connection is equal to the self-dual part of the Levi-Civita connection

$$ {}^{+}\omega^{ab} = {}^{+}\omega^{ab}(e), $$

the variation $\dfrac{\delta}{\delta^{+}\omega}\,{}^{+}S_T[e, {}^{+}\omega] = 0$ is satisfied. Then ${}^{+}S_T[e, {}^{+}\omega] = \frac{1}{2}S_T[e, \omega(e)] = \frac{1}{2}S_E[g]$.

- The equivalence to the Einstein theory requires additional condition. Since

$$ R^{ab}({}^{+}\omega) = {}^{+}R^{ab}(\omega) = \frac{1}{2}\left(R^{ab}(\omega) - i\,{}^{*}R^{ab}(\omega)\right), $$

the action turns out to be

$$ \begin{aligned} {}^{+}S_T[e, {}^{+}\omega(e)] &= \frac{1}{2}\int {}^{*}\left(R_{ab}(\omega(e)) - i\,{}^{*}R_{ab}(\omega(e))\right) \wedge e^a \wedge e^b \\ &= \frac{1}{2}\int \left({}^{*}R_{ab}(\omega(e)) + iR_{ab}(\omega(e))\right) \wedge e^a \wedge e^b \\ &= \frac{1}{2}S_T[e, \omega(e)] + i\frac{1}{2}\int R_{ab}(\omega(e)) \wedge e^a \wedge e^b = \frac{1}{2}S_E[g] + 0, \end{aligned} $$

where the last imaginary term is vanished by virtue of the 1st Bianchi identity

$$ R^a{}_b(\omega(e)) \wedge e^b \equiv 0 $$

which is the cyclic identity $R_{\mu[\nu\alpha\beta]} = 0$ in the tensor form.

テトラド (tetrad), トライアド (triad)

- 各時空点ごとに局所的な 4 次元直交座標系を定義する．直交座標の基底ベクトルを E^I として，これを任意の座標系で表したもの E^I_μ を**テトラド（4 脚場）**と呼ぶ．

$$g_{\mu\nu} = E^I_\mu E^J_\nu \eta_{IJ}, \qquad \eta_{IJ} = \text{diag}\,(-1,1,1,1)$$

- 同様に，3 次元空間で局所的に直交座標を導入した基底ベクトルを**トライアド（3 脚場）**と呼ぶ．

$$g_{ij} = E^a_i E^b_j \delta_{ab}$$

スピン接続 (spin connection)

- 局所直交座標系の成分を持つベクトルに対する共変微分を

$$\nabla_\mu V^I = \partial_\mu V^I + \omega^I_{\mu J} V^J$$

と表すとき，$\omega^I_{\mu J}$ を**スピン接続**　と呼ぶ．具体的には，

$$\omega^{IJ}_\mu = E^{I\nu}\nabla_\mu E^J_\nu = E^{\nu I}\partial_{[\mu}E^J_{\nu]} - E_{\mu K}E^{\rho I}E^{\nu J}\partial_{[\rho}E^K_{\nu]} + E^{\rho J}\partial_{[\rho}E^I_{\mu]}$$

From Einstein to Ashtekar: via transformation of Lagragian (5)

Step 5: New Variables (densitized inverse triad $\tilde{E}^i_a$, self-dual connection ${}^+\mathcal{A}^a_\mu$)

The self-dual action would lead to the same equation of motion as the Einstein equation so far as the tetrad or equivalently the metric is concerned.

- The Ashtekar formalism can be regarded as a canonical theory starting from the self-dual action,

$$ {}^+S_T[e, {}^+\omega] = \int d^4x \, e \, E^\mu_a \, E^\nu_b \, R^{ab}{}_{\mu\nu}({}^+\omega). \tag{8} $$

where E^μ_a is the inverse tetrad, defined as $E^\mu_a := E^b_\nu g^{\mu\nu}\eta_{ab}$, which makes the inverse space-time metric as $q^{\mu\nu} = \eta^{ab}E^\mu_a E^\nu_b$ as we mentioned before.

- $3+1$ decomposition of the self-dual theory in the tetrad form.

The spatial component of the tetrad, E^i_I acts as an inverse triad since it produces the inverse 3-metric, $q^{ij} = E^i_I E^j_I$. We further impose the gauge condition

$$ E^\mu_a = \begin{pmatrix} E^0_0 & E^i_0 \\ E^0_I & E^i_I \end{pmatrix} = \begin{pmatrix} 1/N & -N^i/N \\ 0 & E^i_I \end{pmatrix} \tag{9} $$

Ashtekar variables (New variables)

PRL **57**, 2244 (1986); PRD **36**, 1587 (1987).

The pair of new variables, $(\tilde{E}^i_a, {}^+\mathcal{A}^a_i)$

- Self-dual connection (Ashtekar connection)

  We define so(3,C) connections

$$ {}^{\pm}\mathcal{A}^I_\mu := \omega^{0I}_\mu \mp \frac{i}{2}\epsilon^I{}_{JK}\,\omega^{JK}_\mu, \tag{10} $$

  where ω^{IJ}_μ is a spin connection 1-form (Ricci connection), $\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu E^J_\nu$. Ashtekar's plan is to use only ${}^+\mathcal{A}^a_\mu$. and to use its spatial part ${}^+\mathcal{A}^a_i$ as a dynamical variable. Hereafter, we simply denote ${}^+\mathcal{A}^a_\mu$ as $\mathcal{A}^a_\mu$.

- Densitized inverse triad $\tilde{E}^i_a$

$$ \tilde{E}^i_a := eE^i_a, \tag{11} $$

  where $e := \det E^a_i$ is a density.

This pair forms a canonical set.

- In the case of pure gravitational spacetime, the Hilbert action takes the form

$$ {}^{+}S_A[\tilde{E}, {}^{+}\mathcal{A}] = \int \mathrm{d}^4x[(\partial_t \mathcal{A}_i^a)\tilde{E}_a^i + \underset{\sim}{N}\,\mathcal{C}_H + N^i\,\mathcal{C}_{Mi} + \mathcal{A}_0^a\,\mathcal{C}_{Ga}], \tag{12} $$

where $\underset{\sim}{N} := e^{-1}N$.

- Lagrange multipliers ($\underset{\sim}{N}$, N^i, and $\mathcal{A}_0^a$) their accompanied constraints, $\mathcal{C}_H \approx 0, \mathcal{C}_{Mi} \approx 0$ and $\mathcal{C}_{Ga} \approx 0$.
- The set of $(\tilde{E}_a^i, \mathcal{A}_i^a)$ forms a canonical relation,

$$
\begin{aligned}
\{\tilde{E}_a^i(x), \tilde{E}_b^j(y)\} &= 0, \\
\{\mathcal{A}^a{}_i(x), \tilde{E}_b^j(y)\} &= i\delta^j{}_i\delta^a{}_b\delta(x-y), \\
\{\mathcal{A}^a{}_i(x), \mathcal{A}^b{}_j(y)\} &= 0.
\end{aligned}
$$

The dynamical degrees of freedom

covariant vars.		canonical vars.	gauge conditions	gauge vars.
E_a^μ (16)	$\Longrightarrow$	$\tilde{E}_a^i$ (9)	$E_a^0 = 0$ (3)	N^i (3) $+$ $\underset{\sim}{N}$ (1)
${}^{+}\omega_\mu^{ab}$ (12)	$\Longrightarrow$	$\mathcal{A}_i^a$ (9)		$\mathcal{A}_0^a$ (3)

The Ashtekar formulation:

PRL **57**, 2244 (1986); PRD **36**, 1587 (1987).

- New variables

$$\mathcal{A}^a_i := \omega^{0a}_i - \frac{i}{2}\epsilon^a{}_{bc}\omega^{bc}_i = -K_{ij}E^{ja} - \frac{i}{2}\epsilon^a{}_{bc}\omega^{bc}_i \quad \text{and} \quad \tilde{E}^i_a := eE^i_a$$

- The evolution equations for a set of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ are

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c, \tag{13}$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + 2\Lambda \underset{\sim}{N} \tilde{e}^a_i, \tag{14}$$

where $\mathcal{D}_j X^{ji}_a := \partial_j X^{ji}_a - i\epsilon_{ab}{}^c \mathcal{A}^b_j X^{ji}_c$, and $F^a_{ij} := 2\partial_{[i}\mathcal{A}^a_{j]} - i\epsilon^a{}_{bc}\mathcal{A}^b_i \mathcal{A}^c_j$.

- Constraint equations: (Hamiltonian, momentum and Gauss constraints)

$$\mathcal{C}^{\mathrm{ASH}}_H := (i/2)\epsilon^{ab}{}_c \tilde{E}^i_a \tilde{E}^j_b F^c_{ij} - 2\Lambda \det\tilde{E} \approx 0, \tag{15}$$

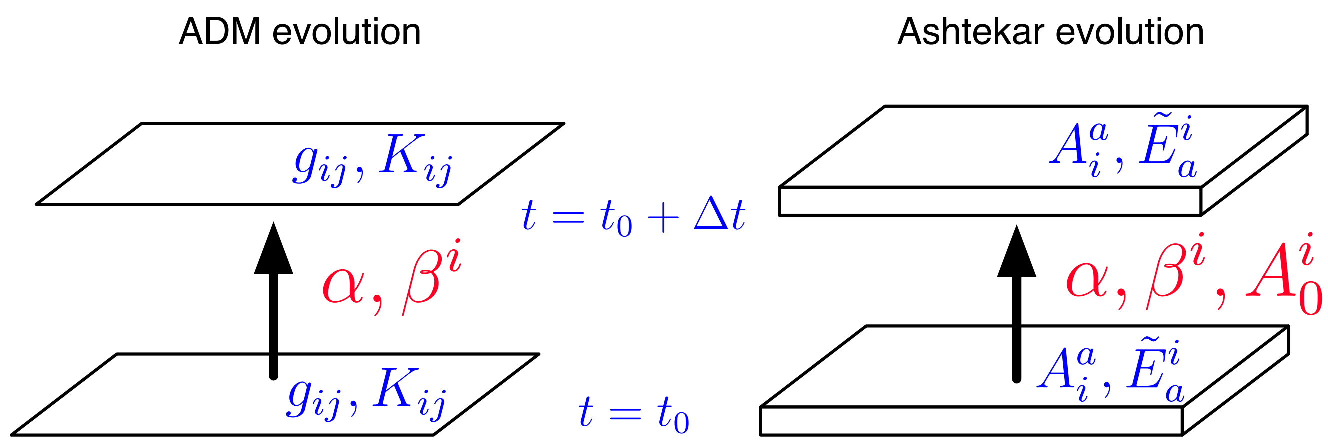
$$\mathcal{C}^{\mathrm{ASH}}_{Mi} := F^a_{ij}\tilde{E}^j_a \approx 0, \tag{16}$$

$$\mathcal{C}^{\mathrm{ASH}}_{Ga} := \mathcal{D}_i \tilde{E}^i_a \approx 0. \tag{17}$$

- Gauge variables: $\underset{\sim}{N}$, N^i, and the "triad lapse" $\mathcal{A}^a_0$.

Einstein vs. Ashtekar

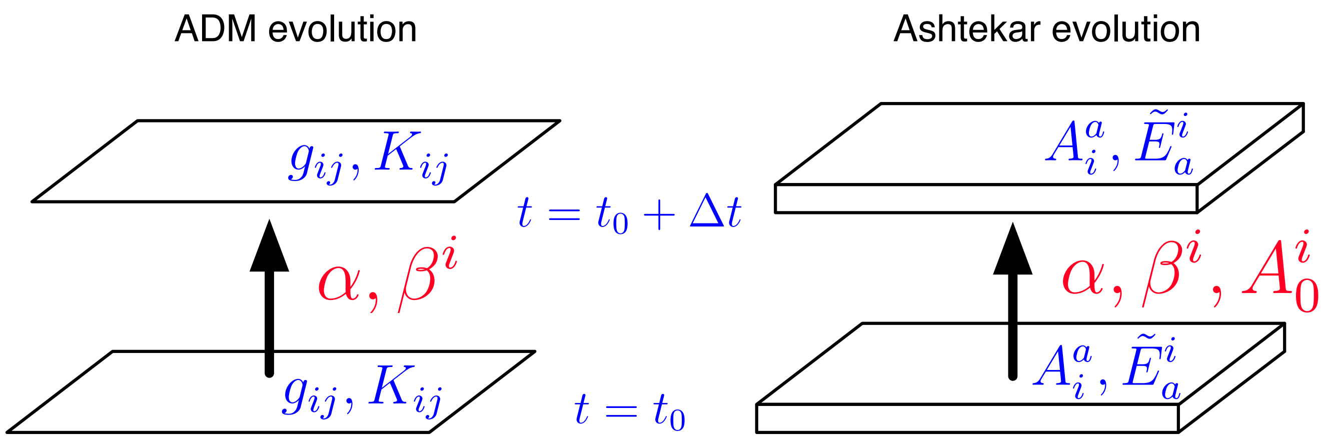
Einstein theory	Ashtekar theory
purely geometrical theory	gauge theoretical features
2nd order derivative theory	1st order derivative theory
dynamical eqs are non-polynomial	dynamical els are polynomial
	dynamical eqs are (weakly) hyperbolic
does contain the inverse of variables	does not contain the inverse of variables
does not admit degenerate metric	does admit degenerate metric
constraints are $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}_{\mathcal{M}}$	additional constraint, $\mathcal{C}_{\mathcal{G}}$
	"reality condition" to recover real geometry



Ashtekar's formulation : From the viewpoint of classical dynamics (1)

If we apply this formulation to the time evolution of Lorenzian space-time, the bottleneck is the additional constraint $\mathcal{C}_G$ and the reality conditions.

- Additional gauge variables ($\mathcal{A}_0^a$)
  In Ashtekar's theory, there is additional gauge variable, $\mathcal{A}_0^a$, which we named "triad lapse".
  This freedom appears due to the introduction of the internal indices.
  We somehow have to spacify $\mathcal{A}_0^a$ in a proper manner.



Ashtekar's formulation : From the viewpoint of classical dynamics (2)

- Additional "Gauss constraint" ($\mathcal{C}_{\mathcal{G}}$)

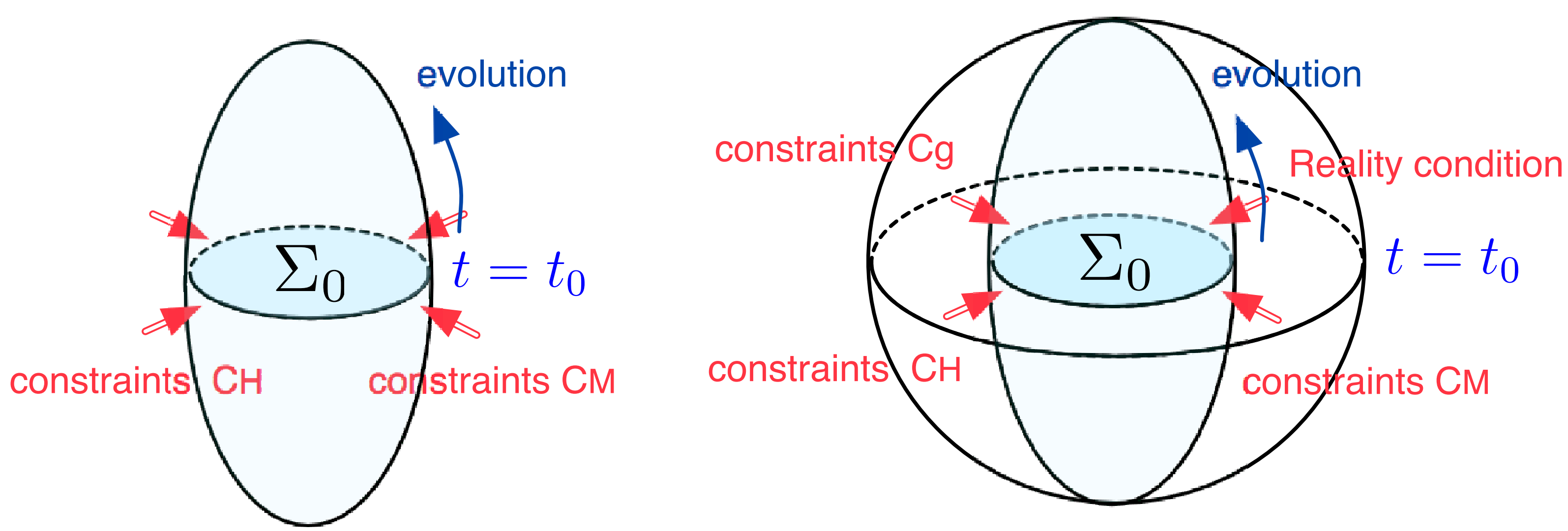
  In Ashtekar's theory, we have additional $\mathcal{C}_{\mathcal{G}}$, which has 3 components.

  The set of constraints forms the first-class, therefore we have to solve them when we prepare the initial data.

- Reality conditions to recover classical GR

  We have to solve the reality conditions when we describe the classical spacetime.

  Fortunately, the metric will remain on its real-valued constraint surface during time evolution automatically if we prepare initial data which satisfies the reality condition (Ashtekar-Romano-Tate, 1989).



Reality conditions

HS, Yoneda, Class. Quant. Grav. 13 (1996) 783

1: Metric Reality Condition

- the primary is that the doubly densitized contravariant metric $\tilde{\tilde{\gamma}}^{ij} := e^2\gamma^{ij}$ is real,

$$\Im(\tilde{E}^i_a \tilde{E}^{ja}) = 0, \tag{1}$$

- the secondary condition is that the time derivative of $\tilde{\tilde{\gamma}}^{ij}$ is real,

$$\Im\{\partial_t(\tilde{E}^i_a \tilde{E}^{ja})\} = 0. \tag{2}$$

These will be hold if the initial data satisfy the metric realicy conditions.

2: Triad Reality Condition

- "primary triad reality condition" and the "secondary triad reality condition"

$$\Im(\tilde{E}^i_a) = 0 \tag{3}$$

$$\text{and} \quad \Im(\dot{\tilde{E}}^i_a) = 0, \tag{4}$$

- Using the equations of motion of $\tilde{E}^i_a$ and $\mathcal{C}_G$, (1)-(4) will be

$$\Re(\mathcal{A}^a_0) = \partial_i(\underset{\sim}{N})\tilde{E}^{ia} + \frac{1}{2}e^{-1}e^b_i \underset{\sim}{N}\tilde{E}^{ja}\partial_j\tilde{E}^i_b + N^i\Re(\mathcal{A}^a_i). \tag{5}$$

This is a kind of slicing condition on $\mathcal{A}^a_0$.

Ashtekar's formulation : From the viewpoint of classical dynamics (3)

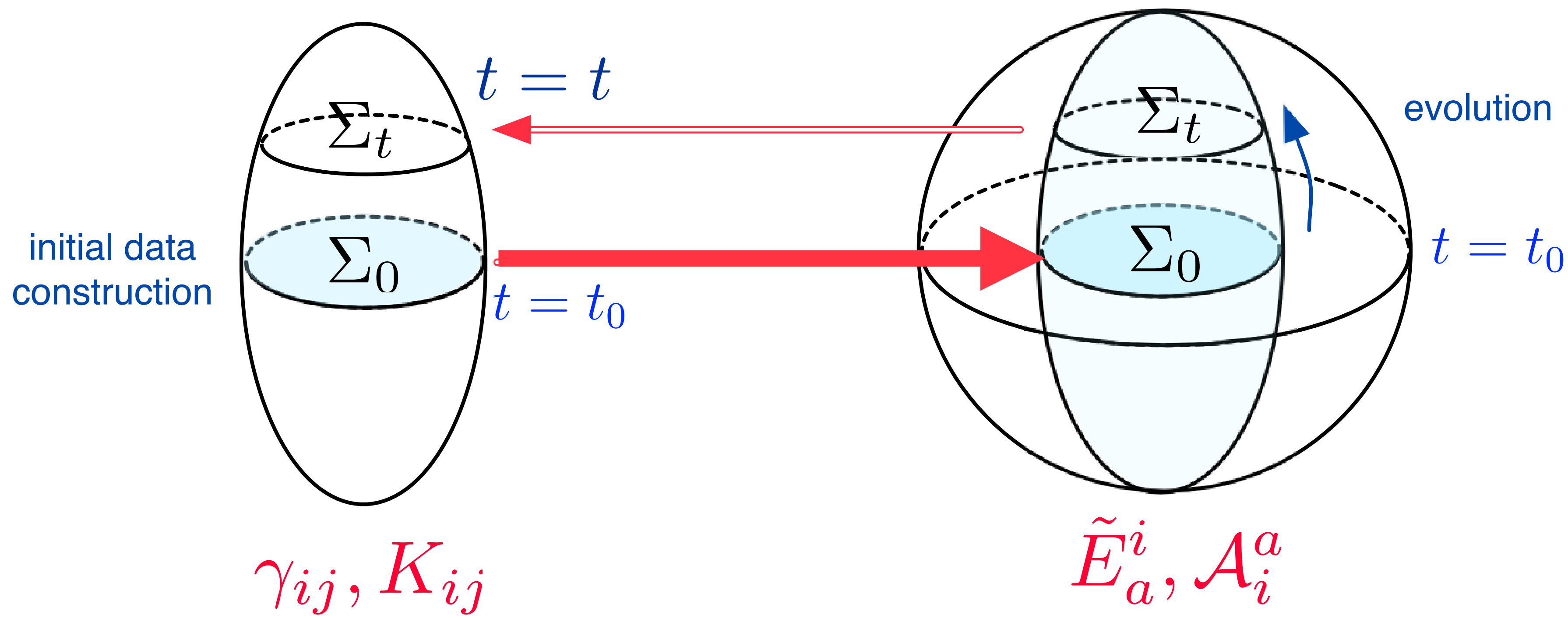
- Reality condition for the metric or for the triad?

  More practically, we further can require that triad is real-valued. But again this reality condition appears as a gauge restriction on the real part of the gauge function ${\mathcal{A}^a}_0$ (Yoneda-Shinkai, 1996)

	ADM formulation		connection formulation			
			Re(metric)		Re(triad)	
						Σ_0 (Σ_t)
variables	γ_{ij}	6	$\tilde{E}^i_a$	18	$\tilde{E}^i_a$	18 (9)
	K_{ij}	6	$\mathcal{A}^a_i$	18	$\mathcal{A}^a_i$	18 (9)
gauge	N	1	N	1	N	1 (1)
	N^i	3	N^i	3	N^i	3 (3)
			$\mathcal{A}^a_0$	6	$\mathcal{A}^a_0$	3 (3)
constraints	$\mathcal{C}_H$	1	$\mathcal{C}_H$	1	$\mathcal{C}_H$	1 (1)
	$\mathcal{C}_{Mi}$	3	$\mathcal{C}_{Mi}$	3	$\mathcal{C}_{Mi}$	3 (3)
			$\mathcal{C}_{Ga}$	6	$\mathcal{C}_{Ga}$	6 (3)
reality condition			primary	6 (Σ_0)	primary	9 (0)
			secondary	6 (Σ_0)	secondary	6 (0)
GW freedom		2×2		2×2		2×2

表 1 Number of components in actual simulations. We here count the numbers of freedom in components, i.e. one complex number has two components.

ADM 2 Ashtekar



$\gamma_{ij} \Longrightarrow \tilde{E}^i_a$

1. Define a triad E^a_i from 3-metric γ_{ij}:

$$E^a_i = \begin{bmatrix} E^1_x & E^1_y & E^1_z \\ E^2_x & E^2_y & E^2_z \\ E^3_x & E^3_y & E^3_z \end{bmatrix} = \begin{bmatrix} \sqrt{\gamma_{xx}} & 0 & 0 \\ 0 & b & d \\ 0 & e & c \end{bmatrix}$$

2. inverse triad E^i_a
3. density e: $\quad e = \det E^a_i$
4. $\tilde{E}^i_a = e E^i_a$

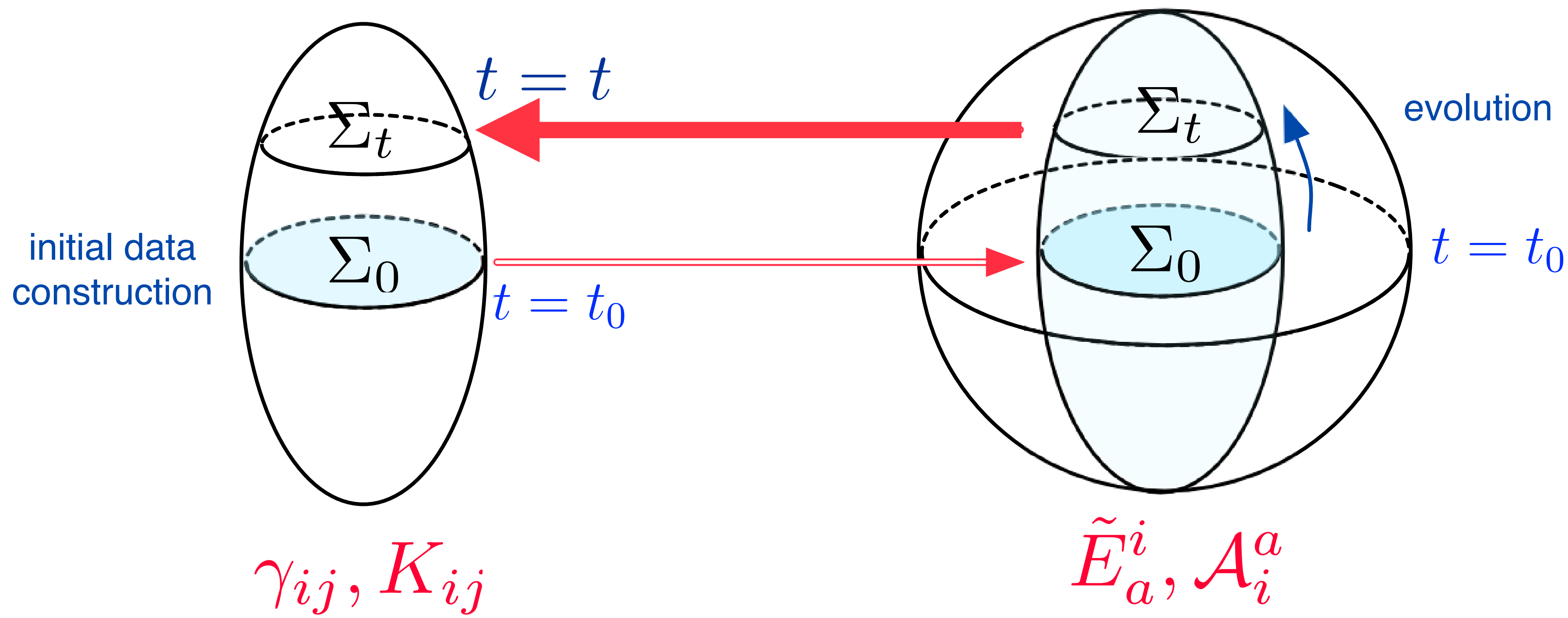
$(\gamma_{ij}, K_{ij}) \Longrightarrow \mathcal{A}^a_i$

1. triad E^a_i
2. inverse triad E^i_a
3. connection 1-form $\omega^{bc}_i = E^{b\mu} \nabla_i E^c_\mu$
   which can be expressed

$$\omega^{IJ}_\mu = E^{\nu I} \partial_{[\mu} E^J_{\nu]} - E_{\mu K} E^{\rho I} E^{\nu J} \partial_{[\rho} E^K_{\nu]} + E^{\rho J} \partial_{[\rho} E^I_{\mu]}$$

4. $\mathcal{A}^a_i = -K_{ij} E^{ja} - \frac{i}{2} \epsilon^a{}_{bc} \omega^{bc}_i$

Ashtekar 2 ADM



$\gamma_{ij} \Longleftarrow \tilde{E}^i_a$

1. density $e = (\det \tilde{E}^i_a)^{1/2}$
2. $\gamma^{ij} = \tilde{E}^i_a \tilde{E}^j_a / e^2$
3. γ_{ij}

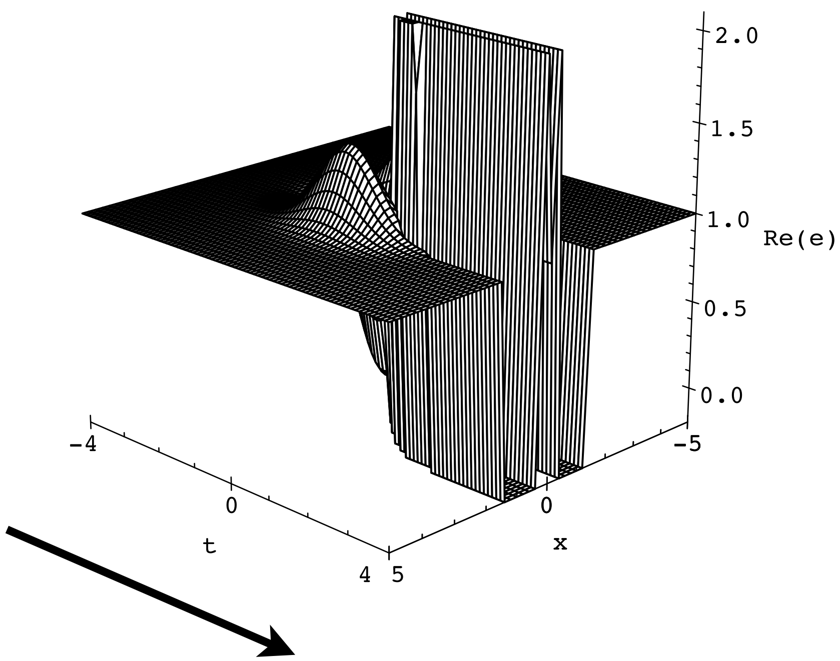
$K_{ij} \Longleftarrow (\tilde{E}^i_a, \mathcal{A}^a_i)$

1. un-densitized inverse triad $E^i_a = \tilde{E}^i_a / e$
2. triad E^a_i
3. connection 1-form $\epsilon^a{}_{bc}\omega^{bc}_i$
4. $K_{ij}E^{ja} = -\mathcal{A}^a_i + \frac{i}{2}\epsilon^a{}_{bc}\omega^{bc}_i \equiv Z^a_i$, then
   $K_{ij} = Z^a_i E_{ja}$

縮退点の通過が可能？

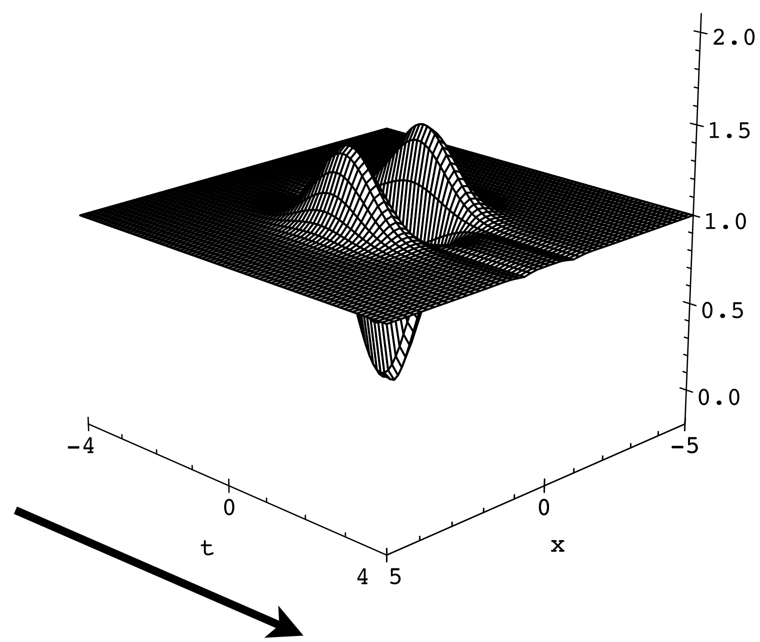
Yoneda, HS, Nakamichi, PRD 56, 2086 (1997)

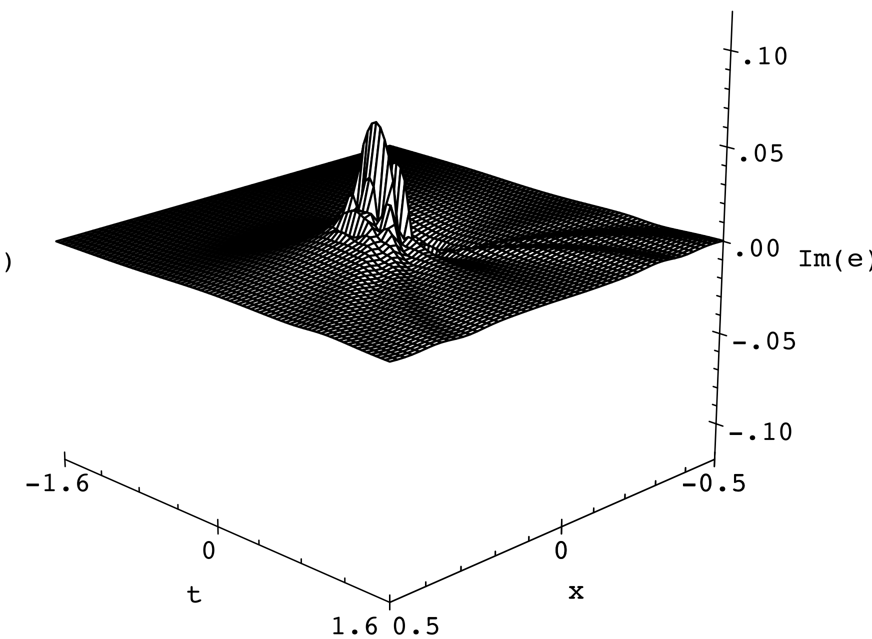
Intersecting Approach



変数の有界性を保ち，有限時間で density→0 を要請するならば，
lapse, density の時間微分は不連続にしないといけない．

Deformed Slicing Approach





実空間にある縮退点を，複素空間に迂回して回避させることが可能
であるが，解の一意性が問題になる．

	ADM	Ashtekar
Intersecting Approach 直接通過	×	×
Deformed Slice Method 複素空間迂回	×	○

The Ashtekar formulation:

PRL **57**, 2244 (1986); PRD **36**, 1587 (1987).

- New variables

$$\mathcal{A}^a_i := \omega^{0a}_i - \frac{i}{2}\epsilon^a{}_{bc}\omega^{bc}_i = -K_{ij}E^{ja} - \frac{i}{2}\epsilon^a{}_{bc}\omega^{bc}_i \quad \text{and} \quad \tilde{E}^i_a := eE^i_a$$

Remark:

最近では，実数条件の困難さを避けるため，Immirzi パラメータ γ を導入し，

$$\mathcal{A}^a_i := P^a_i - \gamma\Gamma^a_i$$

として，変数をあらかじめ実数にしてしまう方法が主流になってきた．
（Hamiltonian constraintは複雑になる．）

一般相対論の数値計算手法

近畿大セミナー
2011/12/9-10

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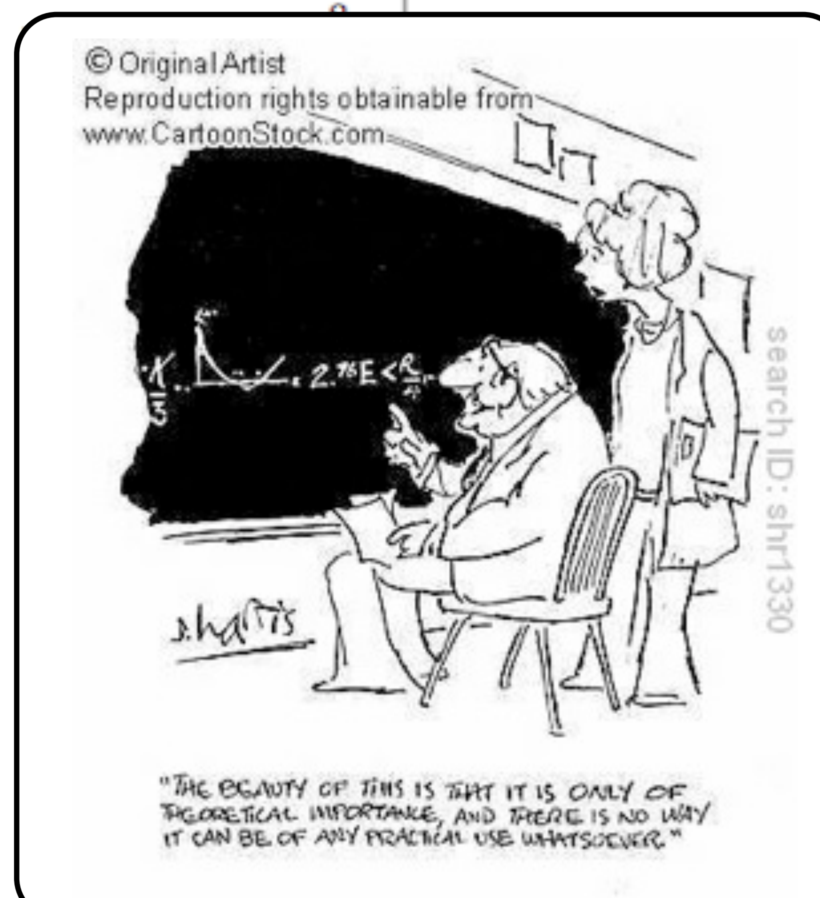
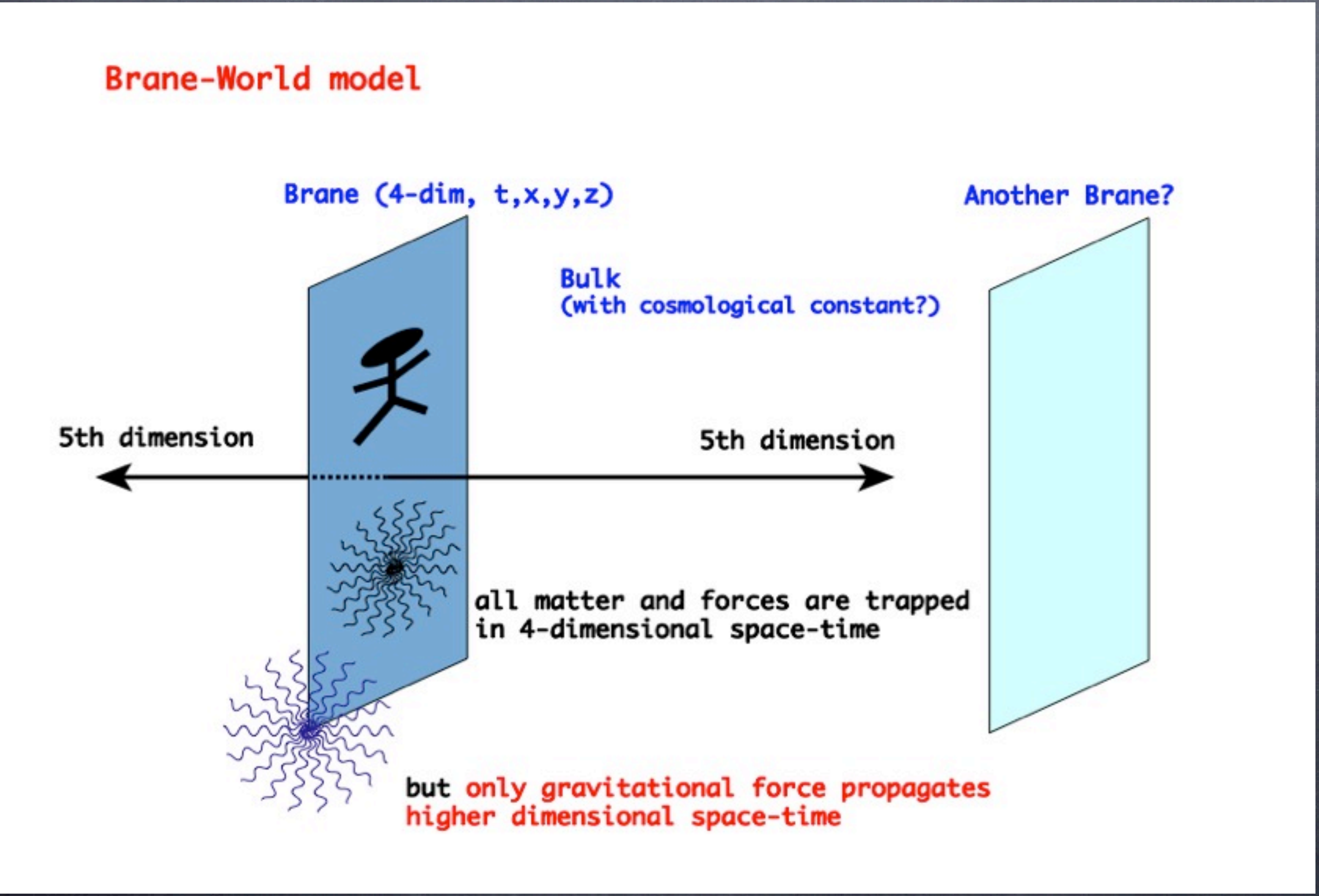
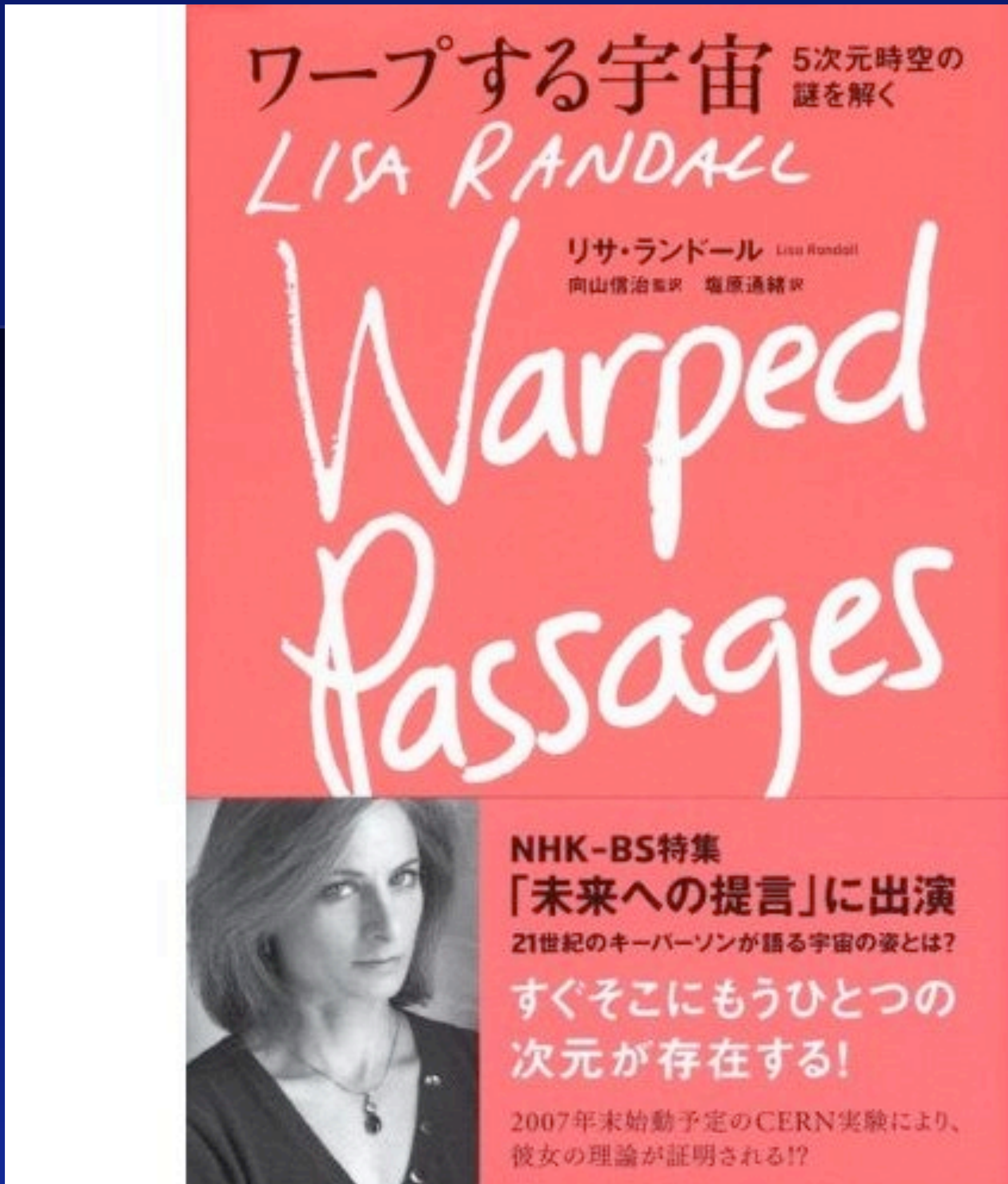


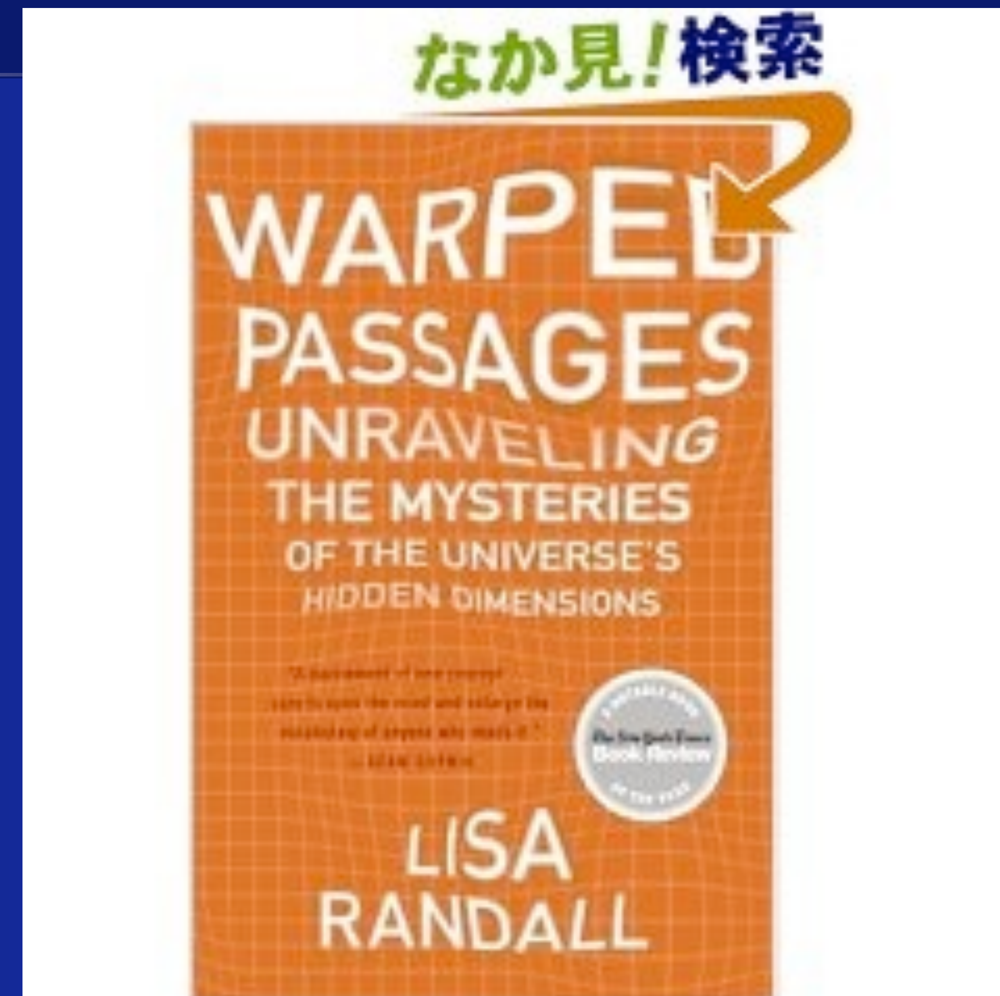
1999〜

ブレーンワールド





＊ファン・ガルシア－ベリド、アンドリュー・シャンブリン、ロベルト・エンパラン、ルース・グレゴリー、スティーヴン・ホーキング、ゲーリー・T・ホロヴィッツ、ネマーニャ・カロパー、ロバート・C・マイヤーズ、ハーヴェイ・S・リオール、真貝寿明、白水徹也、トビー・ワイズマンなど。



*They include Juan Garcia-Bellido, Andrew Chamblin, Roberto Emparan, Ruth Gregory, Stephen Hawking, Gary T. Horowitz, Nemanja Kaloper, Robert C. Myers, Harvey S. Reall, Hisa-aki Shinkai, Tetsuya Shiromizu, and Toby Wiseman.

$N+1$ formalism in Einstein-Gauss-Bonnet Gravity

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- $(N+1)$-dimensional space-time decomposition of Einstein-Gauss-Bonnet gravity
- Due to the quasi-linear property of the Gauss-Bonnet gravity, we find that the evolution equations can be in a treatable form in numerics.
- We also show the conformally-transformed constraint equations for constructing an initial data.
- Both for timelike and spacelike foliations.

Phys Rev D 78, 084037 (2008).

Einstein-Gauss-Bonnet action

- $(N+1)$-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$

$$S = \int_{\mathcal{M}} d^{N+1}X\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(\mathcal{R} - 2\Lambda + \alpha_{GB}\mathcal{L}_{GB}\right) + \mathcal{L}_{\rm matter}\right] \tag{1}$$

$$\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$$

- The action gives the gravitational equation

$$\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu} = \kappa^2\mathcal{T}_{\mu\nu} \tag{2}$$

where

$$\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu},$$

$$\mathcal{H}_{\mu\nu} = 2\left[\mathcal{R}\mathcal{R}_{\mu\nu} - 2\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu} - 2\mathcal{R}^{\alpha\beta}\mathcal{R}_{\mu\alpha\nu\beta} + \mathcal{R}_{\mu}{}^{\alpha\beta\gamma}\mathcal{R}_{\nu\alpha\beta\gamma}\right] - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{GB},$$

$$\mathcal{T}_{\mu\nu} \equiv -2\frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\rm matter}.$$

Projections to Hypersurface Σ_N (spacelike or timelike) (1)

- the projection operator,

$$\perp_{\mu\nu} = g_{\mu\nu} - \varepsilon n_\mu n_\nu, \quad n_\mu n^\mu = \varepsilon \tag{3}$$

where n_μ is the unit-normal vector to Σ with n_μ is timelike (if $\varepsilon = -1$) or spacelike (if $\varepsilon = 1$). Σ is spacelike (timelike) if n_μ is timelike (spacelike).

- The projections of the gravitational equation:

$$(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\, n^\mu\, n^\nu = \kappa^2\, T_{\mu\nu}\, n^\mu\, n^\nu =: \kappa^2 \rho_H, \tag{4}$$

$$(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu})\, n^\mu \perp^\nu_\rho = \kappa^2\, T_{\mu\nu}\, n^\mu \perp^\nu_\rho =: -\kappa^2 J_\rho, \tag{5}$$

$$(\mathcal{G}_{\mu\nu} + \alpha_{GB}\mathcal{H}_{\mu\nu}) \perp^\mu_\rho \perp^\nu_\sigma = \kappa^2\, T_{\mu\nu} \perp^\mu_\rho \perp^\nu_\sigma =: \kappa^2 S_{\rho\sigma}, \tag{6}$$

where we defined

$$T_{\mu\nu} = \rho_H n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\mu\nu}, \;\; T = -\rho_H + S^\ell{}_\ell$$

- Introduce the extrinsic curvature K_{ij}

$$K_{ij} := -\frac{1}{2} \pounds_n h_{ij} = -\perp^\alpha{}_i \perp^\beta{}_j \nabla_\alpha n_\beta, \tag{7}$$

where $\pounds_n$ denotes the Lie derivative in the n-direction and ∇ and D_i is the covariant differentiation with respect to $g_{\mu\nu}$ and γ_{ij}, respectively.

Projections to Hypersurface Σ_N (spacelike or timelike) (2)

- Projection of the $(N+1)$-dimensional Riemann tensor onto Σ_N

$$\text{Gauss eq.} \quad \mathcal{R}_{\alpha\beta\gamma\delta} \perp^{\alpha}{}_{i} \perp^{\beta}{}_{j} \perp^{\gamma}{}_{k} \perp^{\delta}{}_{l} = R_{ijkl} - \varepsilon K_{ik}K_{jl} + \varepsilon K_{il}K_{jk}, \tag{8}$$

$$\text{Codacci eq.} \quad \mathcal{R}_{\alpha\beta\gamma\delta} \perp^{\alpha}{}_{i} \perp^{\beta}{}_{j} \perp^{\gamma}{}_{k} n^{\delta} = -2D_{[i}K_{j]k}, \tag{9}$$

$$\mathcal{R}_{\alpha\beta\gamma\delta} \perp^{\alpha}{}_{i} \perp^{\gamma}{}_{k} n^{\beta} n^{\delta} = \pounds_n K_{ik} + K_{i\ell} K^{\ell}{}_{k}\,, \tag{10}$$

- Curvature relations

$$\begin{aligned}
\mathcal{R}_{\mu\nu\rho\sigma} = {} & R_{\mu\nu\rho\sigma} - \varepsilon(K_{\mu\rho}K_{\nu\sigma} - K_{\mu\sigma}K_{\nu\rho} - n_\mu D_\rho K_{\nu\sigma} + n_\mu D_\sigma K_{\rho\nu} + n_\nu D_\rho K_{\sigma\mu} - n_\nu D_\sigma K_{\rho\mu} \\
& - n_\rho D_\mu K_{\nu\sigma} + n_\rho D_\nu K_{\mu\sigma} + n_\sigma D_\mu K_{\nu\rho} - n_\sigma D_\nu K_{\mu\rho}) \\
& + n_\mu n_\rho K_{\nu\alpha} K^{\alpha}{}_{\sigma} - n_\mu n_\sigma K_{\nu\alpha} K^{\alpha}{}_{\rho} - n_\nu n_\rho K_{\mu\alpha} K^{\alpha}{}_{\sigma} + n_\nu n_\sigma K_{\mu\alpha} K^{\alpha}{}_{\rho} \\
& + n_\mu n_\rho \pounds_n K_{\nu\sigma} - n_\mu n_\sigma \pounds_n K_{\nu\rho} - n_\nu n_\rho \pounds_n K_{\mu\sigma} + n_\nu n_\sigma \pounds_n K_{\mu\rho},
\end{aligned} \tag{11}$$

$$\begin{aligned}
\mathcal{R}_{\mu\nu} = {} & R_{\mu\nu} - \varepsilon[K K_{\mu\nu} - 2K_{\mu\alpha} K^{\alpha}{}_{\nu} + n_\mu \left(D_\alpha K^{\alpha}{}_{\nu} - D_\nu K\right) + n_\nu \left(D_\alpha K^{\alpha}{}_{\mu} - D_\mu K\right)] \\
& + n_\mu n_\nu K_{\alpha\beta} K^{\alpha\beta} + \varepsilon \pounds_n K_{\mu\nu} + n_\mu n_\nu \gamma^{\alpha\beta} \pounds_n K_{\alpha\beta},
\end{aligned} \tag{12}$$

$$\mathcal{R} = R - \varepsilon(K^2 - 3K_{\alpha\beta}K^{\alpha\beta} - 2\gamma^{\alpha\beta}\pounds_n K_{\alpha\beta}). \tag{13}$$

$N+1$ Einstein-Gauss-Bonnet equations

Substituting (11)-(13) into (3) or (4)-(6), we find:

(a) dynamical equations for γ_{ij}:

$$M_{ij} - \frac{1}{2}M\gamma_{ij} - \varepsilon(-K_{ia}K^a{}_j + \gamma_{ij}K_{ab}K^{ab} - \pounds_n K_{ij} + \gamma_{ij}\gamma^{ab}\pounds_n K_{ab})$$
$$+2\alpha_{GB}[H_{ij} + \varepsilon(M\pounds_n K_{ij} - 2M_i{}^a\pounds_n K_{aj} - 2M_j{}^a\pounds_n K_{ai} - W_{ij}{}^{ab}\pounds_n K_{ab})] = \kappa^2\mathcal{T}_{\mu\nu}\gamma^\mu{}_i\gamma^\nu{}_j$$

(b) Hamiltonian constraint equation:

$$M + \alpha_{GB}(M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}) = -2\varepsilon\kappa^2\mathcal{T}_{\mu\nu}n^\mu n^\nu$$

(c) momentum constraint equation:

$$N_i + 2\alpha_{GB}\left(MN_i - 2M_i{}^aN_a + 2M^{ab}N_{iab} - M_i{}^{cab}N_{abc}\right) = -\kappa^2\mathcal{T}_{\mu\nu}n^\mu\gamma^\nu{}_i$$

$$M_{ijkl} = R_{ijkl} - \varepsilon(K_{ik}K_{jl} - K_{il}K_{jk})$$
$$M_{ij} = \gamma^{ab}M_{iajb} = R_{ij} - \varepsilon(KK_{ij} - K_{ia}K^a{}_j)$$
$$M = \gamma^{ab}M_{ab} = R - \varepsilon(K^2 - K_{ab}K^{ab})$$
$$N_{ijk} = D_iK_{jk} - D_jK_{ik}$$
$$N_i = \gamma^{ab}N_{aib} = D_aK_i{}^a - D_iK$$
$$W_{ij}{}^{kl} = M\gamma_{ij}\gamma^{kl} - 2M_{ij}\gamma^{kl} - 2\gamma_{ij}M^{kl} + 2M_{iajb}\gamma^{ak}\gamma^{bl}$$

$$H_{ij} = MM_{ij} - 2(M_{ia}M^a{}_j + M^{ab}M_{iajb}) + M_{iabc}M_j{}^{abc}$$
$$-2\varepsilon\Big[-K_{ab}K^{ab}M_{ij} - \frac{1}{2}MK_{ia}K^a{}_j + K_{ia}K^a{}_bM^b{}_j + K_{ja}K^a{}_bM^b{}_i + K^{ac}K_c{}^bM_{iajb}$$
$$+N_iN_j - N^a(N_{aij} + N_{aji}) - \frac{1}{2}N_{abi}N^{ab}{}_j - N_{iab}N_j{}^{ab}\Big]$$
$$-\frac{1}{4}\gamma_{ij}\left[M^2 - 4M_{ab}M^{ab} + M_{abcd}M^{abcd}\right]$$
$$-\varepsilon\gamma_{ij}\left[K_{ab}K^{ab}M - 2M_{ab}K^{ac}K_c{}^b - 2N_aN^a + N_{abc}N^{abc}\right]$$

$N+1$ Einstein-Gauss-Bonnet evolution equations

$$(1+2\alpha_{GB}M)\pounds_n K_{ij} - (\gamma_{ij}\gamma^{ab} + 2\alpha_{GB}W_{ij}{}^{ab})\pounds_n K_{ab} - 8\alpha_{GB}M_{(i}{}^{a}\pounds_n K_{|a|j)}$$
$$= -\varepsilon\Big(M_{ij} - \frac{1}{2}M\gamma_{ij}\Big) - K_{ia}K^{a}{}_{j} + \gamma_{ij}K_{ab}K^{ab} + \varepsilon\kappa^2 S_{ij} - \varepsilon\gamma_{ij}\Lambda - 2\varepsilon\alpha_{GB}H_{ij}\,, \qquad (20)$$

- $\pounds_n K_{\mu\nu}$ terms appear only in the linear form, due to the quasi-linear property of the Gauss-Bonnet gravity.
- Iterative scheme is necessary, but treatable in numerics.

$$\begin{pmatrix} \pounds_n\gamma_{11} \\ \pounds_n\gamma_{12} \\ \pounds_n\gamma_{13} \\ \vdots \\ \pounds_n K_{11} \\ \pounds_n K_{12} \\ \pounds_n K_{13} \\ \vdots \end{pmatrix} = \begin{pmatrix} O & O \\ O & \text{Mixing} \end{pmatrix} \begin{pmatrix} \pounds_n\gamma_{11} \\ \pounds_n\gamma_{12} \\ \pounds_n\gamma_{13} \\ \vdots \\ \pounds_n K_{11} \\ \pounds_n K_{12} \\ \pounds_n K_{13} \\ \vdots \end{pmatrix} + \begin{pmatrix} K_{11} \\ K_{12} \\ K_{13} \\ \vdots \\ \\ \text{Source} \\ \\ \\ \end{pmatrix}$$

- Coding is in progress, ... but .... Are the evolution eqs always invertible??

Newman-Penrose Formulation (1)

References

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Advantages

- Natural framework for calculations in radiative spacetimes
- Variables have geometrical meanings
- Practical advantages in treating Petrov type-D spacetimes
- Closely related with spinor formalism

Newman-Penrose Formulation (2): Null Tetrad

- Take real-valued null vectors $\mathbf{l}, \mathbf{n}$ and complex conjugate null vectors $\mathbf{m}, \overline{\mathbf{m}}$,

$$\mathbf{l} \cdot \mathbf{n} = l^a n_a = l_a n^a = 1 \tag{1a}$$

$$\mathbf{m} \cdot \overline{\mathbf{m}} = m^a \overline{m}_a = m_a \overline{m}^a = -1 \tag{1b}$$

$$\text{and else } 0 \tag{1c}$$

This null basis $(l^a, n^a, m^a, \overline{m}^a)$ and the orthogonal tetrad basis $(t^{\hat{a}}, x^{\hat{a}}, y^{\hat{a}}, z^{\hat{a}})$ are related by

$$l^a = o^A o^{A'} = \frac{1}{\sqrt{2}}(t^{\hat{a}} + z^{\hat{a}}), \tag{2a}$$

$$n^a = \iota^A \iota^{A'} = \frac{1}{\sqrt{2}}(t^{\hat{a}} - z^{\hat{a}}), \tag{2b}$$

$$m^a = o^A \iota^{A'} = \frac{1}{\sqrt{2}}(x^{\hat{a}} - i y^{\hat{a}}), \tag{2c}$$

$$\overline{m}^a = \iota^A o^{A'} = \frac{1}{\sqrt{2}}(x^{\hat{a}} + i y^{\hat{a}}), \tag{2d}$$

where I also put spinor expressions (o^A, ι^A) of those.

- The indice rules are

$$l_a = g_{ab} l^b, \quad l^a = g^{ab} l_b, \qquad g_{ab} = metric \tag{3}$$

$$x_{\hat{a}} = \eta_{\hat{a}\hat{b}} x^{\hat{b}}, \quad x^{\hat{a}} = \eta^{\hat{a}\hat{b}} x_{\hat{b}}, \qquad \eta_{\hat{a}\hat{b}} = (1, -1, -1, -1) \tag{4}$$

Metric g_{ab} will be recovered by

$$g_{ab} = \eta_{ij} e^i{}_a e^j{}_b \tag{5}$$

$$g_{ab} = 2l_{(a} n_{b)} - 2m_{(a} \overline{m}_{b)} \qquad g^{ab} = 2l^{(a} n^{b)} - 2m^{(a} \overline{m}^{b)} \tag{6}$$

$$g_{\hat{a}\hat{b}} = t_{\hat{a}} t_{\hat{b}} - x_{\hat{a}} x_{\hat{b}} - y_{\hat{a}} y_{\hat{b}} - z_{\hat{a}} z_{\hat{b}} \tag{7}$$

where $\eta_{ij} = \eta^{ij} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix}$.

Spinor Algebra

- A set of non-zero 2-component vector (o, ι) is called *spin basis,* if their skew-scalar product obeys $[o, \iota] = 1$, where $[,]$ is defined by $[\xi, \eta] = -[\eta, \xi] = \epsilon_{AB}\xi^A\eta^B$.
- This condition is also expressed by

$$\epsilon_{AB}o^A o^B = \epsilon_{AB}\iota^A\iota^B = 0, \quad \epsilon_{AB}o^A\iota^B = o_A\iota^A = 1. \tag{1}$$

In components with $o^A = (1, 0), \iota^A = (0, 1)$,

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{AB} = o^A\iota^B - \iota^A o^B, \quad \epsilon^{AB}\epsilon_{BC} = -\delta^A_{\ C} \tag{2}$$

- When raise and lower indices, be careful the position of them.

$$\xi_A \equiv \epsilon_{BA}\xi^B = -\epsilon_{AB}\xi^B = \xi^B\epsilon_{BA} \tag{3a}$$

$$\xi^A = \epsilon^{AB}\xi_B = -\epsilon^{BA}\xi_B = \xi_B\epsilon^{AB} \tag{3b}$$

If the contraction is across adjacent *northwest* and *southeast* indices, no minus signs occur.

Theorem 1 Every non-vanishing real null vector k^a can be written in one are other of the forms

$$k^a = \pm\kappa^A\overline{\kappa}^{A'} \tag{4}$$

A relation to orthonormal tetrad $g_{\hat{a}\hat{b}} = \text{diag}(1, -1, -1, -1)$ is given by

$$\epsilon_{AB}\overline{\epsilon}_{A'B'} = g_{\hat{a}\hat{b}}\sigma^{\hat{a}}{}_{AA'}\sigma^{\hat{b}}{}_{BB'} \tag{5}$$

where $\sigma^{\hat{a}}{}_{AA'}$ are the Infeld-van der Warden symbols, and these are related to the Pauli matrix with $\sigma_{\hat{a}}{}^{AA'} = (1/\sqrt{2})\sigma_{\hat{a}} \quad (a = 0, 1, 2, 3)$ where $\sigma_{\hat{0}} = I$ and

$$\sigma_{\hat{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{\hat{2}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{\hat{3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

Newman-Penrose Formulation (3): Connection coefficients

∇ (deriv. op.)	$m^a \nabla l_a \Leftrightarrow o^A \nabla o_A$	$\frac{1}{2}(n^a \nabla l_a - \overline{m}^a \nabla m_a) \Leftrightarrow o^A \nabla \iota_A = \iota^A \nabla o_A$	$-\overline{m}^a \nabla n_a \Leftrightarrow \iota^A \nabla \iota_A$
$D = l^a \nabla_a \Leftrightarrow o^A \bar{o}^{A'} \nabla_{AA'}$	$\kappa \ \{3, 1\}$	ϵ	$\pi \ (-\tau') \ \{-1, 1\}$
$\Delta = n^a \nabla_a$	$\tau \ \{1, -1\}$	$\gamma \ (-\epsilon')$	$\nu \ (-\kappa') \ \{-3, -1\}$
$\delta = m^a \nabla_a$	$\sigma \ \{3, -1\}$	β	$\mu \ (-\rho') \ \{-1, -1\}$
$\bar{\delta} = \bar{m}^a \nabla_a$	$\rho \ \{1, 1\}$	$\alpha \ (-\beta')$	$\lambda \ (-\sigma') \ \{-3, 1\}$

$$\rho = m^a \bar{m}^b \nabla_b l_a$$

$\{p, q\}$ indicates *spin- and boost-weighted type* and prime-operation will be defined later by R. Gerosh, A. Held and R. Penrose, J. Math. Phys. **14**, 874 (1973).

Newman-Penrose Formulation (4): Weyl scalars

R.K. Sachs, Proc. Roy. Soc. London, **A264**, 309 (1961); *ibid* **A270**, 103 (1962).
P. Szekeres, J. Math. Phys. **6**, 1387 (1965).

- The Weyl curvature C_{abcd} is defined as

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} - \frac{1}{3}Rg_{a[c}g_{d]b}, \tag{1}$$

or alternatively by using $\Lambda = R/24, \Phi_{ab} = -\frac{1}{2}R_{ab} + \frac{1}{8}Rg_{ab}$,

$$C_{abcd} = R_{abcd} + 2\Phi_{a[c}g_{d]b} - 2\Phi_{b[c}g_{d]a} + 8\Lambda g_{a[c}g_{d]b}. \tag{2}$$

- The 10 components of Weyl curvature are expressed by the following 5 complex scalars;

$\Psi_0 \equiv \psi_{ABCD}o^A o^B o^C o^D = C_{abcd}l^a m^b l^c m^d$, n^a-directed transverse component, $\{4,0\}$ (3a)

$\Psi_1 \equiv \psi_{ABCD}o^A o^B o^C \iota^D = C_{abcd}l^a n^b l^c m^d$, n^a-directed longitudinal component $\{2,0\}$ (3b)

$\Psi_2 \equiv \psi_{ABCD}o^A o^B \iota^C \iota^D = C_{abcd}l^a m^b \bar{m}^c n^d$, 'Coulomb' component, $\{0,0\}$ (3c)

$\Psi_3 \equiv \psi_{ABCD}o^A \iota^B \iota^C \iota^D = C_{abcd}l^a n^b \bar{m}^c n^d$, l^a-directed longitudinal component, $\{-2,0\}$ (3d)

$\Psi_4 \equiv \psi_{ABCD}\iota^A \iota^B \iota^C \iota^D = C_{abcd}n^a \bar{m}^b n^c \bar{m}^d$, l^a-directed transverse component, $\{-4,0\}$ (3e)

Newman-Penrose Formulation (5): Ricci tensor components

Real valued $(\Phi_{00}, \Phi_{11}, \Phi_{22}, \Lambda)$ and Complex valued conjugate pairs (Φ_{01}, Φ_{10}), (Φ_{02}, Φ_{20}) and (Φ_{12}, Φ_{21}). Total 10 real components.

$$\Phi_{00} = \Phi_{ABA'B'} o^A o^B \bar{o}^{A'} \bar{o}^{B'} = -\frac{1}{2} R_{ab} l^a l^b, \quad \{2, 2\} \tag{1a}$$

$$\Phi_{01} = \Phi_{ABA'B'} o^A o^B \bar{o}^{A'} \bar{\iota}^{B'} = -\frac{1}{2} R_{ab} l^a m^b, \quad \{2, 0\} \tag{1b}$$

$$\Phi_{02} = \Phi_{ABA'B'} o^A o^B \bar{\iota}^{A'} \bar{\iota}^{B'} = -\frac{1}{2} R_{ab} m^a m^b, \quad \{2, -2\} \tag{1c}$$

$$\Phi_{10} = \Phi_{ABA'B'} o^A \iota^B \bar{o}^{A'} \bar{o}^{B'} = -\frac{1}{2} R_{ab} l^a \bar{m}^b, \quad \{0, 2\} \tag{1d}$$

$$\Phi_{11} = \Phi_{ABA'B'} o^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'} = -\frac{1}{2} R_{ab} l^a n^b + 3\Lambda, \quad \{0, 0\} \tag{1e}$$

$$\Phi_{12} = \Phi_{ABA'B'} o^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'} = -\frac{1}{2} R_{ab} m^a n^b, \quad \{0, -2\} \tag{1f}$$

$$\Phi_{20} = \Phi_{ABA'B'} \iota^A \iota^B \bar{o}^{A'} \bar{o}^{B'} = -\frac{1}{2} R_{ab} \bar{m}^a \bar{m}^b, \quad \{-2, +2\} \tag{1g}$$

$$\Phi_{21} = \Phi_{ABA'B'} \iota^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'} = -\frac{1}{2} R_{ab} \bar{m}^a n^b, \quad \{-2, 0\} \tag{1h}$$

$$\Phi_{22} = \Phi_{ABA'B'} \iota^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'} = -\frac{1}{2} R_{ab} n^a n^b, \quad \{-2, -2\} \tag{1i}$$

Newman-Penrose Formulation (6): Commutators acting on a scalar function

Derived from $\nabla_{[a}\nabla_{b]}\psi = 0$. Useful to reduce derivatives.

$$(\Delta D - D\Delta)\psi = [(\gamma + \overline{\gamma})D + (\epsilon + \overline{\epsilon})\Delta - (\pi + \overline{\tau})\delta - (\overline{\pi} + \tau)\overline{\delta}]\psi$$

$$(\delta D - D\delta)\psi = [(\overline{\alpha} + \beta - \overline{\pi})D + \kappa\Delta - (\overline{\rho} + \epsilon - \overline{\epsilon})\delta - \sigma\overline{\delta}]\psi$$

$$(\delta\Delta - \Delta\delta)\psi = [-\overline{\nu}D + (\tau - \overline{\alpha} - \beta)\Delta + (\mu - \gamma + \overline{\gamma})\delta - \overline{\lambda\delta}]\psi$$

$$(\overline{\delta}\delta - \delta\overline{\delta})\psi = [(\overline{\mu} - \mu)D + (\overline{\rho} - \rho)\Delta + (\alpha - \overline{\beta})\delta - (\overline{\alpha} + \beta)\overline{\delta}]\psi$$

Newman-Penrose Formulation (7): Field equations

Expressions of Riemann tensor. 18 complex equations, but with 16 hidden eliminant relations, in total 20 degrees of freedom.

$$D\rho - \overline{\delta}\kappa = (\rho^2 + \sigma\overline{\sigma}) + (\epsilon + \overline{\epsilon})\rho - \overline{\kappa}\tau - \kappa(3\alpha + \overline{\beta} - \pi) + \Phi_{00} \tag{1}$$

$$D\sigma - \delta\kappa = \cdots \tag{2}$$

$$D\tau - \Delta\kappa = \cdots \tag{3}$$

$$D\alpha - \overline{\delta}\epsilon = \cdots \tag{4}$$

$$D\beta - \delta\epsilon = \cdots \tag{5}$$

$$D\gamma - \Delta\epsilon = \cdots \tag{6}$$

$$D\lambda - \overline{\delta}\pi = \cdots \tag{7}$$

$$D\mu - \delta\pi = \cdots \tag{8}$$

$$D\nu - \Delta\pi = \cdots \tag{9}$$

$$\Delta\lambda - \overline{\delta}\nu = \cdots \tag{10}$$

$$\delta\rho - \overline{\delta}\sigma = \cdots \tag{11}$$

$$\delta\alpha - \overline{\delta}\beta = \cdots \tag{12}$$

$$\delta\lambda - \overline{\delta}\mu = \cdots \tag{13}$$

$$\delta\lambda - \overline{\delta}\mu = \cdots \tag{14}$$

$$\Delta\mu - \delta\nu = \cdots \tag{15}$$

$$\Delta\beta - \delta\gamma = \cdots \tag{16}$$

$$\Delta\sigma - \delta\tau = \cdots \tag{17}$$

$$\Delta\rho - \overline{\delta}\tau = \cdots \tag{18}$$

$$\Delta\alpha - \overline{\delta}\gamma = \cdots \tag{19}$$

Newman-Penrose Formulation (8): Bianchi id.

20 identities. (In vacuum, 16).

$$
\begin{aligned}
& D\Psi_1 - \bar{\delta}\Psi_0 - D\Phi_{01} + \delta\Phi_{00} \\
& \quad = (\pi - 4\alpha)\Psi_0 + 2(2\rho + \epsilon)\Psi_1 - 3\kappa\Psi_2 - (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} \\
& \quad\; - 2(\bar{\rho} + \epsilon)\Phi_{01} - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} + \bar{\kappa}\Phi_{02} \qquad (1a) \\
& \Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = \cdots \qquad (1b) \\
& \cdots = \cdots \qquad (1c)
\end{aligned}
$$

Newman-Penrose Formulation (8): Geometrical meaning of variables

When we consider the propagation of the basis vectors along $\mathbf{l}$ or $\mathbf{n}$, the physical significance of the spin coefficiencies becomes apparent. The propagation properties of null congruence of null geodesics are given by

$$\frac{1}{2}l^i{}_{;i} = -\frac{1}{2}(\rho + \overline{\rho}) = -\Re\rho \equiv \theta \tag{1}$$

$$\frac{1}{2}l_{[i;j]}l^{i;j} = -\frac{1}{4}(\rho - \overline{\rho})^2 = (\Im\rho)^2 \equiv \omega^2 \tag{2}$$

$$\frac{1}{2}l_{(i;j)}l^{i;j} = \theta^2 + |\sigma|^2 \tag{3}$$

θ, ω and σ are called *optical scalars*.

Newman-Penrose Formulation (9): Counting degrees of freedom

Degrees of tetrad transformations. (6 degrees of Lorentz boosts.)

Type I $\mathbf{l} \to \mathbf{l}, \mathbf{m} \to a\mathbf{l}, \overline{\mathbf{m}} \to \overline{\mathbf{m}} + \bar{a}\mathbf{l}$, and $\mathbf{n} \to \mathbf{n} + \bar{a}\mathbf{m} + a\overline{\mathbf{m}} + a\bar{a}\mathbf{l}$.

Type II $\mathbf{n} \to \mathbf{n}, \mathbf{m} \to b\mathbf{n}, \overline{\mathbf{m}} \to \overline{\mathbf{m}} + \bar{b}\mathbf{n}$, and $\mathbf{l} \to \mathbf{l} + \bar{b}\mathbf{m} + b\overline{\mathbf{m}} + b\bar{b}\mathbf{n}$.

Type III $\mathbf{l} \to A^{-1}\mathbf{l}, \mathbf{n} \to A\mathbf{n}, \mathbf{m} \to e^{i\theta}\mathbf{m}$, and $\overline{\mathbf{m}} \to e^{-i\theta}\overline{\mathbf{m}}$.

- Counting degrees of freedom in vacuum.
  variables: Ψ 10, spin coef. 24, and boost 6.
  equation: field eq. 12×2, Bianchi id. 16.
- Counting degrees of freedom with matter.
  variables: curvature 20, spin coef. 24. equation: field eq. 20, Bianchi 20, and $\nabla_a T^a_{\ b} = 0$, 4.

Principal Null Directions (1): Weyl & dual of Weyl tensor

$$C_{abcd} = C_{AA'BB'CC'DD'}$$
$$= \Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} + \bar{\Psi}_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} \quad (1)$$
$$^*C_{abcd} = \frac{1}{2}\varepsilon_{ab}{}^{mn}C_{mncd} \quad (2)$$
$$C_{abcd} + i\ ^*C_{abcd} = 2\Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} \quad (3)$$

Principal Null Directions (2): Canonical decomposition of symmetric spinor

Spinor $\psi_{ABCD} = \psi_{(ABCD)}$ is expressed by principal null spinor $\alpha_A, \beta_A, \ldots$ by using spinor characters,

$$\psi_{ABCD} = \psi_{(ABCD)} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}, \quad (4)$$

and principal null vectors are defined as real vectors constructed by $\alpha_A\bar{\alpha}_{A'}, \beta_A\bar{\beta}_{A'}, \ldots$. PNDs are those directions.

Principal Null Directions (3): Petrov classification

Classification of vacuum spacetime by degeneracy of PNDs. [*1]

$$
\begin{aligned}
0 &= \Psi_{ABCD}(z\iota^A + o^A)(z\iota^B + o^B)(z\iota^C + o^C)(z\iota^D + o^D) \\
0 &= \Psi_4 z^4 + 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0
\end{aligned} \tag{1}
$$

Type I $\{1,1,1,1\}$ no coincide PNDs. algebraically general.

Type II $\{2,1,1\}$ two coincide PNDs. algebraically special (all below).

Type D $\{2,2\}$ two different pairs. Schwarzschild, Kerr, ...

Type III $\{3,1\}$ three coincide PNDs.

Type N $\{4\}$ all four coincide PNDs. Plane-wave

Principal Null Directions (4): Projections of Weyl curvature

Gunnarsen-Shinkai-Maeda [*2] proposed a transformation formula of Weyl scalar Ψ_i from ADM variables (γ_{ij}, K_{ij}), motivated by an application to interpret numerically generated spacetime.

Here, we consider vacuum spacetimes with cosmological constant Λ. Let $(\mathcal{M}, \eta_{ab})$ be real, 4-dimensional Lorentz vector space with volume form ε_{abcd}; $\varepsilon_{abcd}\varepsilon^{abcd} = -4!$. Let (t^a, x^a, y^a, z^a) be orthonormal basis of $(\mathcal{M}, \eta_{ab})$, and define

$$t^a t_a = +s \quad (s = \pm 1), \quad \varepsilon_{abc} = \varepsilon^{abcd} t_d, \tag{1}$$

where the tensor field $\varepsilon_{abc} = \varepsilon_{[abc]}$ satisfies $\varepsilon_{abc}\varepsilon^{abc} = 3!$ We formulate our equations in the signatures both $(+,-,-,-)$ and $(-,+,+,+)$ by choosing $s = 1$ or -1, respectively[*3], because the former notation is common in working with the spinors.

First, we define the Weyl curvature C_{abcd} by

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} + g_{b[c}R_{d]a} - \frac{1}{3}R g_{a[c}g_{d]b}, \tag{2}$$

and decompose those into its electric and a magnetic components,

$$E_{ab} \equiv - \ C_{ambn}t^m t^n, \quad B_{ab} \equiv - \ {}^*C_{ambn}t^n t^m, \tag{3}$$

[*2] L. Gunnarsen, H. Shinkai and K. Maeda, Class. Quantum and Grav. **12**, 133 (1995).

where ${}^{*}C_{abcd}=\frac{1}{2}\varepsilon_{ab}{}^{mn}C_{mncd}$ is a dual of the Weyl tenso. These decomposed elements E_{ab} and B_{ab} are also presented by the 3-metric γ_{ab} and the extrinsic curvature K_{ab} as [*4]

$$E_{ab}=R_{ab}-K_{a}{}^{m}K_{bm}+KK_{ab}-\frac{2}{3}\Lambda\gamma_{ab}, \tag{4}$$

$$B_{ab}=\varepsilon_{a}{}^{mn}D_{m}K_{nb}. \tag{5}$$

This is why we emphasize that our inputs are '3+1' elements. It follows from two constraint equations that the fields E_{ab}, B_{ab} are both trace-free and symmetric. We can reconstruct the Weyl curvature completely from E_{ab} and B_{ab} by

$$C_{abcd}=4t_{[a}E_{b][c}t_{d]}+2\varepsilon_{ab}{}^{m}B_{m[c}t_{d]}+2\varepsilon_{cd}{}^{m}B_{m[a}t_{b]}+\varepsilon_{ab}{}^{m}\varepsilon_{cd}{}^{n}E_{mn}. \tag{6}$$

The next step is to choose a unit vector field $\hat{z}^{a}$ on Σ, and to decompose E_{ab}, B_{ab} into components along and perpendicular to $\hat{z}^{a}$. We set

$$e=E_{ab}\hat{z}^{a}\hat{z}^{b}, \tag{7a}$$

$$e_{a}=E_{bc}\hat{z}^{b}(\delta_{a}{}^{c}+s\hat{z}_{a}\hat{z}^{c}), \tag{7b}$$

$$e_{ab}=E_{cd}(\delta_{a}{}^{c}+s\hat{z}_{a}\hat{z}^{c})(\delta_{b}{}^{d}+s\hat{z}_{b}\hat{z}^{d})+\frac{1}{2}es_{ab}, \tag{7c}$$

$$b=B_{ab}\hat{z}^{a}\hat{z}^{b}, \tag{7d}$$

$$b_{a}=B_{bc}\hat{z}^{b}(\delta_{a}{}^{c}+s\hat{z}_{a}\hat{z}^{c}), \tag{7e}$$

$$b_{ab}=B_{cd}(\delta_{a}{}^{c}+s\hat{z}_{a}\hat{z}^{c})(\delta_{b}{}^{d}+s\hat{z}_{b}\hat{z}^{d})+\frac{1}{2}bs_{ab}, \tag{7f}$$

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where $s_{ab} = \gamma_{ab} - \hat{z}_a \hat{z}_b$. We note that E_{ab}, B_{ab} is again reconstructed from (8a)-(8f)

$$E_{ab} = e\hat{z}_a\hat{z}_b + 2e_{(a}\hat{z}_{b)} + e_{ab} - \frac{1}{2}s_{ab}e. \tag{8}$$

$$B_{ab} = b\hat{z}_a\hat{z}_b + 2b_{(a}\hat{z}_{b)} + b_{ab} - \frac{1}{2}s_{ab}b. \tag{9}$$

Such decompositions will be useful to discuss the effects of curvatures on the transversal plane to the $\hat{z}^a$ direction.

We put a rotation operator on the plane spanned by $\hat{x}_a$ and $\hat{y}_a$ as,

$$J_a{}^b \equiv \varepsilon_a{}^{bcd}\hat{z}_c t_d. \tag{10}$$

It is easy to check this mapping preserves s_{ab}, and is also easy to check $J_a{}^c J_c{}^b = -(\delta_a{}^b + s\hat{z}_a\hat{z}^b)$, which shows us $J_a{}^b$ has a complex structure, i.e., $J_a{}^b$ lets us define complex multiples of vectors $x^a \in P_z$, according to the formula $(m+in)x^a = mx^a + nJ_b{}^a x^b$. In short, $J_a{}^b$ expresses a rotation by 90 degrees in the plane orthogonal to $\hat{z}^a$.

The 10 components of Weyl curvature are expressed by the following 5 complex numbers in the NP formalism;

$$\begin{aligned}
\Psi_0 &\equiv C_{abcd}l^a m^b l^c m^d &&= \psi_{ABCD}o^A o^B o^C o^D, \\
\Psi_1 &\equiv C_{abcd}l^a n^b l^c m^d &&= \psi_{ABCD}o^A o^B o^C \iota^D, \\
\Psi_2 &\equiv C_{abcd}l^a m^b \bar{m}^c n^d &&= \psi_{ABCD}o^A o^B \iota^C \iota^D, \\
\Psi_3 &\equiv C_{abcd}l^a n^b \bar{m}^c n^d &&= \psi_{ABCD}o^A \iota^B \iota^C \iota^D, \\
\Psi_4 &\equiv C_{abcd}n^a \bar{m}^b n^c \bar{m}^d &&= \psi_{ABCD}\iota^A \iota^B \iota^C \iota^D
\end{aligned} \tag{11}$$

where $l^a, n^a, m^a, \bar{m}^a$ are null tetrad and o^A, ι^A are spinor basis defined as

$$\begin{aligned} l^a &= \tfrac{1}{\sqrt{2}}(t^a + z^a) &= o^A o^{A'}, \\ n^a &= \tfrac{1}{\sqrt{2}}(t^a - z^a) &= \iota^A \iota^{A'}, \\ m^a &= \tfrac{1}{\sqrt{2}}(x^a - iy^a) &= o^A \iota^{A'}, \\ \bar{m}^a &= \tfrac{1}{\sqrt{2}}(x^a + iy^a) &= \iota^A o^{A'}. \end{aligned} \tag{12}$$

Spinor $\psi_{ABCD} = \psi_{(ABCD)}$ is expressed by principal null spinor $\alpha_A, \beta_A, ...$ by using spinor characters,

$$\psi_{ABCD} = \psi_{(ABCD)} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}, \tag{13}$$

and principal null vectors are defined as real vectors constructed by $\alpha_A\bar{\alpha}_{A'}, \beta_A\bar{\beta}_{A'}, ...$ PNDs are those directions.

Finally, substitute (7) and (13) into (12), we get Ψ_i using (8a)-(8f) and (11):

$$\Psi_0 = -(e_{ab} + sJ_a{}^c b_{bc})m^a m^b, \tag{14a}$$
$$\Psi_1 = -\frac{s}{\sqrt{2}}(e_a + sJ_a{}^c b_c)m^a, \tag{14b}$$
$$\Psi_2 = -\frac{1}{2}(e + ib), \tag{14c}$$
$$\Psi_3 = -\frac{s}{\sqrt{2}}(e_a - sJ_a{}^c b_c)\bar{m}^a, \tag{14d}$$
$$\Psi_4 = -(e_{ab} - sJ_a{}^c b_{bc})\bar{m}^a \bar{m}^b. \tag{14e}$$