一般相対論の数値計算手法

近畿大セミナー
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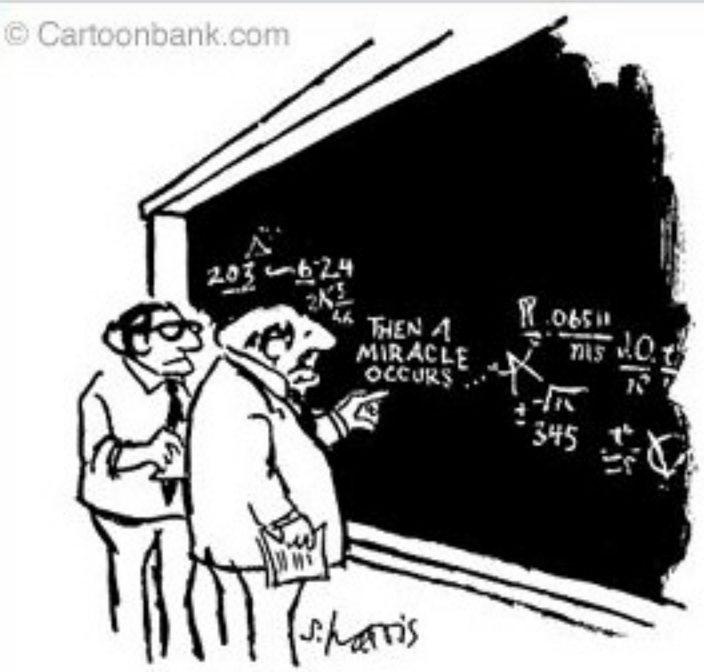

"I think you should be more explicit here in step two."

数値シミュレーションのための定式化

- 現在の一般相対性理論の数値シミュレーションは，ベストな方程式か？
- 使っている方程式が，不安定な発展をする可能性が十分にある．
- Lagrange乗数法で，拘束条件を付加する自由度を使え．

数値シミュレーションのための定式化

Journal of the Korean Physical Society, Vol. 54, No. 6, June 2009, pp. 2513~2528

Formulations of the Einstein Equations for Numerical Simulations

Hisaaki SHINKAI*

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(Received 24 January 2008)

We review recent efforts to re-formulate the Einstein equations for fully relativistic numerical simulations. The so-called numerical relativity is a promising research field matching with ongoing gravitational wave observations. In order to complete long-term and accurate simulations of binary compact objects, people seek a robust set of equations against the violation of constraints. Many trials have revealed that mathematically equivalent sets of evolution equations show different numerical stabilities in free evolution schemes. In this article, we overview the efforts of the community, categorizing them into three directions: (1) modifying of the standard Arnowitt-Deser-Misner (ADM) equations initiated by the Kyoto group [the so-called Baumgarte-Shapiro-Shibata-Nakamura (BSSN) equations], (2) rewriting the evolution equations in a hyperbolic form and (3) constructing an "asymptotically constrained" system. We then introduce our series of works that tries to explain these evolution behaviors in a unified way by using an eigenvalue analysis of the constraint-propagation equations. The modifications of (or adjustments to) the evolution equations change the character of constraint propagation and several particular adjustments using constraints are expected to damp the constraint-violating modes. We show several sets of adjusted ADM and BSSN equations, together with their numerical demonstrations.

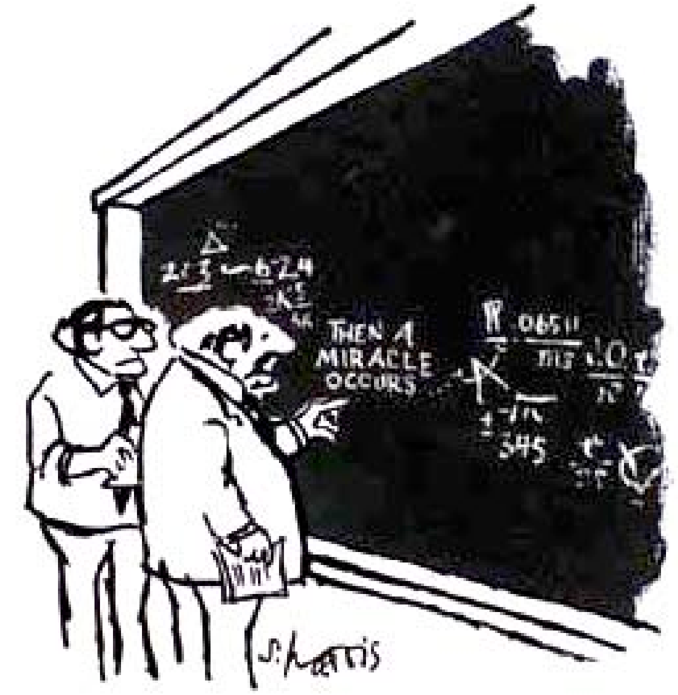
arXiv:0805.0068

Goals of the Lecture

What is the guiding principle for selecting evolution equations for simulations in GR?

Why many groups use the BSSN equations?

Are there an alternative formulation better than the BSSN?



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Procedure of the Standard Numerical Relativity

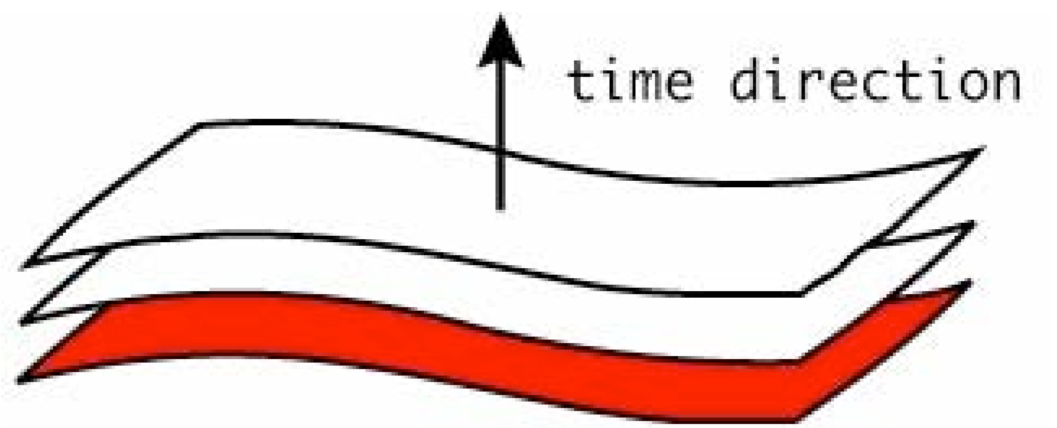
- 3+1 (ADM) formulation

- Preparation of the Initial Data
  - Assume the background metric
  - Solve the constraint equations

- Time Evolution

  do time=1, time_end
  - Specify the slicing condition
  - Evolve the variables
  - Check the accuracy
  - Extract physical quantities

  end do



Σ: Initial 3-dimensional Surface

The Standard ADM formulation (aka York 1978):

The fundamental dynamical variables are (γ_{ij}, K_{ij}), the three-metric and extrinsic curvature. The three-hypersurface Σ is foliated with gauge functions, (α, β^i), the lapse and shift vector.

- The evolution equations:

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \\
\partial_t K_{ij} &= \alpha\ {}^{(3)}R_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - D_i D_j \alpha \\
&\quad +(D_i\beta^k)K_{kj} + (D_j\beta^k)K_{ki} + \beta^k D_k K_{ij} \\
&\quad -8\pi G\alpha\{S_{ij} + (1/2)\gamma_{ij}(\rho_H - \mathrm{tr}S)\},
\end{aligned}$$

where $K = K^i{}_i$, and ${}^{(3)}R_{ij}$ and D_i denote three-dimensional Ricci curvature, and a covariant derivative on the three-surface, respectively.

- Constraint equations:

$$\begin{aligned}
&\text{Hamiltonian constr.} \qquad \mathcal{H}^{ADM} := {}^{(3)}R + K^2 - K_{ij}K^{ij} \approx 0, \\
&\text{momentum constr.} \qquad \mathcal{M}_i^{ADM} := D_j K^j{}_i - D_i K \approx 0,
\end{aligned}$$

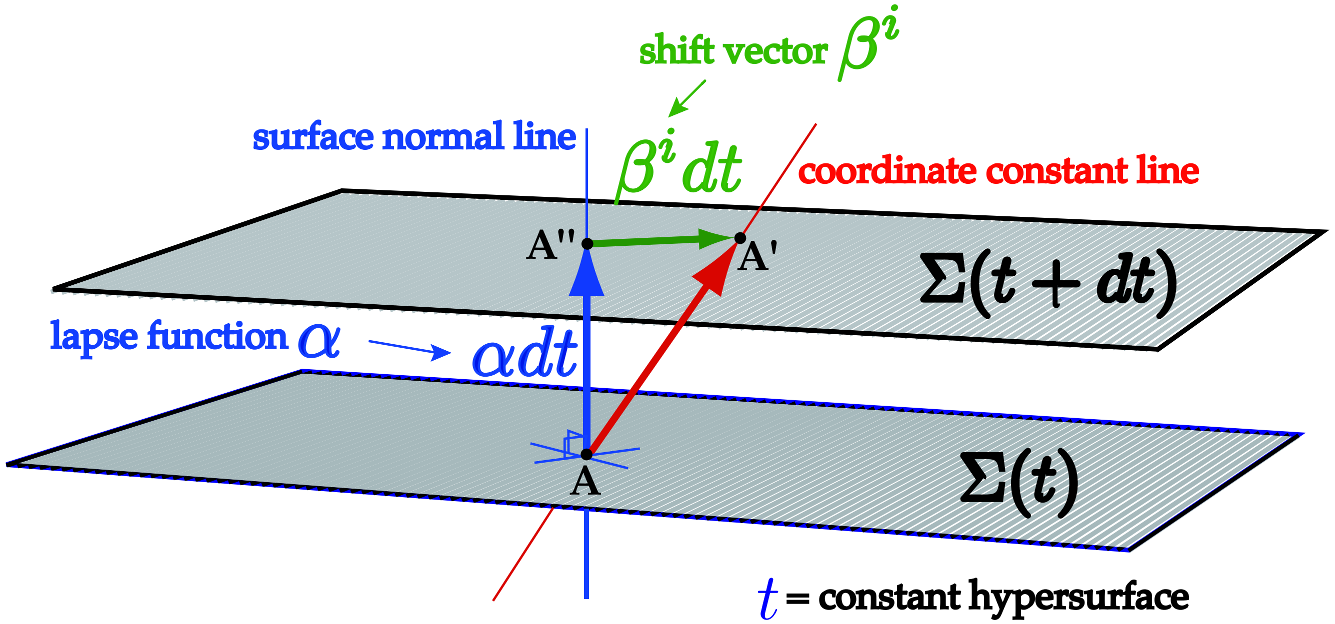
where ${}^{(3)}R = {}^{(3)}R^i{}_i$.

strategy 0 The standard approach :: Arnowitt-Deser-Misner (ADM) formulation (1962)

3+1 decomposition of the spacetime.

Evolve 12 variables (γ_{ij}, K_{ij})

with a choice of gauge condition.



	Maxwell eqs.	ADM Einstein eq.
constraints	div $\mathbf{E} = 4\pi\rho$ div $\mathbf{B} = 0$	${}^{(3)}R + (\mathrm{tr}K)^2 - K_{ij}K^{ij} = 2\kappa\rho_H + 2\Lambda$ $D_j K^j{}_i - D_i \mathrm{tr}K = \kappa J_i$
evolution eqs.	$\frac{1}{c}\partial_t \mathbf{E} = \mathrm{rot}\ \mathbf{B} - \frac{4\pi}{c}\mathbf{j}$ $\frac{1}{c}\partial_t \mathbf{B} = -\mathrm{rot}\ \mathbf{E}$	$\partial_t \gamma_{ij} = -2NK_{ij} + D_j N_i + D_i N_j,$ $\partial_t K_{ij} = N({}^{(3)}R_{ij} + \mathrm{tr}K K_{ij}) - 2NK_{il}K^l{}_j - D_i D_j N$ $+ (D_j N^m)K_{mi} + (D_i N^m)K_{mj} + N^m D_m K_{ij} - N\gamma_{ij}\Lambda$ $- \kappa\alpha\{S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - \mathrm{tr}S)\}$

S. Frittelli, Phys. Rev. D55, 5992 (1997)
HS and G. Yoneda, Class. Quant. Grav. 19, 1027 (2002)

The Constraint Propagations of the Standard ADM:

$$
\begin{aligned}
\partial_t \mathcal{H} &= \beta^j(\partial_j \mathcal{H}) + 2\alpha K \mathcal{H} - 2\alpha \gamma^{ij}(\partial_i \mathcal{M}_j) \\
&\quad + \alpha(\partial_l \gamma_{mk})(2\gamma^{ml}\gamma^{kj} - \gamma^{mk}\gamma^{lj})\mathcal{M}_j - 4\gamma^{ij}(\partial_j \alpha)\mathcal{M}_i, \\
\partial_t \mathcal{M}_i &= -(1/2)\alpha(\partial_i \mathcal{H}) - (\partial_i \alpha)\mathcal{H} + \beta^j(\partial_j \mathcal{M}_i) \\
&\quad + \alpha K \mathcal{M}_i - \beta^k \gamma^{jl}(\partial_i \gamma_{lk})\mathcal{M}_j + (\partial_i \beta_k)\gamma^{kj}\mathcal{M}_j.
\end{aligned}
$$

From these equations, we know that

if the constraints are satisfied on the initial slice Σ,
then the constraints are satisfied throughout evolution (in principle).

Primary / Secondary constraint
First-class / Second-class constraint

- Primary Constraints

$$\text{constraint } C_1(q,p) \approx 0$$
$$\text{constraint } C_2(q,p) \approx 0$$

- Secondary Constraints
  = when propagation of constraints require additional constraints

$$\begin{aligned}\dot{C}_i &= \{C_1, H\}_P = \{C_i, H'(q,p) + \lambda^k C_k\}_P \\ &= \{C_i, H'\}_P + \lambda^k \{C_i, C_k\}_P \approx 0\end{aligned}$$

- First-Class Constraints
  = $\text{set of constraints } C_i \text{ satisfy } \{C_i, C_k\}_P \approx 0$

Numerical Relativity in the 20th century

1960s	Hahn-Lindquist	2 BH head-on collision	AnaPhys29(1964)304
	May-White	spherical grav. collapse	PR141(1966)1232
1970s	ÓMurchadha-York	conformal approach to initial data	PRD10(1974)428
	Smarr	3+1 formulation	PhD thesis (1975)
	Smarr-Cades-DeWitt-Eppley	2 BH head-on collision	PRD14(1976)2443
	Smarr-York	gauge conditions	PRD17(1978)2529
	ed. by L.Smarr	"Sources of Grav. Radiation"	Cambridge(1979)
1980s	Nakamura-Maeda-Miyama-Sasaki	axisym. grav. collapse	PTP63(1980)1229
	Miyama	axisym. GW collapse	PTP65(1981)894
	Bardeen-Piran	axisym. grav. collapse	PhysRep96(1983)205
	Stark-Piran	axisym. grav. collapse	unpublished
1990	Shapiro-Teukolsky	naked singularity formation	PRL66(1991)994
	Oohara-Nakamura	3D post-Newtonian NS coalesence	PTP88(1992)307
	Seidel-Suen	BH excision technique	PRL69(1992)1845
	Choptuik	critical behaviour	PRL70(1993)9
	NCSA group	axisym. 2 BH head-on collision	PRL71(1993)2851
	Cook et al	2 BH initial data	PRD47(1993)1471
	Shibata-Nakao-Nakamura	BransDicke GW collapse	PRD50(1994)7304
	Price-Pullin	close limit approach	PRL72(1994)3297
1995	NCSA group	event horizon finder	PRL74(1995)630
	NCSA group	hyperbolic formulation	PRL75(1995)600
	Anninos *et al*	close limit vs full numerical	PRD52(1995)4462
	Scheel-Shapiro-Teukolsky	BransDicke grav. collapse	PRD51(1995)4208
	Shibata-Nakamura	3D grav. wave collapse	PRD52(1995)5428
	Gunnersen-Shinkai-Maeda	ADM to NP	CQG12(1995)133
	Wilson-Mathews	NS binary inspiral, prior collapse?	PRL75(1995)4161
	Pittsburgh group	Cauchy-characteristic approach	PRD54(1996)6153
	Brandt-Brügmann	BH puncture data	PRL78(1997)3606
	Illinois group	synchronized NS binary initial data	PRL79(1997)1182
	Shibata-Baumgarte-Shapiro	2 NS inspiral, PN to GR	PRD58(1998)023002
	BH Grand Challenge Alliance	characteristic matching	PRL80(1998)3915
	Baumgarte-Shapiro	Shibata-Nakamura formulation	PRD59(1998)024007
	Brady-Creighton-Thorne	intermediate binary BH	PRD58(1998)061501
	Meudon group	irrotational NS binary initial data	PRL82(1999)892
	Shibata	2 NS inspiral coalesence	PRD60(1999)104052

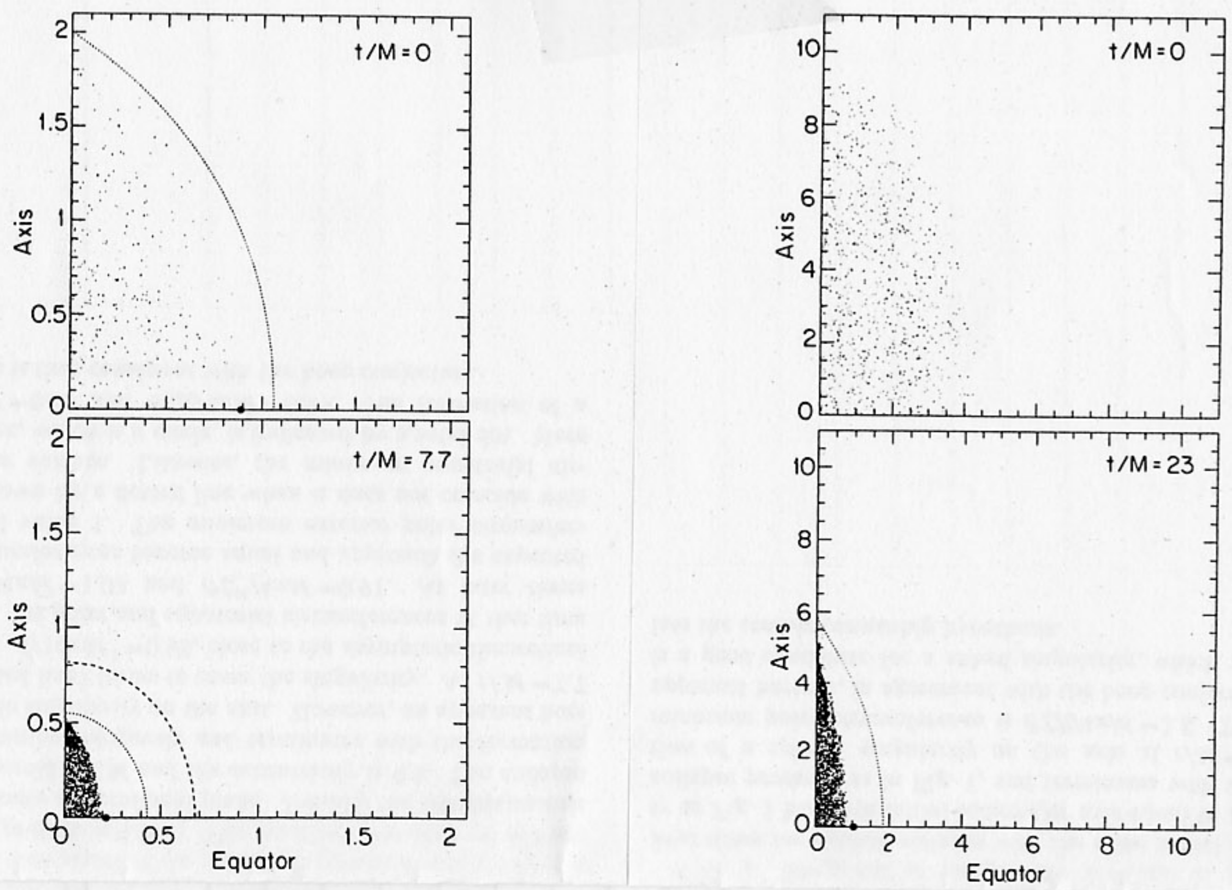
Formation of Naked Singularities: The Violation of Cosmic Censorship

Stuart L. Shapiro and Saul A. Teukolsky

Center for Radiophysics and Space Research and Departments of Astronomy and Physics, Cornell University, Ithaca, New York 14853

(Received 7 September 1990)

We use a new numerical code to evolve collisionless gas spheroids in full general relativity. In all cases the spheroids collapse to singularities. When the spheroids are sufficiently compact, the singularities are hidden inside black holes. However, when the spheroids are sufficiently large, there are no apparent horizons. These results lend support to the hoop conjecture and appear to demonstrate that naked singularities can form in asymptotically flat spacetimes.



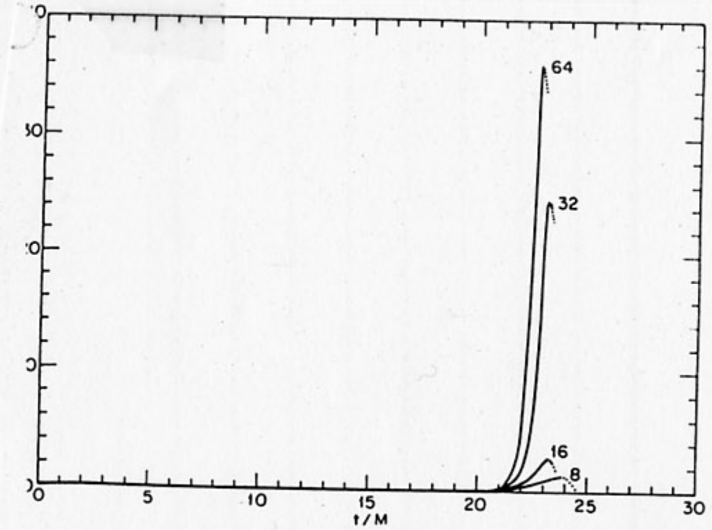


FIG. 3. Growth of the Riemann invariant I (in units of M^{-4}) vs time for the collapse shown in Fig. 2. The simulation was repeated with various angular grid resolutions. Each curve is labeled by the number of angular zones used. We use dots to show where the singularity has caused the code to become inaccurate.

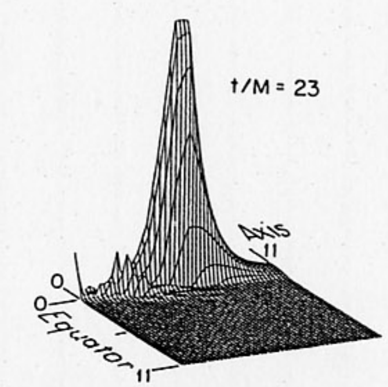


FIG. 4. Profile of I in a meridional plane for the collapse shown in Fig. 2. For the case of 32 angular zones shown here, the peak value of I is $24/M^4$ and occurs on the axis just outside the matter.

Critical Phenomena in Gravitational Collapse

Choptuik, Phys. Rev. Lett. 70 (1993) 9

TABLE I. Initial data specification for various one-parameter families discussed in text. For families (a)–(c), I specified the initial pulses to be purely in-going. For family (d), the functions $X_>(r)$, $Y_<(r)$ and $X_>(r)$, $Y_>(r)$ are late-time fits to subcritical and supercritical evolutions, respectively, of the pulse shape shown in Fig. 1(d).

Family	Form of initial data	p
(a)	$\phi(r) = \phi_0 r^3 \exp(-[(r-r_0)/\delta]^q)$	ϕ_0, r_0, δ, q
(b)	$\phi(r) = \phi_0 \tanh[(r-r_0)/\delta]$	ϕ_0
(c)	$\phi(r+r_0) = \phi_0 r^{-5}[\exp(1/r)-1]^{-1}$	ϕ_0
(d)	$X(r) = (1-\eta)X_<(r) + \eta X_>(r)$ $Y(r) = (1-\eta)Y_<(r) + \eta Y_>(r)$	η

TABLE II. Numerically determined values of the scaling exponent γ in the conjectured relationship $\boxed{M_{\rm BH} \simeq c_f\,|p-p^*|^{\gamma}}$. $\mu_{\min}$ and $\mu_{\max}$ are the minimum and maximum mass fractions ($\mu \equiv M_{\rm BH}/M$) of the black holes computed in the simulation and γ is the least-squares estimate of the scaling exponent.

Family	Parameter	$\mu_{\min}$	$\mu_{\max}$	γ
(a)	ϕ_0	7.9×10^{-3}	8.9×10^{-1}	0.376
(a)	δ	1.3×10^{-3}	9.4×10^{-1}	0.372
(a)	q	3.1×10^{-3}	9.8×10^{-1}	0.372
(a)	r_0	1.3×10^{-2}	9.2×10^{-1}	0.379
(b)	ϕ_0	2.8×10^{-3}	4.0×10^{-1}	0.372
(c)	ϕ_0	4.9×10^{-3}	9.9×10^{-1}	0.366
(d)	η	2.2×10^{-5}	1.7×10^{-2}	0.380

Spherical Sym., Massless Scalar Field

(1) scaling

(2) echoing

(3) universality

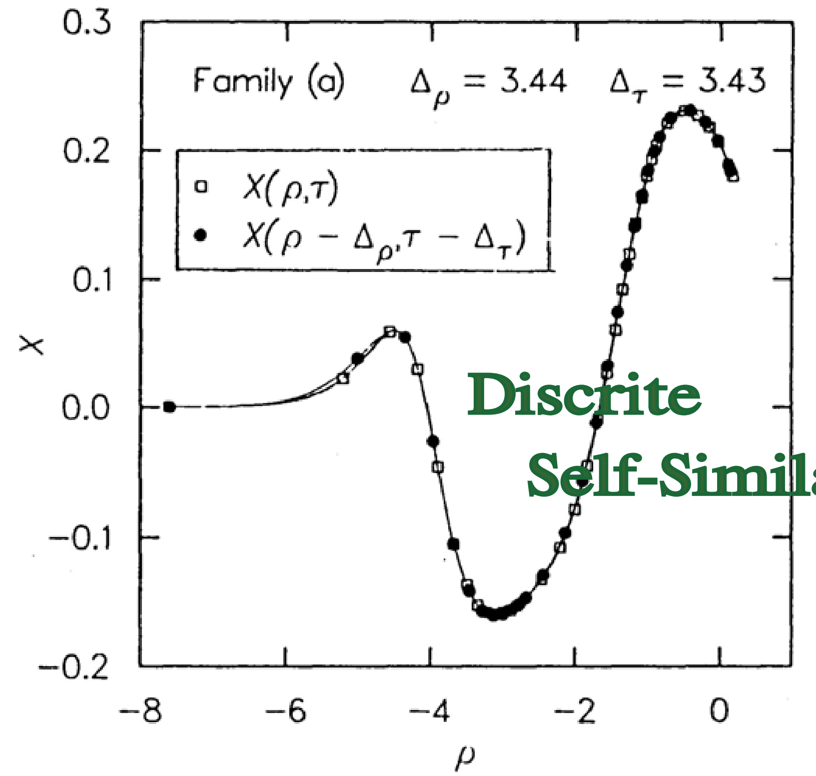


FIG. 2. Illustration of the rescaling or echoing property observed in near-critical evolution of the scalar field. The curve marked with open squares shows the profile of the scalar field variable, X, at some proper central time T_0. The curve marked with solid circles is the profile at a later time $T_0 + e^{\Delta_\tau}$ but on a scale $e^{\Delta_\rho} \approx 30$ times smaller.

Head-on Collision of 2 Black-Holes (Misner initial data)

NCSA group 1995

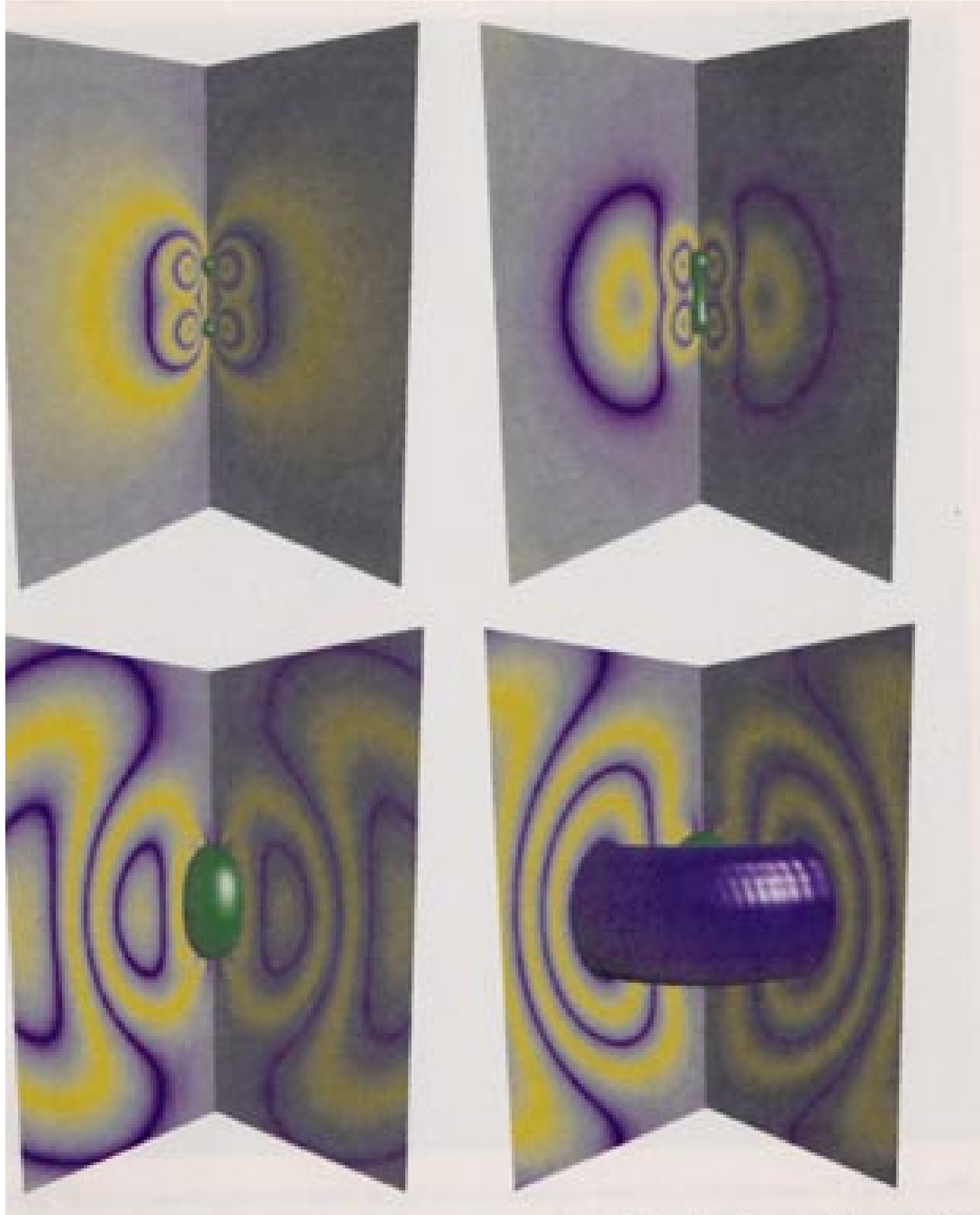


Fig. C.6. 3D evolution of the radiation field Ψ_4 of the head-on collision of two equal-mass black holes shown as a blue and yellow color-map

S. Frittelli, Phys. Rev. D55, 5992 (1997)
HS and G. Yoneda, Class. Quant. Grav. 19, 1027 (2002)

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\end{aligned}
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From these equations, we know that

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\end{aligned}
$$

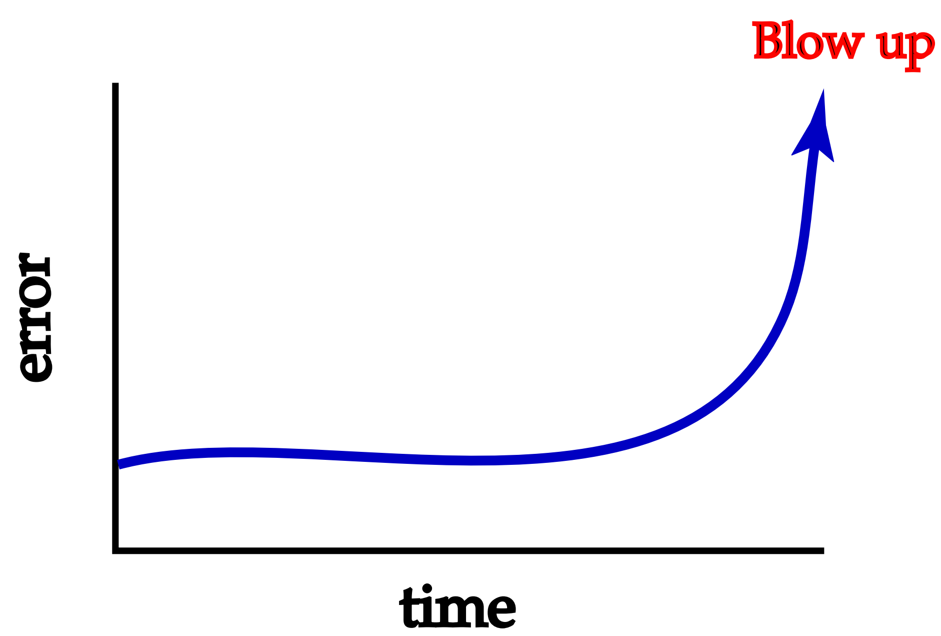
From these equations, we know that

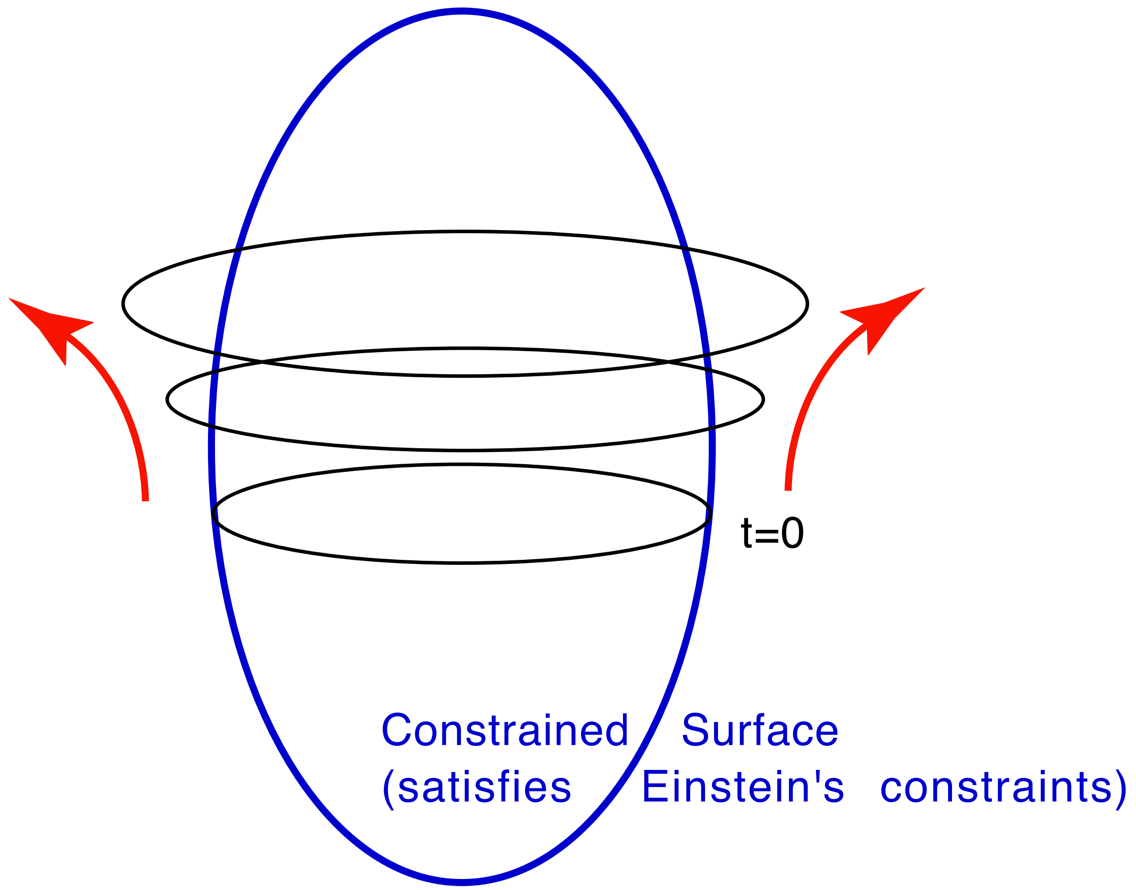
if the constraints are satisfied on the initial slice Σ,
then the constraints are satisfied throughout evolution (in principle).

But this is NOT TRUE in NUMERICS....

- By the period of 1990s, NR had provided a lot of physics:
  Gravitational Collapse, Critical Behavior, Naked Singularity, Event Horizons,
  Head-on Collision of BH-BH and Gravitational Wavve, Cosmology, $\cdots$
- However, for the BH-BH/NS-NS inspiral coalescence problem, $\cdots$ why ???

  Many (too many) trials and errors, hard to find a definit recipe.



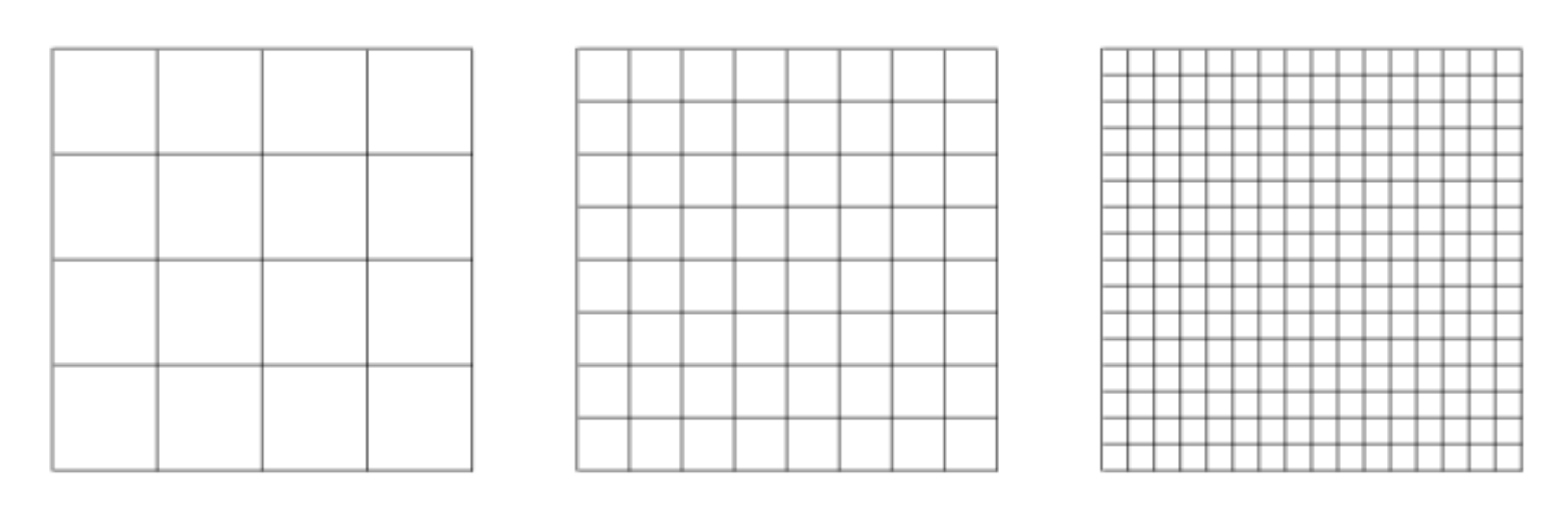


Best formulation of the Einstein eqs. for long-term stable & accurate simulation?

“Convergence”

= higher resolution runs approach to the continuum limit.
(All numerical codes must have this property.)

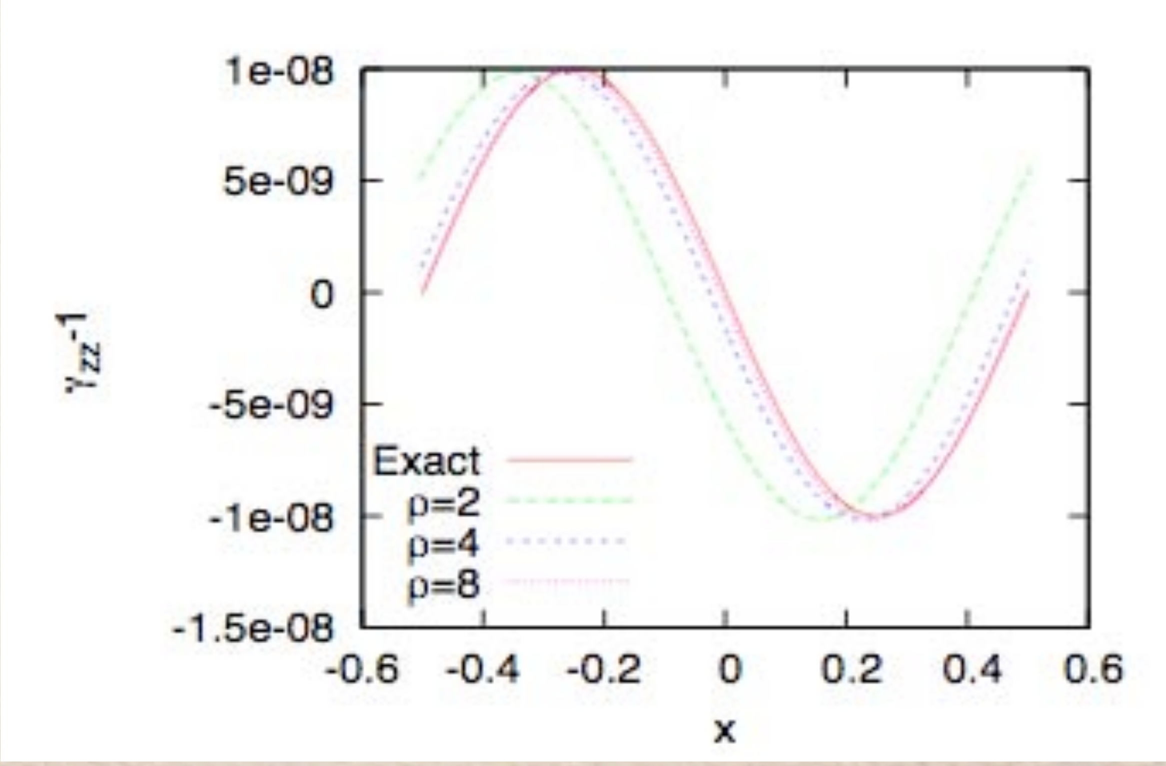
- When the code has 2nd order finite difference scheme, $O\left((\Delta x)^2\right)$ then the error should be scaled with $O\left((\Delta x)^2\right)$
- “Consistency”, Choptuik, PRD 44 (1991) 3124



“Accuracy”

= The numerical results represent the actual solutions.
(All numerical codes must have this property.)

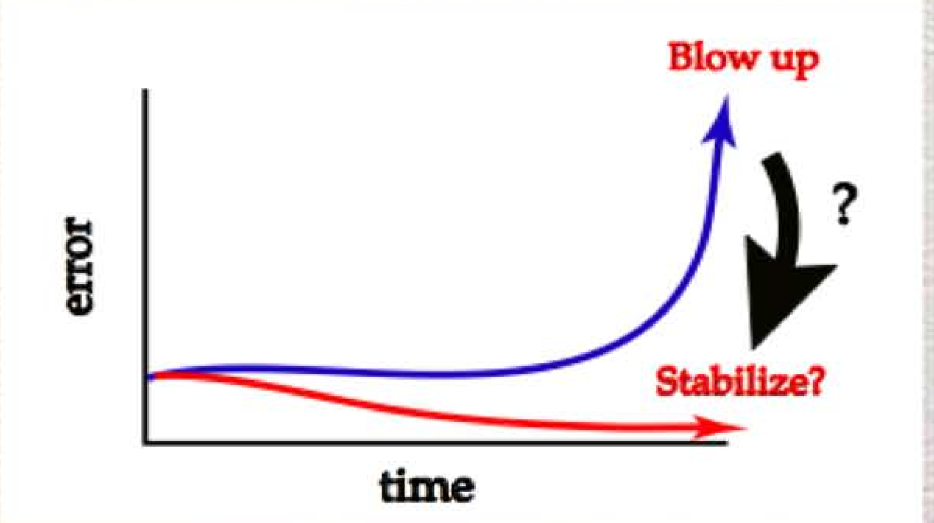
- Check the code with known results.



Gauge wave test in BSSN;
Kiuchi, HS, PRD (2008)

"Stability"

- **We** mean that a numerical simulation continues without any blow-ups and data remains on the constrained surface.



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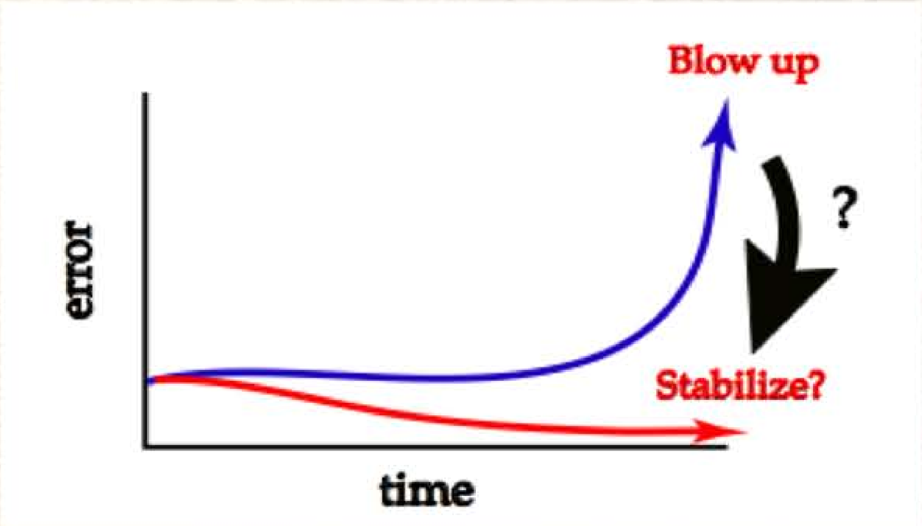

- **Mathematicians** define in terms of the PDE well-posedness.

$$||u(t)|| \leq e^{\kappa t}||u(0)||$$

“Stability”

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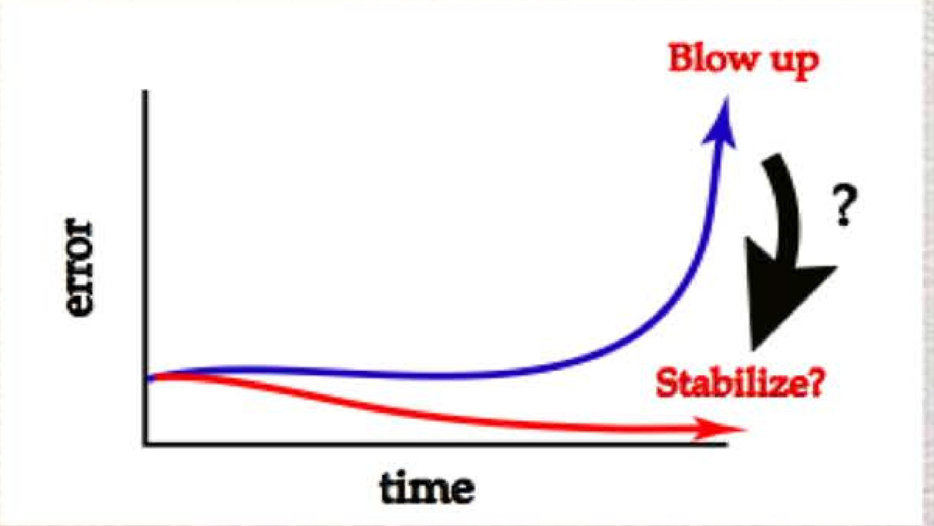


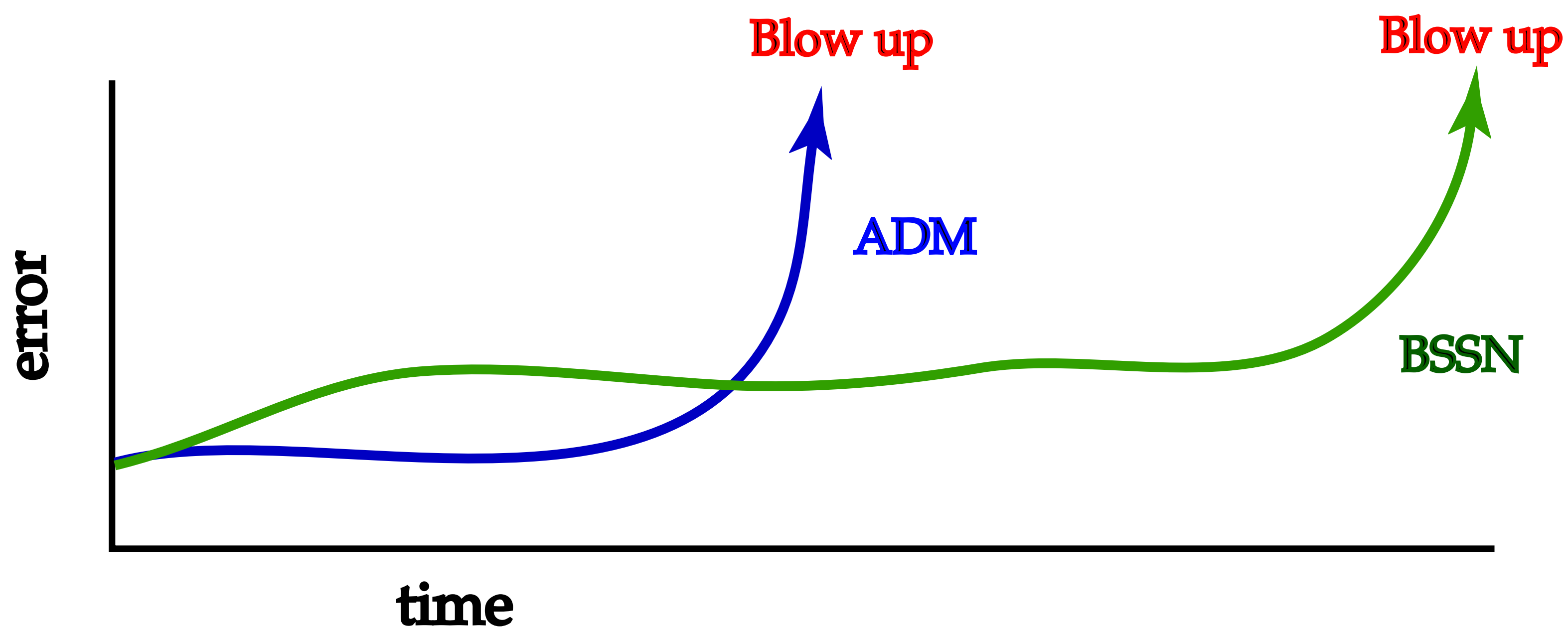
- **Mathematicians** define in terms of the PDE well-posedness.

$$||u(t)|| \leq e^{\kappa t}||u(0)||$$

- **Programmers** define for selecting a finite differencing scheme (judged by von Neumann's analysis).
  Lax's equivalence theorem says that if a numerical scheme is consistent (converging) and stable, then the simulation represents the right (converging) solution.

Best formulation of the Einstein eqs. for long-term stable & accurate simulation?

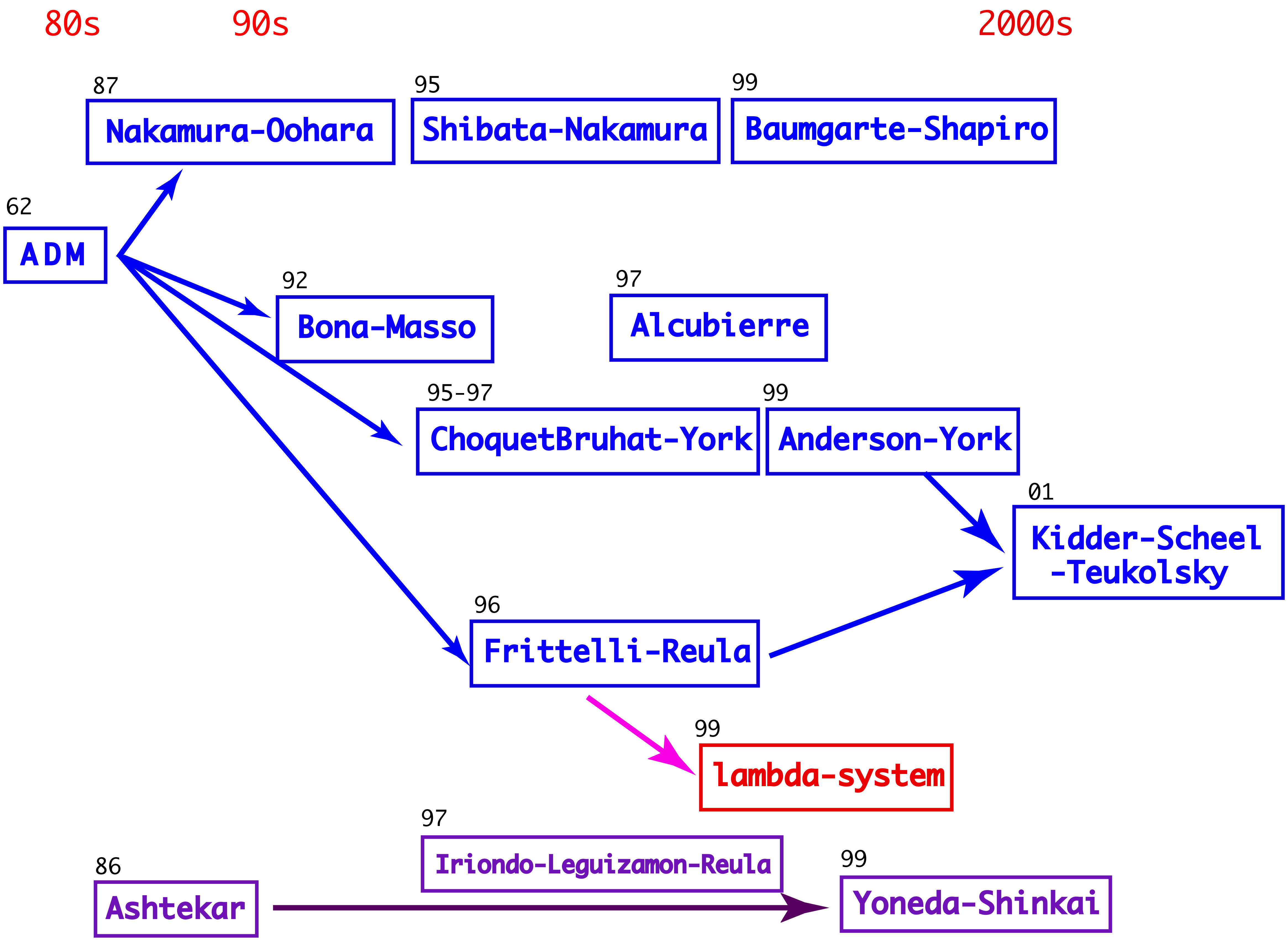
- Many (too many) trials and errors, hard to find a definit recipe.



Mathematically equivalent formulations, but differ in its stability!

strategy 0: Arnowitt-Deser-Misner (ADM) formulation
strategy 1: Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation
strategy 2: Hyperbolic formulations
strategy 3: "Asymptotically constrained" against a violation of constraints

By adding constraints in RHS, we can kill error-growing modes
$\Rightarrow$ How can we understand the features systematically?


80s
90s
2000s
87
Nakamura-Oohara
95
Shibata-Nakamura
99
Baumgarte-Shapiro
62
ADM
92
Bona-Masso
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Alcubierre
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Iriondo-Leguizamon-Reula
86
Ashtekar
99
Yoneda-Shinkai

strategy 1 **Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation**

T. Nakamura, K. Oohara and Y. Kojima, Prog. Theor. Phys. Suppl. **90**, 1 (1987)
M. Shibata and T. Nakamura, Phys. Rev. D **52**, 5428 (1995)
T.W. Baumgarte and S.L. Shapiro, Phys. Rev. D **59**, 024007 (1999)

The popular approach. Nakamura's idea in 1980s.
BSSN is a tricky nickname. BS (1999) introduced a paper of SN (1995).

- define new set of variables $(\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$, instead of the ADM's (γ_{ij}, K_{ij}) where

$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},$$

  and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

- The set of evolution equations become

$$\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}$$

Momentum constraint was used in Γ^i-eq.

- Calculate Riemann tensor as

$$
\begin{aligned}
R_{ij} &= \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^{\phi}_{ij} \\
& R^{\phi}_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi) \\
& \tilde{R}_{ij} = -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}^{lm}\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj}
\end{aligned}
$$

- Constraints are $\mathcal{H}, \mathcal{M}_i$.
  But thre are additional ones, $\mathcal{G}^i, \mathcal{A}, \mathcal{S}$.

Hamiltonian and the momentum constraint equations

$$
\begin{aligned}
\mathcal{H}^{BSSN} &= R^{BSSN} + K^2 - K_{ij}K^{ij}, & (1) \\
\mathcal{M}_i^{BSSN} &= \mathcal{M}_i^{ADM}, & (2)
\end{aligned}
$$

Additionally, we regard the following three as the constraints:

$$
\begin{aligned}
\mathcal{G}^i &= \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, & (3) \\
\mathcal{A} &= \tilde{A}_{ij}\tilde{\gamma}^{ij}, & (4) \\
\mathcal{S} &= \tilde{\gamma} - 1, & (5)
\end{aligned}
$$

Why BSSN better than ADM?
Is the BSSN best? Are there any alternatives?

Some known fact (technical):

- Trace-out A_{ij} at every time step helps the stability.
  Alcubierre, et al, [PRD 62 (2000) 044034]

- "The essential improvement is in the process of replacing terms by the momentum constraints",
  Alcubierre, et al, [PRD 62 (2000) 124011]

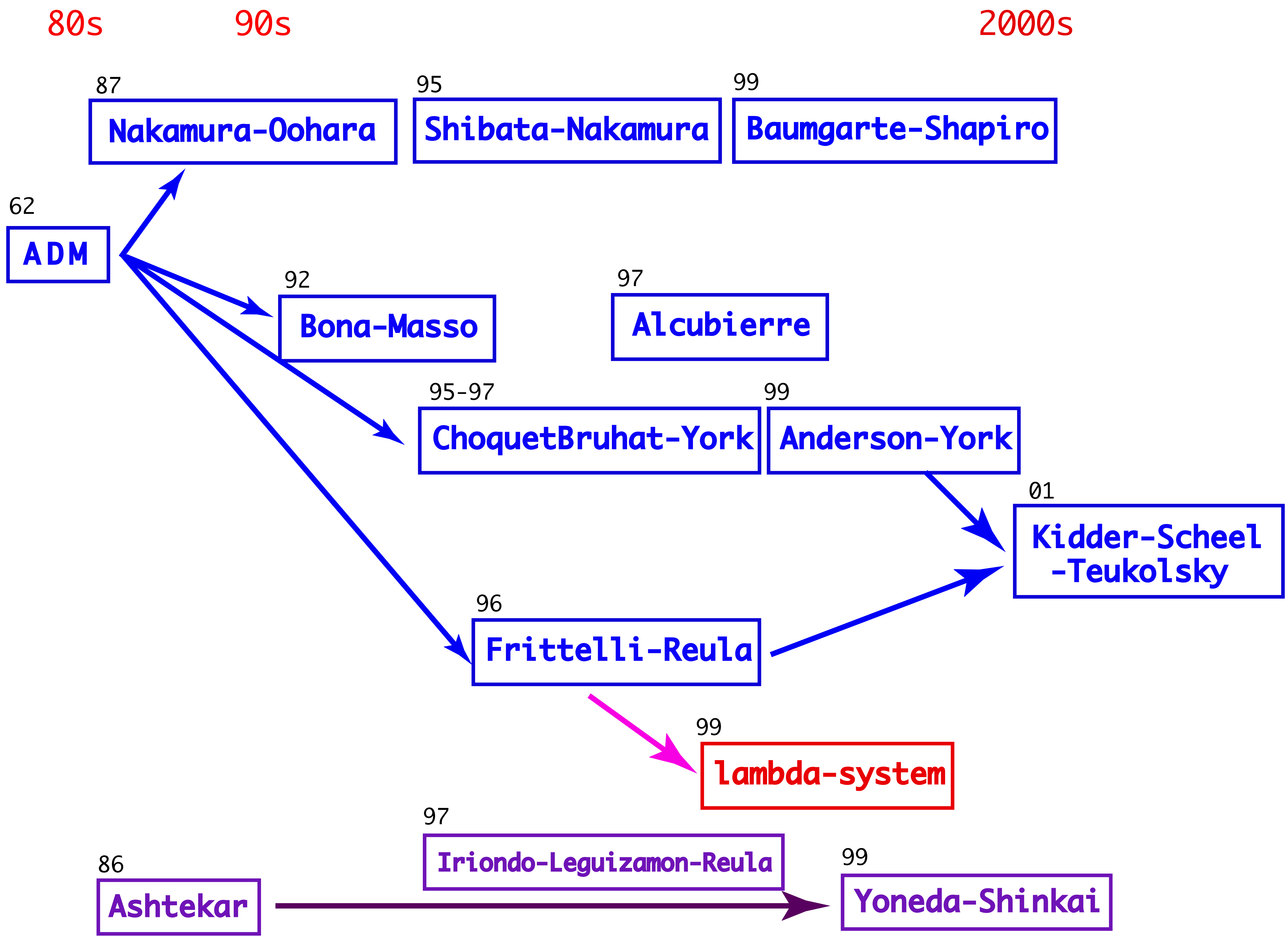
- $\tilde{\Gamma}^i$ is replaced by $-\partial_j \tilde{\gamma}^{ij}$ where it is not differentiated,
  Campanelli, et al, [PRL96 (2006) 111101; PRD 73 (2006) 061501R]

- $\tilde{\Gamma}^i$-equation has been modified as suggested in Yo-Baumgarte-Shapiro [PRD 66 (2002) 084026]
  Baker et al, [PRL96 (2006) 111102; PRD73 (2006) 104002]

Some guesses:

- BSSN has a wider range of parameters that give us stable evolutions in von Neumann's stability analysis. Miller, [gr-qc/0008017]

- The eigenvalues of BSSN *evolution equations* has fewer "zero eigenvalues" than those of ADM, and they conjectured that the instability can be caused by "zero eigenvalues" that violate "gauge mode".
  M. Alcubierre, et al, [PRD 62 (2000) 124011]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Shibata-Nakamura
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Baumgarte-Shapiro
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ADM
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Bona-Masso
97
Alcubierre
95-97
ChoquetBruhat-York
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Anderson-York
01
Kidder-Scheel
-Teukolsky
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Frittelli-Reula
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lambda-system
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Iriondo-Leguizamon-Reula
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Ashtekar
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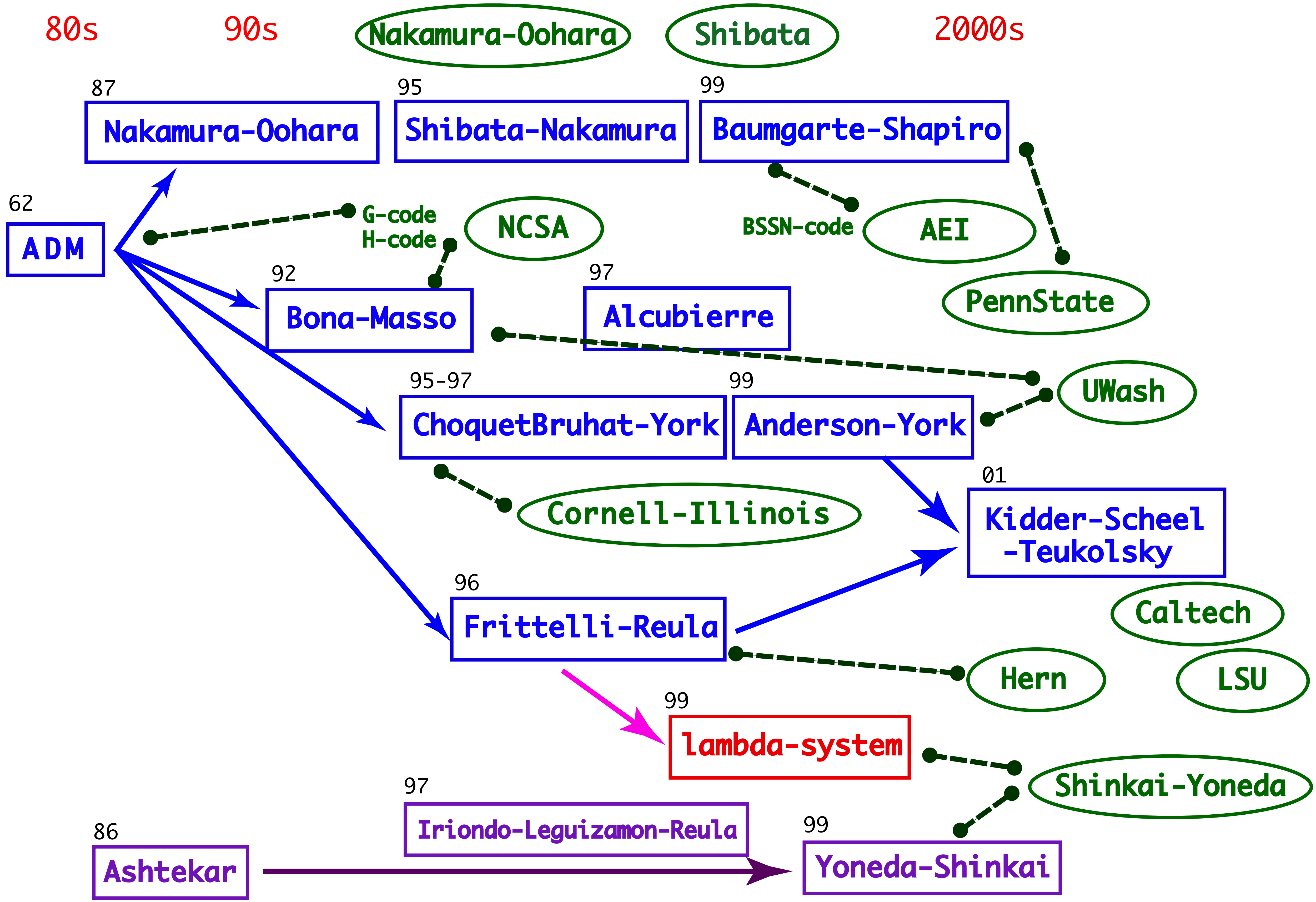
Yoneda-Shinkai



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Alcubierre
PennState
UWash
95-97
ChoquetBruhat-York
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Anderson-York
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Kidder-Scheel
-Teukolsky
Cornell-Illinois
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Frittelli-Reula
Caltech
Hern
LSU
99
lambda-system
Shinkai-Yoneda
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Ashtekar
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Yoneda-Shinkai

strategy 2 **Hyperbolic formulation**

Construct a formulation which reveals a hyperbolicity explicitly.
For a first order partial differential equations on a vector u,

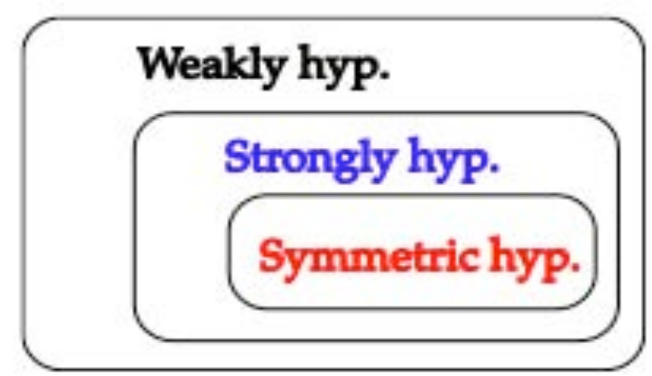
$$\partial_t \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{characteristic part}} + \underbrace{B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{lower order part}}$$

Hyperbolic Formulation
(1) Definition

For a first order partial differential equations on a vector u,

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{characteristic part}} + \underbrace{B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{lower order part}}$$

	if the eigenvalues of A are
weakly hyperbolic	all real.
strongly hyperbolic	all real and $\exists$ a complete set of eigenvalues.
symmetric hyperbolic	if A is real and symmetric (Hermitian).

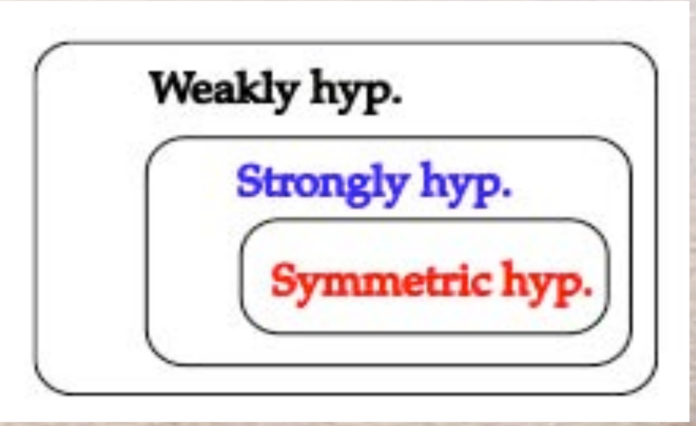


Hyperbolic Formulation (2) Expectations

- if strongly/symmetric hyperbolic ==> well-posed system
  - Given initial data + source terms -> a unique solution exists
  - The solution depends continuously on the data
  - Exists an upper bound on (unphysical) energy norm

$$||u(t)|| \leq e^{\kappa t} ||u(0)||$$

- Better boundary treatments
  <== existence of characteristic field
- Known numerical techniques in
  Newtonian hydro-dynamics



strategy 2 Hyperbolic formulation

Construct a formulation which reveals a hyperbolicity explicitly.
For a first order partial differential equations on a vector u,

$$\partial_t \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \partial_x \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{characteristic part}} + \underbrace{B \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix}}_{\text{lower order part}}$$

However,

- ADM is not hyperbolic.
- BSSN is not hyperbolic.
- Many many hyperbolic formulations are presented. Why many? $\Rightarrow$ Exercise.

One might ask ...

Are they actually helpful?

Which level of hyperbolicity is necessary?

Exercise 1 of hyperbolic formulation

Wave equation $(\partial_t \partial_t - c^2 \partial_x \partial_x) u = 0$

Exercise 1 of hyperbolic formulation

Wave equation $(\partial_t \partial_t - c^2 \partial_x \partial_x) u = 0$

[1a] use u as one of the fundamental variables.

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} \tag{6}$$

Eigenvalues $= \pm c$. Not a symmetric hyperbolic, but a kind of strongly hyperbolic.

[1b]

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} \tag{7}$$

Eigenvalues $= \pm c$. Symmetric hyperbolic.

[2a] Let $U = \dot{u}, V = u'$,

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} U \\ V \end{pmatrix} \tag{8}$$

Eigenvalues $= \pm c$. Not a symmetric hyperbolic, but a kind of strongly hyperbolic.

[2b] Let $U = \dot{u}, V = cu'$,

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \partial_x \begin{pmatrix} U \\ V \end{pmatrix} \tag{9}$$

Eigenvalues $= \pm c$. Symmetric hyperbolic.

Exercise 1 of hyperbolic formulation

Wave equation $(\partial_t\partial_t - c^2\partial_x\partial_x)u = 0$

$[\mathbf{3a}]$ Let $v = \dot{u}, w = v'$,

$$\partial_t \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c^2 \\ 0 & 1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Eigenvalues $= 0, \pm c$. Not a symmetric hyperbolic, nor a strongly hyperbolic.

$[\mathbf{3b}]$ Let $v = \dot{u}, w = cv'$,

$$\partial_t \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & c & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

Eigenvalues $= 0, \pm c$. Not a symmetric hyperbolic, nor a strongly hyperbolic.

$[4]$ Let $f = \dot{u} - cu', g = \dot{u} + cu'$,

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \partial_x \begin{pmatrix} f \\ g \end{pmatrix} \tag{12}$$

Eigenvalues $= \pm c$. Symmetric hyperbolic, de-coupled.

Exercise 2 of hyperbolic formulation | Maxwell equations

Consider the Maxwell equations in the vacuum space,

$$\operatorname{div}\mathbf{E} = 0, \tag{1a}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{1b}$$

$$\operatorname{rot}\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0, \tag{1c}$$

$$\operatorname{rot}\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0. \tag{1d}$$

Exercise 2 of hyperbolic formulation | Maxwell equations | (cont.)

- Take a pair of variables as $u^i = (E_1, E_2, E_3, B_1, B_2, B_3)^T$, and write (1c) and (1d) in the matrix form

$$\partial_t \begin{bmatrix} E_i \\ B_i \end{bmatrix} \cong \underbrace{\begin{bmatrix} A^{l}{}_i{}^j & B^{l}{}_i{}^j \\ C^{l}{}_i{}^j & D^{l}{}_i{}^j \end{bmatrix}}_{\text{Hermitian?}} \partial_l \begin{bmatrix} E_j \\ B_j \end{bmatrix}. \tag{2}$$

- In the Maxwell case, we see immediately

$$\partial_t u_i = c \begin{pmatrix} 0 & \epsilon_i{}^{lm} \\ -\epsilon_i{}^{lm} & 0 \end{pmatrix} \partial_l u_m$$

or with the actual components

$$\partial_t \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = c \begin{pmatrix} 0 & \begin{pmatrix} 0 & -\delta_3^l & \delta_2^l \\ \delta_3^l & 0 & -\delta_1^l \\ -\delta_2^l & \delta_1^l & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \delta_3^l & -\delta_2^l \\ -\delta_3^l & 0 & \delta_1^l \\ \delta_2^l & -\delta_1^l & 0 \end{pmatrix} & 0 \end{pmatrix} \partial_l \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

That is, symmetric hyperbolic system.

Exercise 2 of hyperbolic formulation | Maxwell equations | (cont.)

- The eigen-equation of the characteristic matrix becomes

$$\det\begin{pmatrix} A^{l}{}_{i}{}^{j} - {\color{blue}\lambda^l}\delta_i^j & B^{l}{}_{i}{}^{j} \\ C^{l}{}_{i}{}^{j} & D^{l}{}_{i}{}^{j} - {\color{blue}\lambda^l}\delta_i^j \end{pmatrix} = \det\begin{pmatrix} \begin{pmatrix} -\lambda^l & 0 & 0 \\ 0 & -\lambda^l & 0 \\ 0 & 0 & -\lambda^l \end{pmatrix} & c\begin{pmatrix} 0 & -\delta_3^l & \delta_2^l \\ \delta_3^l & 0 & -\delta_1^l \\ -\delta_2^l & \delta_1^l & 0 \end{pmatrix} \\ c\begin{pmatrix} 0 & \delta_3^l & -\delta_2^l \\ -\delta_3^l & 0 & \delta_1^l \\ \delta_2^l & -\delta_1^l & 0 \end{pmatrix} & \begin{pmatrix} -\lambda^l & 0 & 0 \\ 0 & -\lambda^l & 0 \\ 0 & 0 & -\lambda^l \end{pmatrix} \end{pmatrix} = 0$$

We therefore obtain the eigenvalues as

$$0 \text{ (2 multi)}, \qquad \pm c\sqrt{(\delta_1^l)^2 + (\delta_2^l)^2 + (\delta_3^l)^2} \equiv \pm c \text{ (2 each)}$$

Exercise 3 of hyperbolic formulation

Adjusted Maxwell equations

By adding constraints (1a) and (1b) in the RHS of equations, and see what will be happend.

$$\partial_t u_i = c \begin{pmatrix} 0 & -\epsilon_i{}^{lm} \\ \epsilon_i{}^{lm} & 0 \end{pmatrix} \partial_l u_m + c \begin{pmatrix} x \\ y \end{pmatrix} \partial_k E_k + c \begin{pmatrix} z \\ w \end{pmatrix} \partial_k B_k, \tag{3}$$

where x, y, z, w are parameters.

Exercise 3 of hyperbolic formulation | Adjusted Maxwell equations (cont.)

By adding constraints (1a) and (1b) in the RHS of equations, and see what will be happend.

$$\partial_t u_i = c \begin{pmatrix} 0 & -{\epsilon_i}^{lm} \\ {\epsilon_i}^{lm} & 0 \end{pmatrix} \partial_l u_m + c \begin{pmatrix} x \\ y \end{pmatrix} {\color{blue}\partial_k E_k} + c \begin{pmatrix} z \\ w \end{pmatrix} {\color{blue}\partial_k B_k,} \tag{3}$$

where x, y, z, w are parameters.

- The actual components are

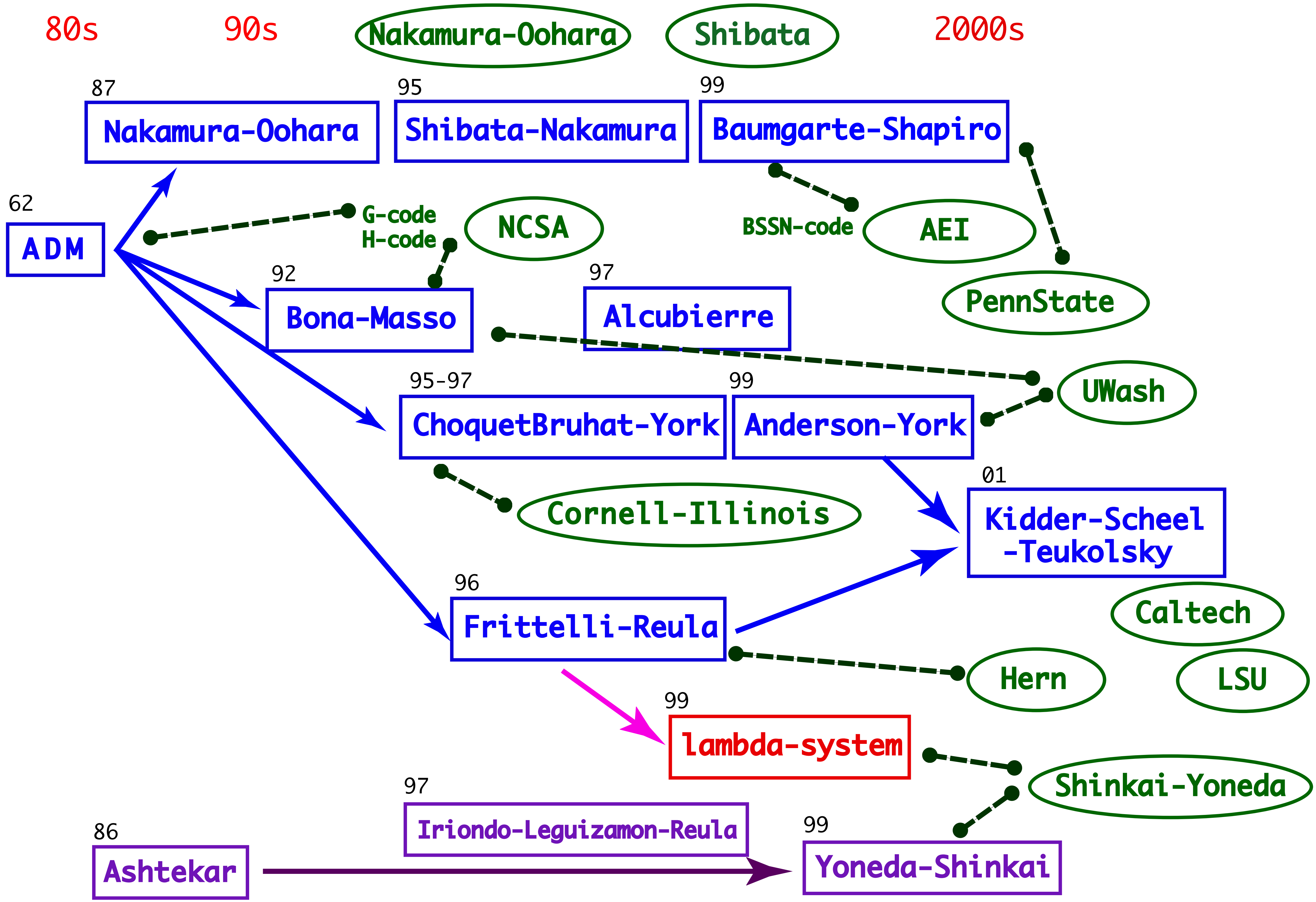
$$\partial_t \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} = c \begin{pmatrix} x \begin{pmatrix} \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \end{pmatrix} & z \begin{pmatrix} \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \end{pmatrix} + \begin{pmatrix} 0 & -\delta_3^l & \delta_2^l \\ \delta_3^l & 0 & -\delta_1^l \\ -\delta_2^l & \delta_1^l & 0 \end{pmatrix} \\ y \begin{pmatrix} \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \end{pmatrix} + \begin{pmatrix} 0 & \delta_3^l & -\delta_2^l \\ -\delta_3^l & 0 & \delta_1^l \\ \delta_2^l & -\delta_1^l & 0 \end{pmatrix} & w \begin{pmatrix} \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \\ \delta_1^l & \delta_2^l & \delta_3^l \end{pmatrix} \end{pmatrix} \partial_l \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

  We see that adding constraint terms break the symmetricity of the characteristic matrix.

- The eigenvalues will be changed as

  $$\frac{c}{2}\left(x + w \pm \sqrt{x^2 - 2xw + w^2 + 4yz}\right)(\delta_1^l + \delta_2^l + \delta_3^l) \text{ (1 each)}, \qquad \pm c \text{ (2 each)}.$$

  The zero eigenvalues disappear by adding constraints, and they can be also $|c|$ if the parameters have the relation $(yz - xw - 1)^2 = (x + w)^2$.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86
Ashtekar
99
Yoneda-Shinkai

Kidder-Scheel-Teukolsky hyperbolic formulation (Anderson-York + Frittelli-Reula)

Phys. Rev. D. 64 (2001) 064017

- Construct a First-order form using variables $(K_{ij}, g_{ij}, d_{kij})$ where $d_{kij} \equiv \partial_k g_{ij}$
  Constraints are $(\mathcal{H}, \mathcal{M}_i, \mathcal{C}_{kij}, \mathcal{C}_{klij})$ where $\mathcal{C}_{kij} \equiv d_{kij} - \partial_k g_{ij}$, and $\mathcal{C}_{klij} \equiv \partial_{[k} d_{l]ij}$
- Densitize the lapse, $Q = \log(N g^{-\sigma})$
- Adjust equations with constraints

$$\begin{aligned}
\hat{\partial}_0 g_{ij} &= -2NK_{ij} \\
\hat{\partial}_0 K_{ij} &= (\cdots) + \gamma N g_{ij}\mathcal{H} + \zeta N g^{ab}\mathcal{C}_{a(ij)b} \\
\hat{\partial}_0 d_{kij} &= (\cdots) + \eta N g_{k(i}\mathcal{M}_{j)} + \chi N g_{ij}\mathcal{M}_k
\end{aligned}$$

- Re-deining the variables $(P_{ij}, g_{ij}, M_{kij})$

$$\begin{aligned}
P_{ij} &\equiv K_{ij} + \hat{z} g_{ij} K, \\
M_{kij} &\equiv (1/2)[\hat{k} d_{kij} + \hat{e} d_{(ij)k} + g_{ij}(\hat{a} d_k + \hat{b} b_k) + g_{k(i}(\hat{c} d_{j)} + \hat{d} b_{j)})], \quad d_k = g^{ab} d_{kab}, b_k = g^{ab} d_{abk}
\end{aligned}$$

The redefinition parameters

- do not change the eigenvalues of evolution eqs.
- do not effect on the principal part of the constraint evolution eqs.
- do affect the eigenvectors of evolution system.
- do affect nonlinear terms of evolution eqs/constraint evolution eqs.

Numerical experiments of KST hyperbolic formulation

Weak wave on flat spacetime.
-> No non-principal part.

-> We can observe the
features of hyperbolicity.

-> Using constraints in RHS
may improve the blow-up.

PHYSICAL REVIEW D **66**, 064011 (2002)

Stability properties of a formulation of Einstein's equations

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(Received 27 May 2002; published 19 September 2002)

We study the stability properties of the Kidder-Scheel-Teukolsky (KST) many-parameter formulation of Einstein's equations for weak gravitational waves on flat space-time from a continuum and numerical point of view. At the continuum, performing a linearized analysis of the equations around flat space-time, it turns out that they have, essentially, no non-principal terms. As a consequence, in the weak field limit the stability properties of this formulation depend only on the level of hyperbolicity of the system. At the discrete level we present some simple one-dimensional simulations using the KST family. The goal is to analyze the type of instabilities that appear as one changes parameter values in the formulation. Lessons learned in this analysis can be applied in other formulations with similar properties.

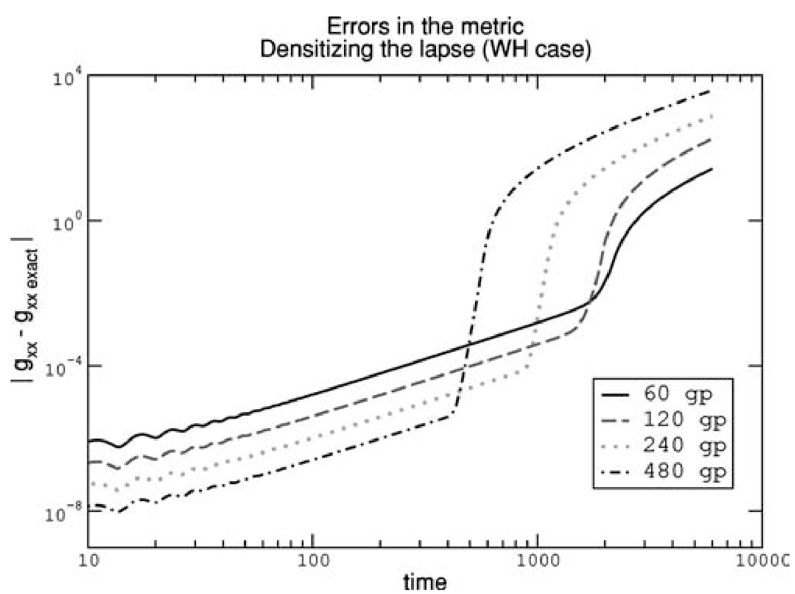

FIG. 7. L_2 norms of the errors for the metric.

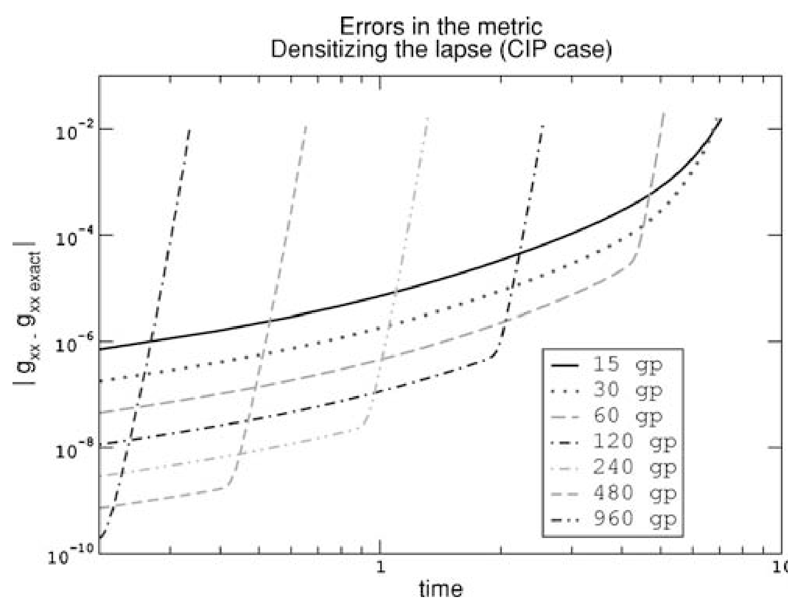

FIG. 9. L_2 norm of the errors for the metric.

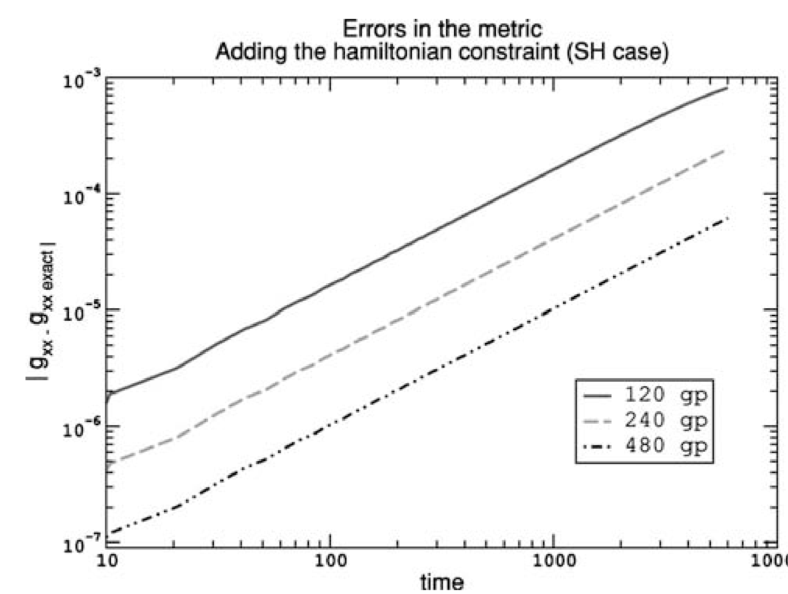

FIG. 12. L_2 norm of the errors for the metric.

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Hyperbolic formulations and numerical relativity: experiments using Ashtekar's connection variables

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Received 3 May 2000, in final form 13 September 2000

Abstract. In order to perform accurate and stable long-time numerical integration of the Einstein equation, several hyperbolic systems have been proposed. Here we present a numerical comparison between weakly hyperbolic, strongly hyperbolic and symmetric hyperbolic systems based on Ashtekar's connection variables. The primary advantage for using this connection formulation in this experiment is that we can keep using the same dynamical variables for all levels of hyperbolicity. Our numerical code demonstrates gravitational wave propagation in plane-symmetric spacetimes, and we compare the accuracy of the simulation by monitoring the violation of the constraints. By comparing with results obtained from the weakly hyperbolic system, we observe that the strongly and symmetric hyperbolic system show better numerical performance (yield less constraint violation), but not so much difference between the latter two. Rather, we find that the symmetric hyperbolic system is not always the best in terms of numerical performance.

This study is the first to present full numerical simulations using Ashtekar's variables. We also describe our procedures in detail.

$$\partial_t \tilde{E}^i_a = -\mathrm{i}\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + \mathrm{i}\mathcal{A}^b_0 \epsilon_{ab}{}^c\, \tilde{E}^i_c,$$

$$\partial_t \mathcal{A}^a_i = -\mathrm{i}\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0,$$

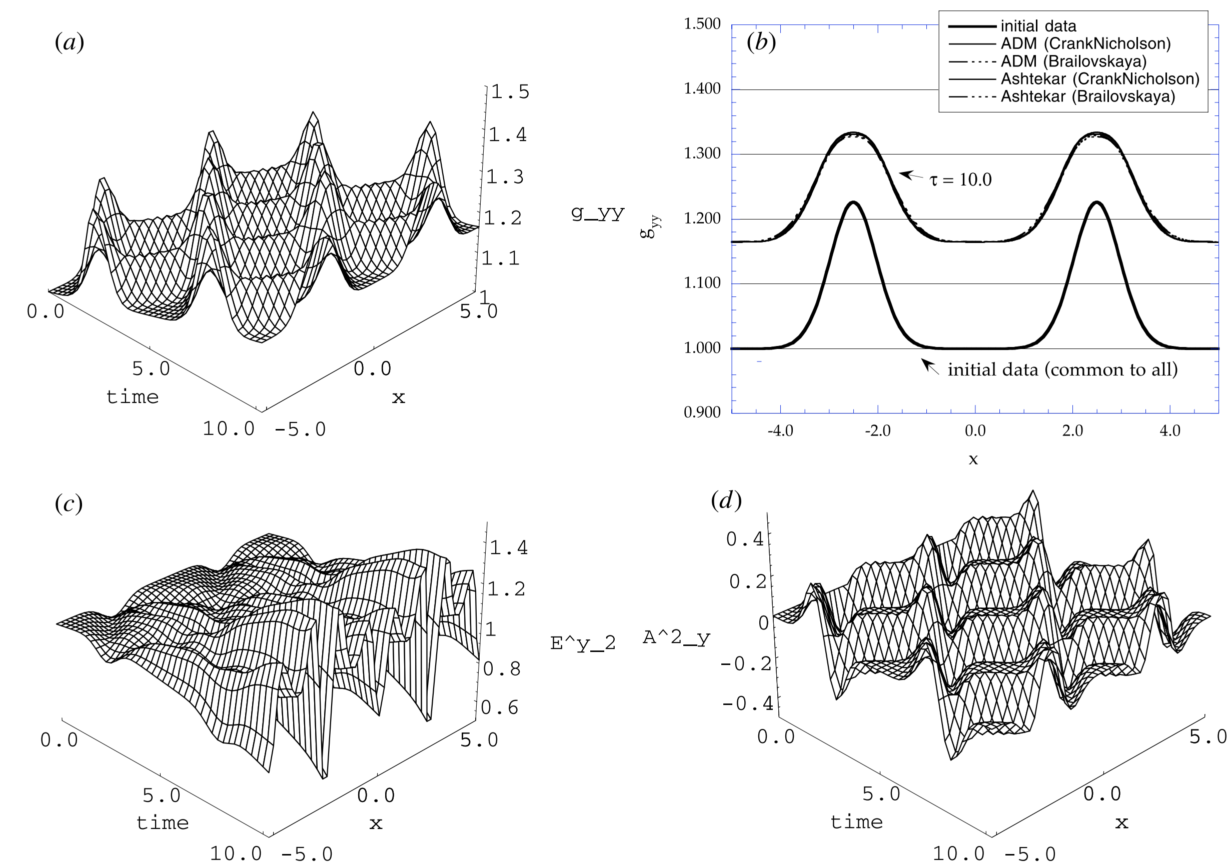

Figure 2. Images of gravitational wave propagation and comparisons of dynamical behaviour of Ashtekar's variables and ADM variables. We applied the same initial data of two +-mode pulse waves ($a = 0.2, b = 2.0, c = \pm 2.5$ in equation (21) and $K_0 = -0.025$), and the same slicing condition, the standard geodesic slicing condition ($N = 1$). (*a*) Image of the 3-metric component g_{yy} of a function of proper time τ and coordinate x. This behaviour can be seen identically both in ADM and Ashtekar evolutions, and both with the Brailovskaya and Crank–Nicholson time-integration scheme. Part (*b*) explains this fact by comparing the snapshot of g_{yy} at the same proper time slice ($\tau = 10$), where four lines at $\tau = 10$ are looked at identically. Parts (*c*) and (*d*) are of the real part of the densitized triad $\tilde{E}^y_2$, and the real part of the connection $\mathcal{A}^2_y$, respectively, obtained from the evolution of the Ashtekar variables.

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Hyperbolic formulations and numerical relativity: experiments using Ashtekar's connection variables

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Abstract. In order to perform accurate and stable long-time numerical integration of the Einstein equation, several hyperbolic systems have been proposed. Here we present a numerical comparison between weakly hyperbolic, strongly hyperbolic and symmetric hyperbolic systems based on Ashtekar's connection variables. The primary advantage for using this connection formulation in this experiment is that we can keep using the same dynamical variables for all levels of hyperbolicity. Our numerical code demonstrates gravitational wave propagation in plane-symmetric spacetimes, and we compare the accuracy of the simulation by monitoring the violation of the constraints. By comparing with results obtained from the weakly hyperbolic system, we observe that the strongly and symmetric hyperbolic system show better numerical performance (yield less constraint violation), but not so much difference between the latter two. Rather, we find that the symmetric hyperbolic system is not always the best in terms of numerical performance.

This study is the first to present full numerical simulations using Ashtekar's variables. We also describe our procedures in detail.

$$\partial_t \tilde{E}^i_a = -\mathrm{i}\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + \mathrm{i}\mathcal{A}^b_0 \epsilon_{ab}{}^c\, \tilde{E}^i_c + \kappa\, P^i{}_{ab}\, \mathcal{C}_G^{\mathrm{ASH}b},$$

$$\text{where} \qquad P^i{}_{ab} \equiv N^i \delta_{ab} + \mathrm{i}\underset{\sim}{N} \epsilon_{ab}{}^c \tilde{E}^i_c,$$

$$\partial_t \mathcal{A}^a_i = -\mathrm{i}\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \kappa\, Q^a_i \mathcal{C}_H^{\mathrm{ASH}} + \kappa\, R_i{}^{ja}\, \mathcal{C}_{Mj}^{\mathrm{ASH}},$$

$$\text{where} \qquad Q^a_i \equiv e^{-2}\underset{\sim}{N}\tilde{E}^a_i, \qquad R_i{}^{ja} \equiv \mathrm{i}e^{-2}\underset{\sim}{N}\epsilon^{ac}{}_b \tilde{E}^b_i \tilde{E}^j_c.$$

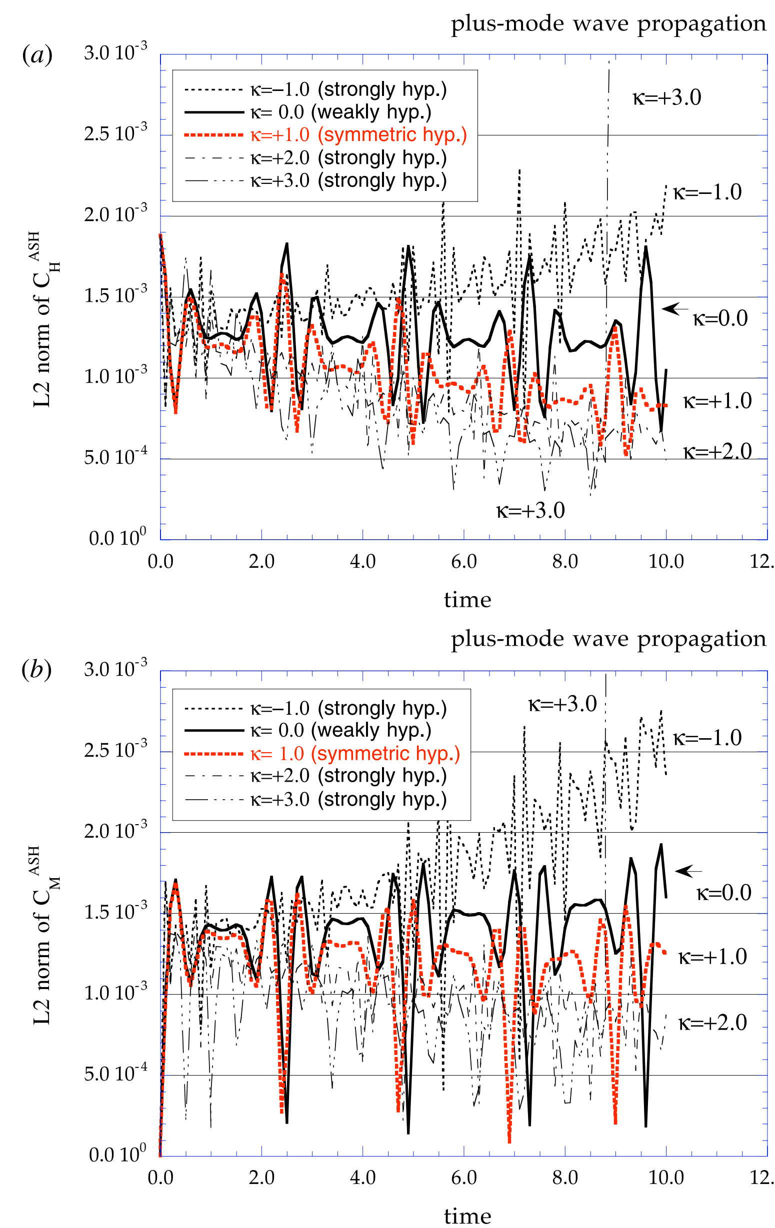


Figure 6. Comparisons of the 'adjusted' system with the different multiplier, κ, in equations (31) and (32). The model uses +-mode pulse waves ($a = 0.1$, $b = 2.0$, $c = \pm 2.5$) in equation (21) in a background $K_0 = -0.025$. Plots are of the L2 norm of the Hamiltonian and momentum constraint equations, $\mathcal{C}_H^{\mathrm{ASH}}$ and $\mathcal{C}_M^{\mathrm{ASH}}$ ((*a*) and (*b*), respectively). We see some κ produce a better performance than the symmetric hyperbolic system.

No drastic differences in stability between 3 levels of hyperbolicity.

BSSN Pros:

- With Bona-Masso-type α (1+log), and frozon β ($\partial_t \Gamma^i \sim 0$), BSSN plus auxiliary variables form a 1st-order symmetric hyperbolic system,
  Heyer-Sarbach, [PRD 70 (2004) 104004]
- If we define 2nd order symmetric hyperbolic form, principal part of BSSN can be one of them,
  Gundlach-MartinGarcia, [PRD 70 (2004) 044031, PRD 74 (2006) 024016]

BSSN Cons:

- Existence of an ill-posed solution in BSSN (as well in ADM)
  Frittelli-Gomez [JMP 41 (2000) 5535]
- Gauge shocks in Bona-Masso slicing is inevitable. Current 3D BH simulation is lack of resolution.
  Garfinke-Gundlach-Hilditch [arXiv:0707.0726]

strategy 2 Hyperbolic formulation (cont.)

Are they actually helpful?

"YES" group

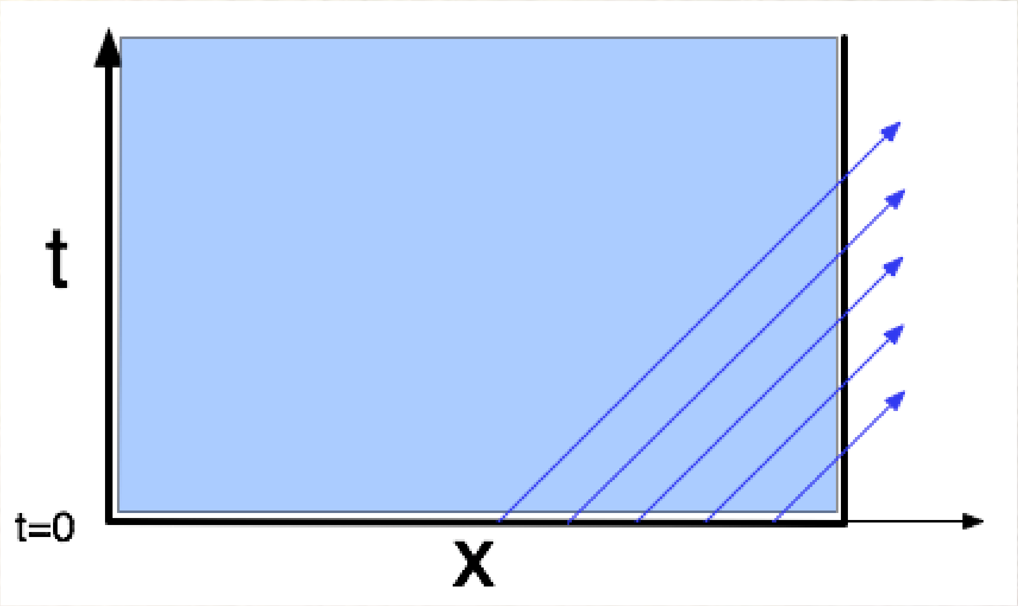
"Well-posed!", $||u(t)|| \le e^{\kappa t}||u(0)||$

Mathematically Rigorous Proofs

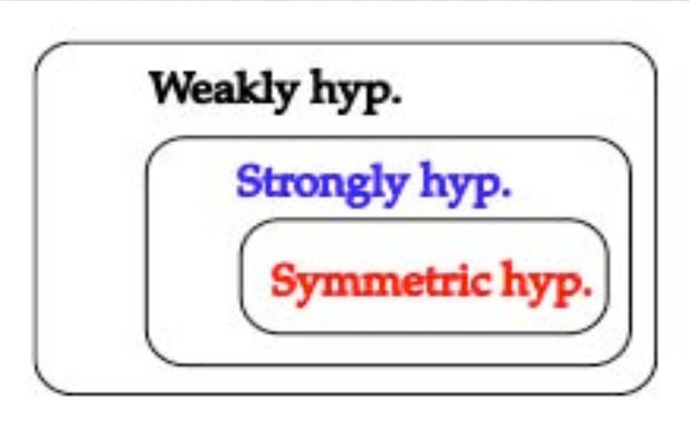
IBVP in future

Initial Boundary Value Problem

Consistent treatment is available only for symmetric hyperbolic systems.



- GR-IBVP
  - Stewart, CQG15 (98) 2865
- Tetrad formalism
  - Friedrich & Nagy, CMP201 (99) 619
- Linearized Bianchi eq.
  - Buchman & Sarbach, CQG 23 (06) 6709
- Constraint-preserving BC
  - Kreiss, Reula, Sarbach & Winicour, CQG 24 (07) 5973
- Higher-order absorbing BC
  - Ruiz, Rinne & Sarbach, CQG 24 (07) 6349



strategy 2 Hyperbolic formulation (cont.)

Are they actually helpful?

"YES" group	"Really?" group
"Well-posed!", $\|\|u(t)\|\| \leq e^{\kappa t}\|\|u(0)\|\|$	"not converging", still blow-up
Mathematically Rigorous Proofs	Proofs are only simple eqs. Discuss only characteristic part. Ignore non-principal part.
IBVP in future	...

strategy 2 Hyperbolic formulation (cont.)

Are they actually helpful?

"YES" group	"Really?" group
"Well-posed!", $\|\|u(t)\|\| \leq e^{\kappa t}\|\|u(0)\|\|$	"not converging", still blow-up
Mathematically Rigorous Proofs	Proofs are only simple eqs. Discuss only characteristic part. Ignore non-principal part.
IBVP in future	...

Which level of hyperbolicity is necessary?

symmetric hyperbolic $\subset$ strongly hyperbolic $\subset$ weakly hyperbolic systems,

Advantages in Numerics (90s)	
Advantages in sym. hyp. – KST formulation by LSU	

strategy 2 Hyperbolic formulation (cont.)

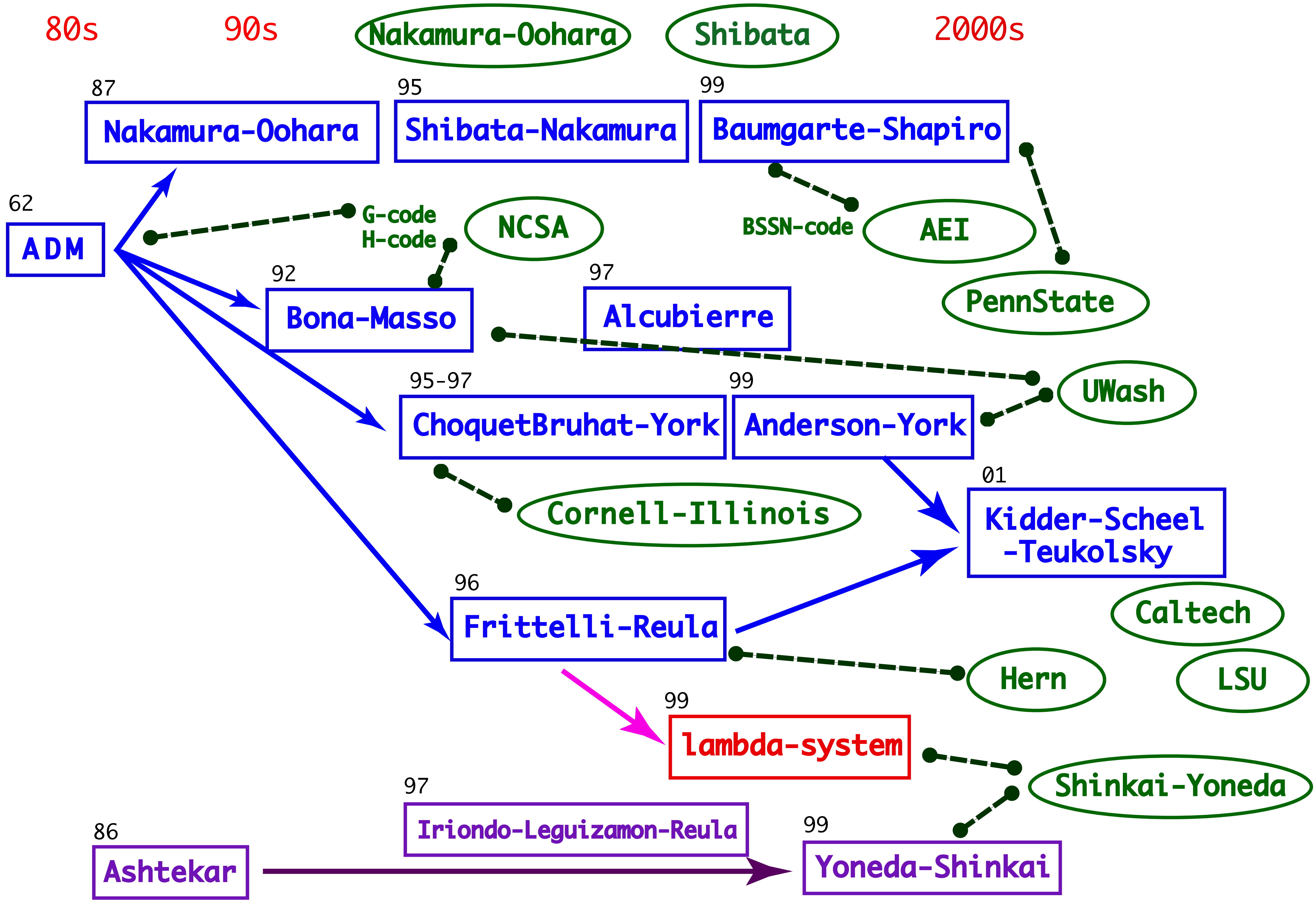
Are they actually helpful?

"YES" group	"Really?" group
"Well-posed!", $\|\|u(t)\|\| \le e^{\kappa t}\|\|u(0)\|\|$	"not converging", still blow-up
Mathematically Rigorous Proofs	Proofs are only simple eqs. Discuss only characteristic part. Ignore non-principal part.
IBVP in future	...

Which level of hyperbolicity is necessary?

symmetric hyperbolic $\subset$ strongly hyperbolic $\subset$ weakly hyperbolic systems,

Advantages in Numerics (90s)	These were vs. ADM
Advantages in sym. hyp. – KST formulation by LSU	Not much differences in hyperbolic 3 levels – FR formulation, by Hern – Ashtekar formulation, by HS-Yoneda sym. hyp. is not always the best



80s
90s
Nakamura-Oohara
Shibata
2000s
87
Nakamura-Oohara
95
Shibata-Nakamura
99
Baumgarte-Shapiro
62
ADM
G-code
H-code
NCSA
BSSN-code
AEI
PennState
92
Bona-Masso
97
Alcubierre
UWash
95-97
ChoquetBruhat-York
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Anderson-York
01
Kidder-Scheel
-Teukolsky
Cornell-Illinois
96
Frittelli-Reula
Caltech
Hern
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lambda-system
Shinkai-Yoneda
97
Iriondo-Leguizamon-Reula
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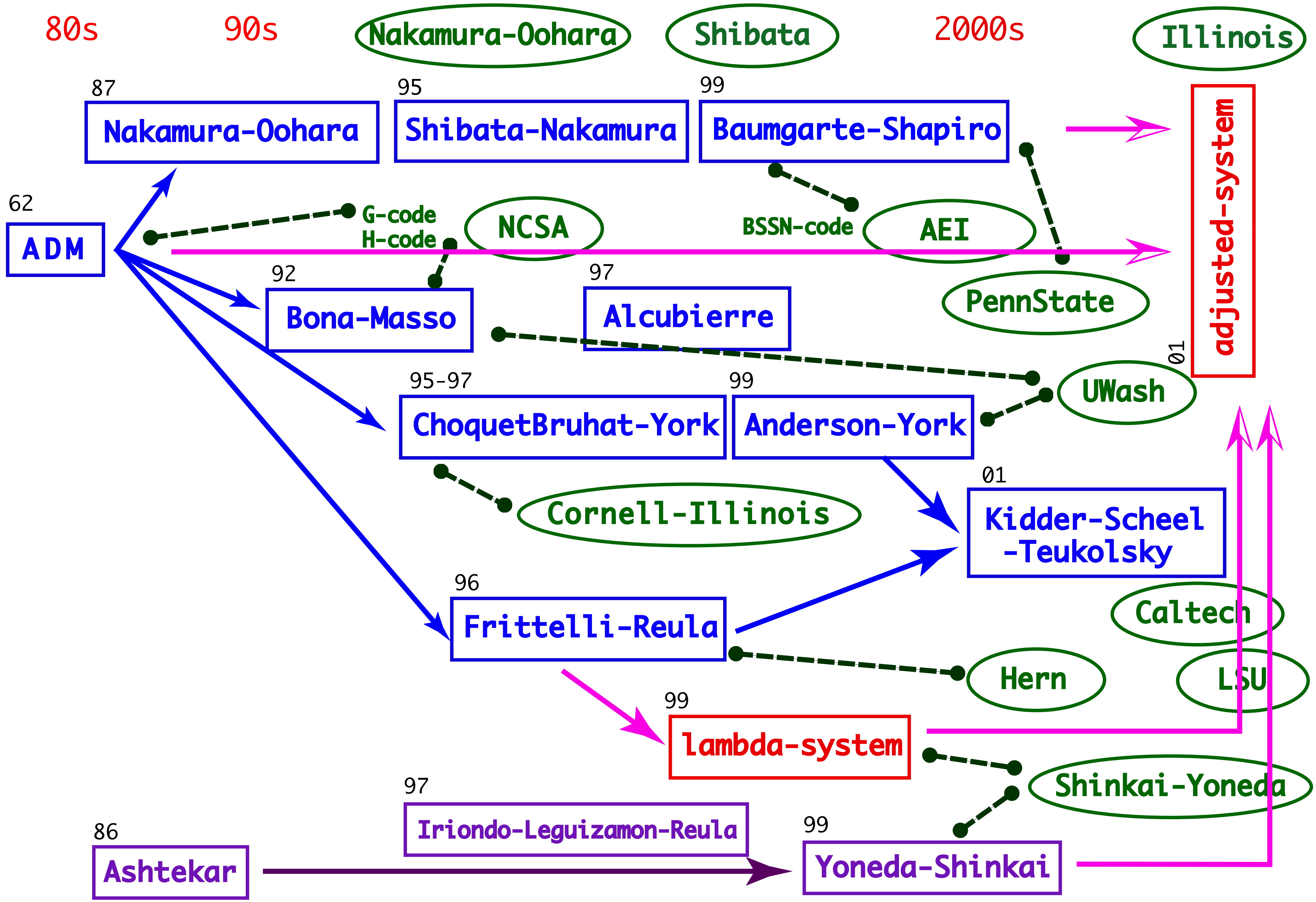
Ashtekar
99
Yoneda-Shinkai


80s
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Shibata
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Nakamura-Oohara
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Shibata-Nakamura
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Baumgarte-Shapiro
adjusted-system
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ADM
G-code
H-code
NCSA
BSSN-code
AEI
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Bona-Masso
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Alcubierre
PennState
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95-97
ChoquetBruhat-York
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Anderson-York
UWash
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Kidder-Scheel
-Teukolsky
Cornell-Illinois
96
Frittelli-Reula
Caltech
Hern
LSU
99
lambda-system
Shinkai-Yoneda
97
Iriondo-Leguizamon-Reula
86
Ashtekar
99
Yoneda-Shinkai

Summary up to here (1st half)

[Keyword 1] Formulation Problem

Although mathematically equivalent, different set of equations shows different numerical stability.

[Keyword 2] ADM formulation

The starting formulation (Historically & Numerically).
Successes in 90s, but not for binary BH-BH/NS-NS problems.

[Keyword 3] BSSN formulation

New variables and gauge fixing to ADM, shows better stability.
The reason why it is better was not known at first.
Many simulation groups uses BSSN. Technical tips are accumulated.

[Keyword 4] hyperbolic formulations

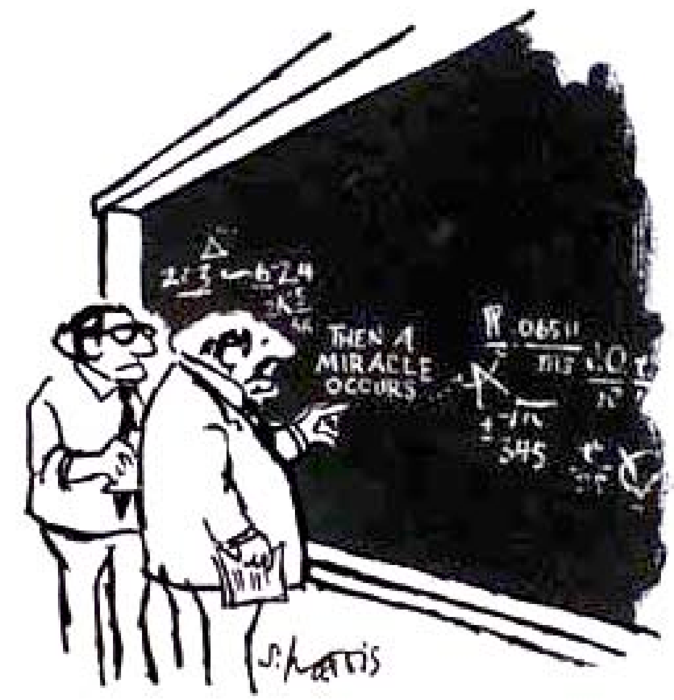
Mathematical classification of PDE shows "well-posedness", but its meaning is limited.
Many versions of hyperbolic Einstein equations are available.
Some group try to show the advantage of BSSN using "hyperbolicity".
But are they really helpful in numerics?

Goals of the Lecture

What is the guiding principle for selecting evolution equations for simulations in GR?

Why many groups use the BSSN equations?

Are there an alternative formulation better than the BSSN?

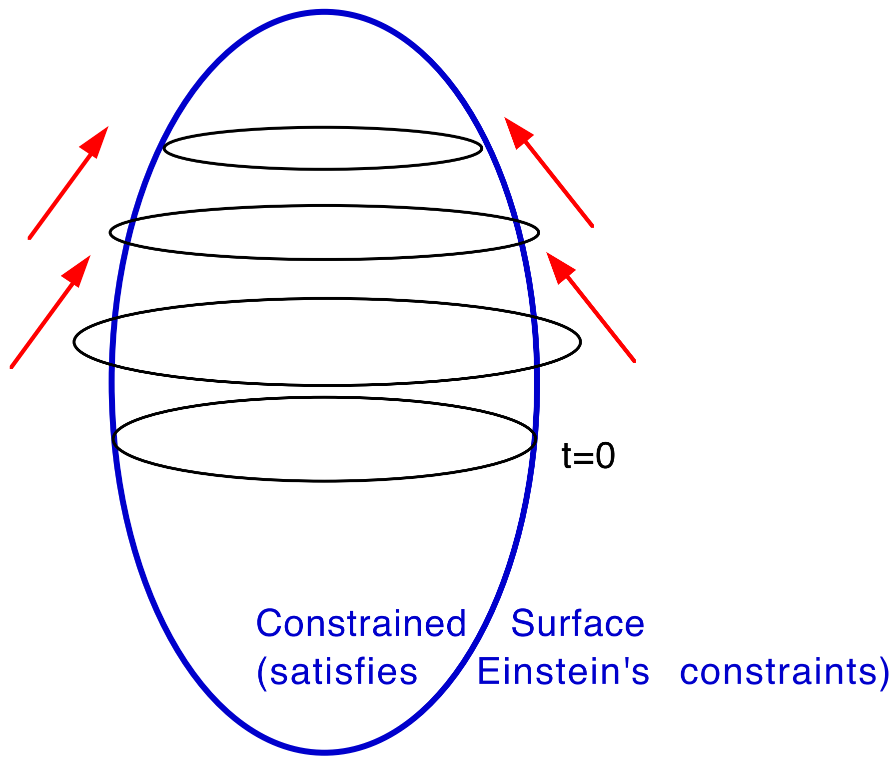


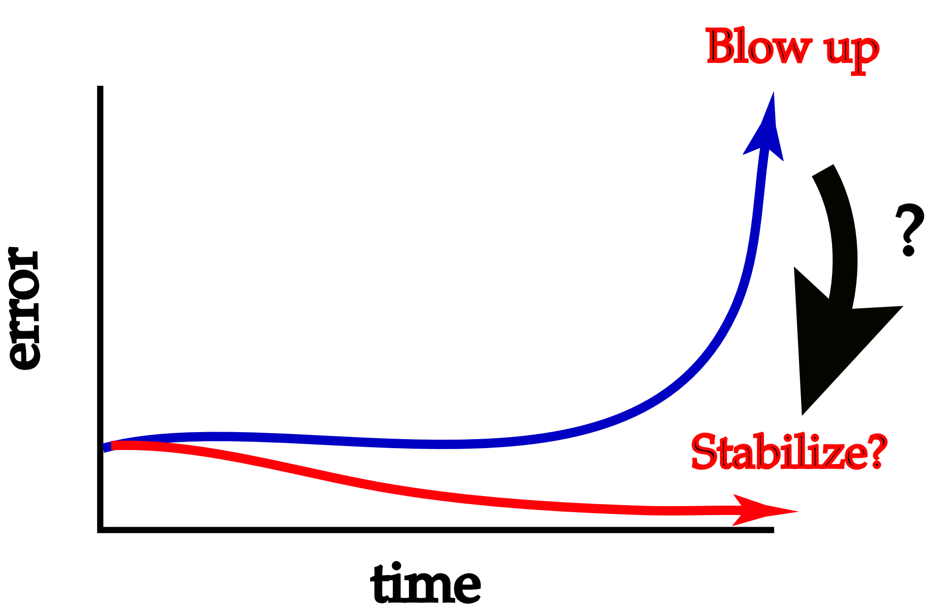
"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

strategy 3 "Asymptotically Constrained" system / "Constraint Damping" system

Formulate a system which is "asymptotically constrained" against a violation of constraints

Constraint Surface as an Attractor





method 1: λ-system (Brodbeck et al, 2000)

- Add aritificial force to reduce the violation of constraints
- To be guaranteed if we apply the idea to a symmetric hyperbolic system.

method 2: Adjusted system (Yoneda HS, 2000, 2001)

- We can control the violation of constraints by adjusting constraints to EoM.
- Eigenvalue analysis of constraint propagation equations may prodict the violation of error.
- This idea is applicable even if the system is not symmetric hyperbolic. $\Rightarrow$ for the ADM/BSSN formulation, too!!

Idea of λ-system

Brodbeck, Frittelli, Hübner and Reula, JMP40(99)909

We expect a system that is robust for controlling the violation of constraints

Recipe

1. Prepare a symmetric hyperbolic evolution system $\qquad \partial_t u = J\partial_i u + K$
2. Introduce λ as an indicator of violation of constraint which obeys dissipative eqs. of motion $\qquad \partial_t \lambda = \alpha C - \beta\lambda$ $\quad (\alpha \neq 0, \beta > 0)$
3. Take a set of (u, λ) as dynamical variables $\qquad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} \simeq \begin{pmatrix} A & 0 \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$
4. Modify evolution eqs so as to form a symmetric hyperbolic system $\qquad \partial_t \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} A & \bar{F} \\ F & 0 \end{pmatrix} \partial_i \begin{pmatrix} u \\ \lambda \end{pmatrix}$

Remarks

- BFHR used a sym. hyp. formulation by Frittelli-Reula [PRL76(96)4667]
- The version for the Ashtekar formulation by HS-Yoneda [PRD60(99)101502] for controlling the constraints or reality conditions or both.
- Succeeded in evolution of GW in planar spacetime using Ashtekar vars. [CQG18(2001)441]
- Do the recovered solutions represent true evolution? by Siebel-Hübner [PRD64(2001)024021]
- The version for Z4 hyperbolic system by Gundlach-Calabrese-Hinder-MartinGarcia [CQG22(05)3767] $\Rightarrow$ Pretorius noticed the idea of "constraint damping" [PRL95(05)121101]

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Hyperbolic formulations and numerical relativity: II. asymptotically constrained systems of Einstein equations

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Maxwell-lambda system works as expected.

$$\partial_t \begin{pmatrix} E^i \\ B^i \\ \lambda_E \\ \lambda_B \end{pmatrix} = \begin{pmatrix} 0 & -c\epsilon^i{}_j{}^l & \alpha_1\delta^{li} & 0 \\ c\epsilon^i{}_j{}^l & 0 & 0 & \alpha_2\delta^{li} \\ \alpha_1\delta^l_j & 0 & 0 & 0 \\ 0 & \alpha_2\delta^l_j & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} E^j \\ B^j \\ \lambda_E \\ \lambda_B \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\beta_1\lambda_E \\ -\beta_2\lambda_B \end{pmatrix}.$$

$$\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \\ \hat{\lambda}_E \\ \hat{\lambda}_B \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\alpha_1 k^2 & 0 \\ 0 & 0 & 0 & -\alpha_2 k^2 \\ \alpha_1 & 0 & -\beta_1 & 0 \\ 0 & \alpha_2 & 0 & -\beta_2 \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \\ \hat{\lambda}_E \\ \hat{\lambda}_B \end{pmatrix},$$

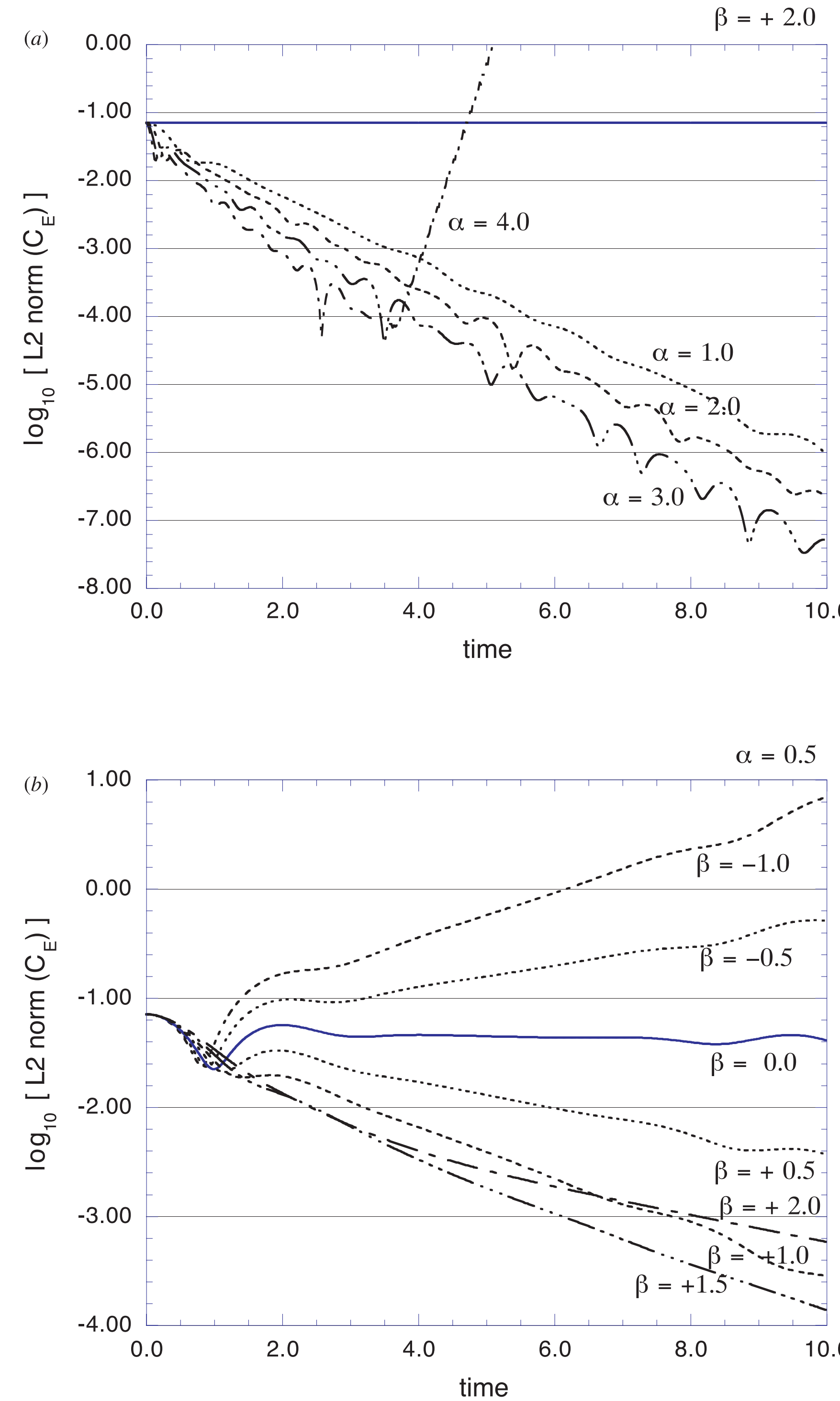

Figure 1. Demonstration of the λ system in the Maxwell equation. (*a*) Constraint violation (L2 norm of C_E) versus time with constant β (= 2.0) but changing α. Here $\alpha = 0$ means no λ system. (*b*) The same plot with constant α (= 0.5) but changing β. We see better performance for $\beta > 0$, which is the case of negative eigenvalues of the constraint propagation equation. The constants in (2.18) were chosen as $A = 200$ and $B = 1$.

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Ashtekar-lambda system works as expected, as well.

$$\partial_t \begin{pmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \\ \lambda \\ \lambda_i \\ \lambda_a \end{pmatrix} \cong \begin{pmatrix} \mathcal{M}^l{}_a{}^{bi}{}_j & 0 & 0 & 0 & \bar{\alpha}_3 \gamma^{il} \delta_a{}^b \\ 0 & \mathcal{N}^{la}{}_{ib}{}^j & \mathrm{i}\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d & \bar{\alpha}_2 e(\delta^j_i \tilde{E}^{la} - \gamma^{lj} \tilde{E}^a_i) & 0 \\ 0 & -\mathrm{i}\alpha_1 \epsilon_b{}^{cd} \tilde{E}^j_c \tilde{E}^l_d & 0 & 0 & 0 \\ 0 & \alpha_2 e(\delta^j_i \tilde{E}^l_b - \delta^l_i \tilde{E}^j_b) & 0 & 0 & 0 \\ \alpha_3 \delta^b_a \delta^l_j & 0 & 0 & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} \tilde{E}^j_b \\ \mathcal{A}^b_j \\ \lambda \\ \lambda_j \\ \lambda_b \end{pmatrix}$$

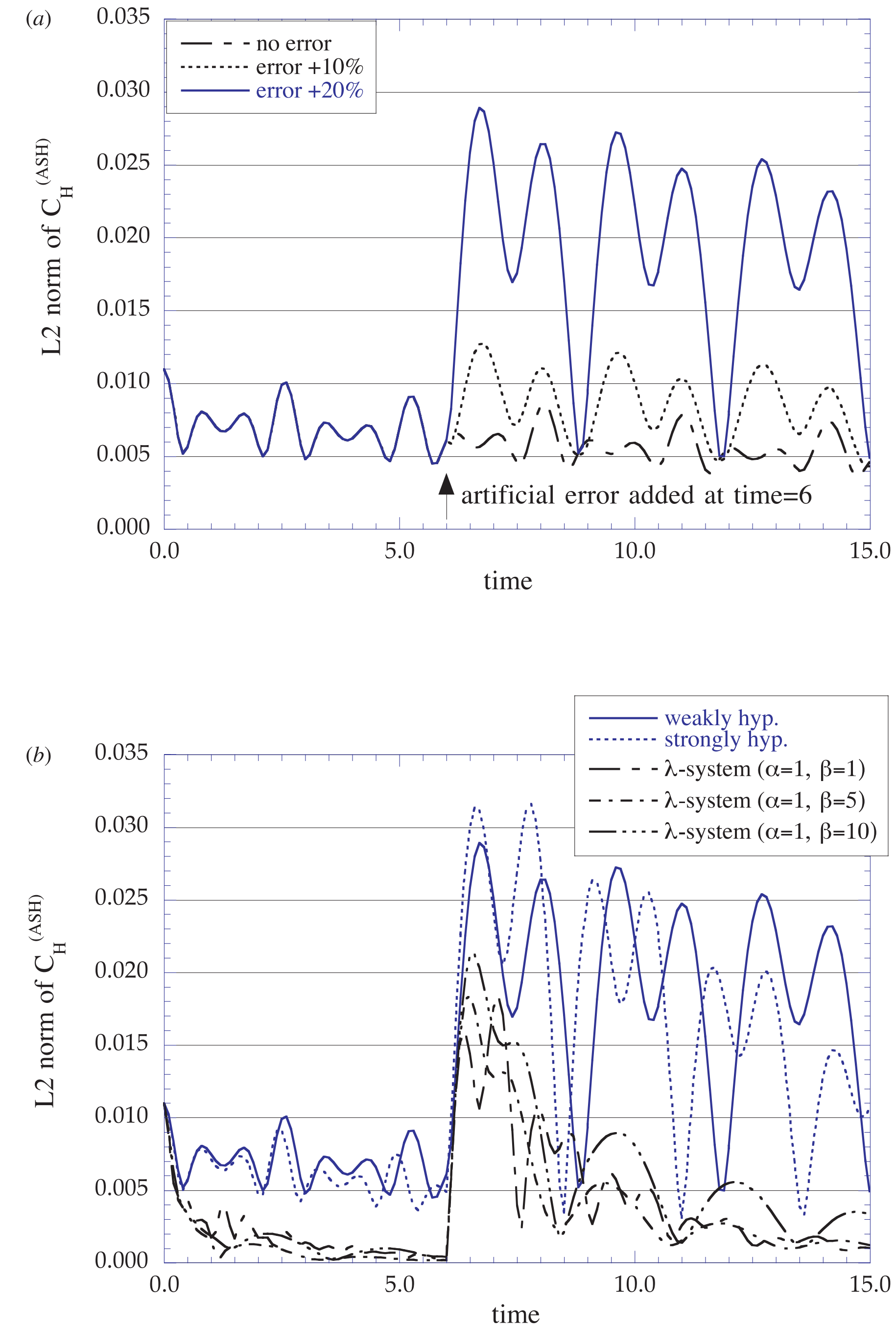

Figure 3. Demonstration of the λ system in the Ashtekar equation. We plot the violation of the constraint (the L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$) for the cases of plane-wave propagation under the periodic boundary. To see the effect more clearly, we added an artificial error at $t = 6$. Part (*a*) shows how the system goes bad depending on the amplitude of artificial error. The error was of the form $\mathcal{A}^2_y \to \mathcal{A}^2_y(1 + \text{error})$. All the curves are of the evolution of Ashtekar's original equation (no λ system). Part (*b*) shows the effect of the λ system. All the curves have 20% error amplitude, but show the difference of the evolution equations. The full curve is for Ashtekar's original equation (the same as in (*a*)), the dotted curve is for the strongly hyperbolic Ashtekar equation. Other curves are of λ systems, which produce a better performance than that of the strongly hyperbolic system.

Idea of “Adjusted system” and Our Conjecture

CQG18 (2001) 441, PRD 63 (2001) 120419, CQG 19 (2002) 1027

General Procedure

1. prepare a set of evolution eqs. $\partial_t u^a = f(u^a, \partial_b u^a, \cdots)$
2. add constraints in RHS $\partial_t u^a = f(u^a, \partial_b u^a, \cdots) \underbrace{+F(C^a, \partial_b C^a, \cdots)}$
3. choose appropriate $F(C^a, \partial_b C^a, \cdots)$ to make the system stable evolution

How to specify $F(C^a, \partial_b C^a, \cdots)$?

4. prepare constraint propagation eqs. $\partial_t C^a = g(C^a, \partial_b C^a, \cdots)$
5. and its adjusted version $\partial_t C^a = g(C^a, \partial_b C^a, \cdots) \underbrace{+G(C^a, \partial_b C^a, \cdots)}$
6. Fourier transform and evaluate eigenvalues $\partial_t \hat{C}^k = \underbrace{A(\hat{C}^a)}\ \hat{C}^k$

Conjecture: Evaluate eigenvalues of (Fourier-transformed) constraint propagation eqs.
If their (1) real part is non-positive, or (2) imaginary part is non-zero, then the system is more stable.

Example: the Maxwell equations

Yoneda HS, CQG 18 (2001) 441

Maxwell evolution equations.

$$\begin{aligned} \partial_t E_i &= c\epsilon_i{}^{jk}\partial_j B_k + P_i\, C_E + Q_i\, C_B, \\ \partial_t B_i &= -c\epsilon_i{}^{jk}\partial_j E_k + R_i\, C_E + S_i\, C_B, \\ & C_E = \partial_i E^i \approx 0, \quad C_B = \partial_i B^i \approx 0, \end{aligned} \qquad \begin{cases} \text{sym. hyp} & \Leftrightarrow \quad P_i = Q_i = R_i = S_i = 0, \\ \text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_i Q_i > 0, \\ \text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_i Q_i \geq 0 \end{cases}$$

Constraint propagation equations

$$\begin{aligned} \partial_t C_E &= (\partial_i P^i) C_E + P^i(\partial_i C_E) + (\partial_i Q^i) C_B + Q^i(\partial_i C_B), \\ \partial_t C_B &= (\partial_i R^i) C_E + R^i(\partial_i C_E) + (\partial_i S^i) C_B + S^i(\partial_i C_B), \end{aligned}$$

$$\begin{cases} \text{sym. hyp} & \Leftrightarrow \quad Q_i = R_i, \\ \text{strongly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_i Q_i > 0, \\ \text{weakly hyp} & \Leftrightarrow \quad (P_i - S_i)^2 + 4R_i Q_i \geq 0 \end{cases}$$

CAFs?

$$\partial_t \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} = \begin{pmatrix} \partial_i P^i + P^i k_i & \partial_i Q^i + Q^i k_i \\ \partial_i R^i + R^i k_i & \partial_i S^i + S^i k_i \end{pmatrix} \partial_l \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} \approx \begin{pmatrix} P^i k_i & Q^i k_i \\ R^i k_i & S^i k_i \end{pmatrix} \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix} =: T \begin{pmatrix} \hat{C}_E \\ \hat{C}_B \end{pmatrix}$$

$$\Rightarrow \text{ CAFs} = (P^i k_i + S^i k_i \pm \sqrt{(P^i k_i + S^i k_i)^2 + 4(Q^i k_i R^j k_j - P^i k_i S^j k_j)})/2$$

Therefore CAFs become negative-real when

$$P^i k_i + S^i k_i < 0, \qquad \text{and} \qquad Q^i k_i R^j k_j - P^i k_i S^j k_j < 0$$

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Adjusted-Maxwell system works as well.

3.2.1. Adjusted system. Here we again consider the Maxwell equations (2.9)–(2.11). We start from the adjusted dynamical equations

$$\partial_t E_i = c\epsilon_i{}^{jk}\partial_j B_k + P_i C_E + p^j{}_i(\partial_j C_E) + Q_i C_B + q^j{}_i(\partial_j C_B), \tag{3.7}$$

$$\partial_t B_i = -c\epsilon_i{}^{jk}\partial_j E_k + R_i C_E + r^j{}_i(\partial_j C_E) + S_i C_B + s^j{}_i(\partial_j C_B), \tag{3.8}$$

where P, Q, R, S, p, q, r and s are multipliers. These dynamical equations adjust the constraint propagation equations as

$$\partial_t C_E = (\partial_i P^i)C_E + P^i(\partial_i C_E) + (\partial_i Q^i)C_B + Q^i(\partial_i C_B)$$
$$\quad +(\partial_i p^{ji})(\partial_j C_E) + p^{ji}(\partial_i\partial_j C_E) + (\partial_i q^{ji})(\partial_j C_B) + q^{ji}(\partial_i\partial_j C_B), \tag{3.9}$$

$$\partial_t C_B = (\partial_i R^i)C_E + R^i(\partial_i C_E) + (\partial_i S^i)C_B + S^i(\partial_i C_B)$$
$$\quad +(\partial_i r^{ji})(\partial_j C_E) + r^{ji}(\partial_i\partial_j C_E) + (\partial_i s^{ji})(\partial_j C_B) + s^{ji}(\partial_i\partial_j C_B). \tag{3.10}$$

This will be expressed using Fourier components by

$$\partial_t \begin{pmatrix}\hat{C}_E\\ \hat{C}_B\end{pmatrix} = \begin{pmatrix}\partial_i P^i + \mathrm{i}P^i k_i + \mathrm{i}k_j(\partial_i p^{ji}) - k_i k_j p^{ji} & \partial_i Q^i + \mathrm{i}Q^i k_i + \mathrm{i}k_j(\partial_i q^{ji}) - k_i k_j q^{ji}\\ \partial_i R^i + \mathrm{i}R^i k_i + \mathrm{i}k_j(\partial_i r^{ji}) - k_i k_j r^{ji} & \partial_i S^i + \mathrm{i}S^i k_i + \mathrm{i}k_j(\partial_i s^{ji}) - k_i k_j s^{ji}\end{pmatrix}$$
$$\times\begin{pmatrix}\hat{C}_E\\ \hat{C}_B\end{pmatrix} =: T\begin{pmatrix}\hat{C}_E\\ \hat{C}_B\end{pmatrix}. \tag{3.11}$$

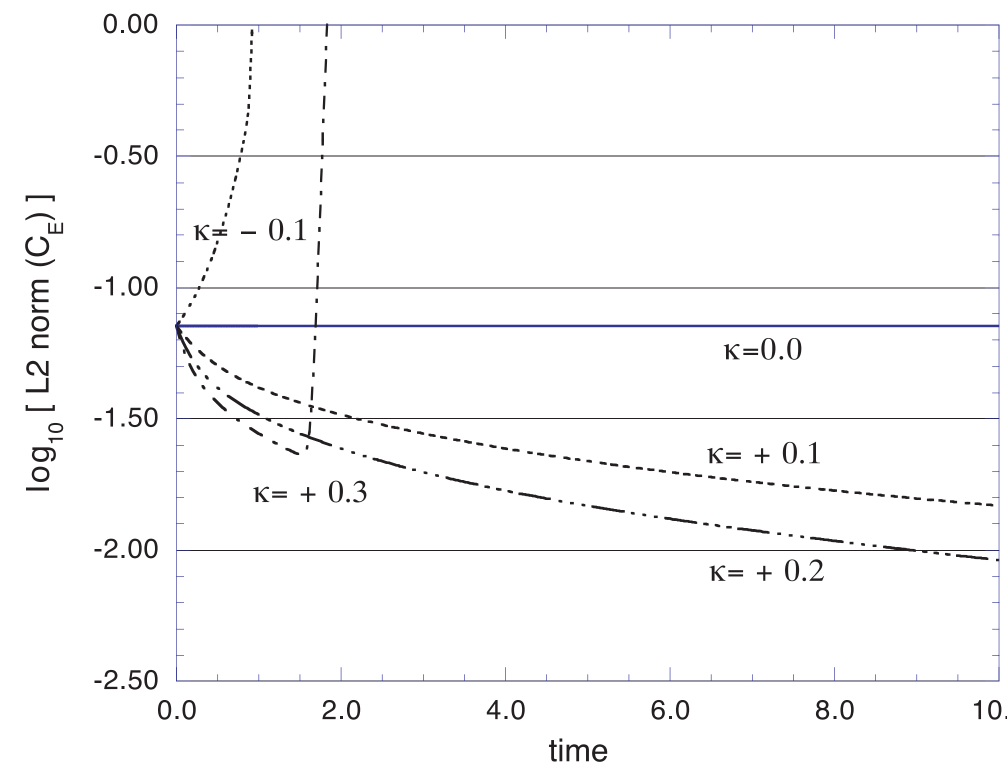

Figure 4. Demonstrations of the adjusted system in the Maxwell equation. We perform the same experiments with section 2.2.3 (figure 1). Constraint violation (L2 norm of C_E) versus time are plotted for various κ $(= p^j{}_i = s^j{}_i)$. We see that $\kappa > 0$ gives a better performance (i.e. negative real part eigenvalues for the constraint propagation equation), while excessively large positive κ makes the system divergent again.

Example: the Ashtekar equations

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Adjusted dynamical equations:

$$\begin{aligned}
\partial_t \tilde{E}^i_a &= -i\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c \underbrace{+X^i_a \mathcal{C}_H + Y^{ij}_a \mathcal{C}_{Mj} + P^{ib}_a \mathcal{C}_{Gb}}_{adjust} \\
\partial_t \mathcal{A}^a_i &= -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i \underbrace{+Q^a_i \mathcal{C}_H + R^{aj}_i \mathcal{C}_{Mj} + Z^{ab}_i \mathcal{C}_{Gb}}_{adjust}
\end{aligned}$$

Adjusted and linearized:

$$X = Y = Z = 0,\ P^{ia}_b = \kappa_1(iN^i\delta^a_b),\ Q^a_i = \kappa_2(e^{-2}\underset{\sim}{N}\tilde{E}^a_i),\ R^{aj}{}_i = \kappa_3(-ie^{-2}\underset{\sim}{N}\epsilon^{ac}{}_d \tilde{E}^d_i \tilde{E}^j_c)$$

Fourier transform and extract 0th order of the characteristic matrix:

$$\partial_t \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mi} \\ \hat{\mathcal{C}}_{Ga} \end{pmatrix} = \begin{pmatrix} 0 & i(1+2\kappa_3)k_j & 0 \\ i(1-2\kappa_2)k_i & \kappa_3 \epsilon^{kj}{}_i k_k & 0 \\ 0 & 2\kappa_3 \delta^j_a & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathcal{C}}_H \\ \hat{\mathcal{C}}_{Mj} \\ \hat{\mathcal{C}}_{Gb} \end{pmatrix}$$

Eigenvalues:

$$\left(0, 0, 0, \pm\kappa_3\sqrt{-kx^2 - ky^2 - kz^2}, \pm\sqrt{(-1+2\kappa_2)(1+2\kappa_3)(kx^2+ky^2+kz^2)}\right)$$

In order to obtain non-positive real eigenvalues:

$$(-1+2\kappa_2)(1+2\kappa_3) < 0$$

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Hyperbolic formulations and numerical relativity: II. asymptotically constrained systems of Einstein equations

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Adjusted-Ashtekar system works as well.

3.3.1. Adjusted system for controlling constraint violations. Here we only consider the adjusted system which controls the departures from the constraint surface. In the appendix, we present an advanced system which controls the violation of the reality condition together with a numerical demonstration.

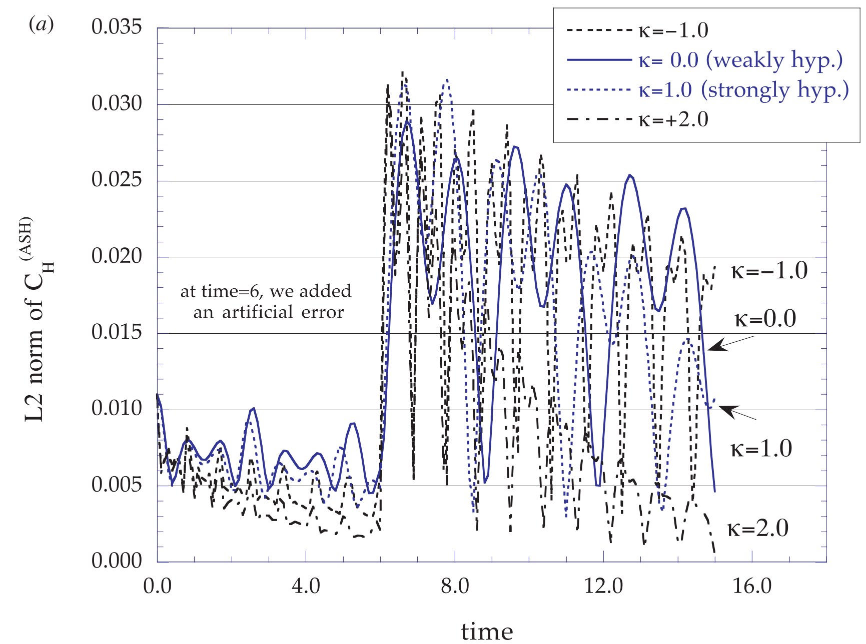
Even if we restrict ourselves to adjusted equations of motion for $(\tilde{E}^i_a, \mathcal{A}^a_i)$ with constraint terms (no adjustment with derivatives of constraints), generally, we could adjust them as

$$\partial_t \tilde{E}^i_a = -\mathrm{i}\mathcal{D}_j(\epsilon^{cb}{}_a \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + \mathrm{i}\mathcal{A}^b_0 \epsilon_{ab}{}^c \tilde{E}^i_c + X^i_a \mathcal{C}_H + Y^{ij}_a \mathcal{C}_{Mj} + P^{ib}_a \mathcal{C}_{Gb}, \tag{3.14}$$

$$\partial_t \mathcal{A}^a_i = -\mathrm{i}\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \Lambda \underset{\sim}{N} \tilde{E}^a_i + Q^a_i \mathcal{C}_H + R_i{}^{ja} \mathcal{C}_{Mj} + Z^{ab}_i \mathcal{C}_{Gb}, \tag{3.15}$$

where X^i_a, Y^{ij}_a, Z^{ab}_i, P^{ib}_a, Q^a_i and R^{aj}_i are multipliers. However, in order to simplify the discussion, we restrict multipliers so as to reproduce the symmetric hyperbolic equations of motion [10, 11], i.e.

$$\begin{aligned} X &= Y = Z = 0, \\ P^{ib}_a &= \kappa_1 (N^i \delta^b_a + \mathrm{i}\underset{\sim}{N} \epsilon_a{}^{bc} \tilde{E}^i_c), \\ Q^a_i &= \kappa_2 (e^{-2} \underset{\sim}{N} \tilde{E}^a_i), \\ R_i{}^{ja} &= \kappa_3 (\mathrm{i} e^{-2} \underset{\sim}{N} \epsilon^{ac}{}_b \tilde{E}^b_i \tilde{E}^j_c). \end{aligned} \tag{3.16}$$



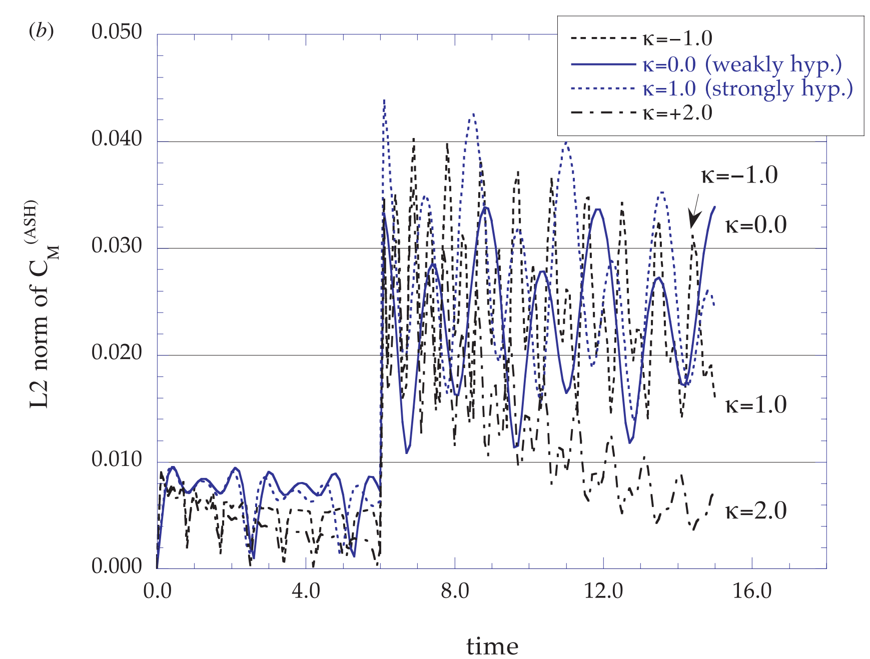


Figure 5. Demonstration of the adjusted system in the Ashtekar equation. We plot the violation of the constraint for the same model as figure 3(*b*). An artificial error term was added at $t = 6$, in the form of $\mathcal{A}^2_y \to \mathcal{A}^2_y(1+\text{ error})$, where error is + 20% as before. (*a*), (*b*) L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$, and momentum constraint equation, $\mathcal{C}_{Mx}$, respectively. The full curve is the case of $\kappa = 0$, that is the case of 'no adjusted' original Ashtekar equation (weakly hyperbolic system). The dotted curve is for $\kappa = 1$, equivalent to the symmetric hyperbolic system. We see that the other curve ($\kappa = 2.0$) shows better performance than the symmetric hyperbolic case.

The Adjusted system (essentials):

Purpose:	Control the violation of constraints by reformulating the system so as to have a constrained surface an attractor.
Procedure:	Add a particular combination of constraints to the evolution equations, and adjust its multipliers.
Theoretical support:	Eigenvalue analysis of the constraint propagation equations.
Advantages:	Available even if the base system is not a symmetric hyperbolic.
Advantages:	Keep the number of the variable same with the original system.

Conjecture on Constraint Amplification Factors (CAFs):

(A) If CAF has a negative real-part (the constraints are forced to be diminished), then we see more stable evolution than a system which has positive CAF.

(B) If CAF has a non-zero imaginary-part (the constraints are propagating away), then we see more stable evolution than a system which has zero CAF.

Adjusted ADM systems

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We adjust the standard ADM system using constraints as:

$$\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, & (1)\\
&\quad + P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), & (2)\\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik}K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, & (3)\\
&\quad + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), & (4)
\end{aligned}$$

with constraint equations

$$\begin{aligned}
\mathcal{H} &:= R^{(3)} + K^2 - K_{ij}K^{ij}, & (5)\\
\mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K. & (6)
\end{aligned}$$

We can write the adjusted constraint propagation equations as

$$\begin{aligned}
\partial_t \mathcal{H} &= \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], & (7)\\
\partial_t \mathcal{M}_i &= \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. & (8)
\end{aligned}$$

Original ADM The original construction by ADM uses the pair of (h_{ij}, π^{ij}).

$$\mathcal{L} \;=\; \sqrt{-g}R = \sqrt{h}N[{}^{(3)}R - K^2 + K_{ij}K^{ij}], \quad \text{where } K_{ij} = \frac{1}{2}\pounds_n h_{ij}$$

$$\text{then} \quad \pi^{ij} \;=\; \frac{\partial \mathcal{L}}{\partial \dot{h}_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}),$$

The Hamiltonian density gives us constraints and evolution eqs.

$$\mathcal{H} \;=\; \pi^{ij}\dot{h}_{ij} - \mathcal{L} = \sqrt{h}\left\{N\mathcal{H}(h,\pi) - 2N_j\mathcal{M}^j(h,\pi) + 2D_i(h^{-1/2}N_j\pi^{ij})\right\},$$

$$\begin{cases} \partial_t h_{ij} \;=\; \dfrac{\delta \mathcal{H}}{\delta \pi^{ij}} \;=\; 2\dfrac{N}{\sqrt{h}}(\pi_{ij} - \dfrac{1}{2}h_{ij}\pi) + 2D_{(i}N_{j)}, \\ \partial_t \pi^{ij} \;=\; -\dfrac{\delta \mathcal{H}}{\delta h_{ij}} \;=\; -\sqrt{h}N({}^{(3)}R^{ij} - \dfrac{1}{2}{}^{(3)}Rh^{ij}) + \dfrac{1}{2}\dfrac{N}{\sqrt{h}}h^{ij}(\pi_{mn}\pi^{mn} - \dfrac{1}{2}\pi^2) - 2\dfrac{N}{\sqrt{h}}(\pi^{in}\pi_n{}^j - \dfrac{1}{2}\pi\pi^{ij}) \\ \qquad\qquad + \sqrt{h}(D^iD^jN - h^{ij}D^mD_mN) + \sqrt{h}D_m(h^{-1/2}N^m\pi^{ij}) - 2\pi^{m(i}D_mN^{j)} \end{cases}$$

Standard ADM (by York) NRists refer ADM as the one by York with a pair of (h_{ij}, K_{ij}).

$$\begin{cases} \partial_t h_{ij} \;=\; -2NK_{ij} + D_jN_i + D_iN_j, \\ \partial_t K_{ij} \;=\; N({}^{(3)}R_{ij} + KK_{ij}) - 2NK_{il}K^l{}_j - D_iD_jN + (D_jN^m)K_{mi} + (D_iN^m)K_{mj} + N^mD_mK_{ij} \end{cases}$$

In the process of converting, $\mathcal{H}$ was used, i.e. the standard ADM has already adjusted.

3 Constraint propagation of ADM systems

3.1 Original ADM vs Standard ADM

Try the adjustment $\underline{R_{ij} = \kappa_1 \alpha \gamma_{ij} \text{ and other multiplier zero,}}$ where $\kappa_1 = \begin{cases} 0 & \text{the standard ADM} \\ -1/4 & \text{the original ADM} \end{cases}$

- The constraint propagation eqs keep the first-order form (cf Frittelli, PRD55(97)5992):

$$\partial_t \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} \simeq \begin{pmatrix} \beta^l & -2\alpha\gamma^{jl} \\ -(1/2)\alpha\delta^l_i + R^l{}_i - \delta^l_i R & \beta^l \delta^j_i \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{H} \\ \mathcal{M}_j \end{pmatrix}. \tag{5}$$

The eigenvalues of the characteristic matrix:

$$\lambda^l = (\beta^l, \beta^l, \beta^l \pm \sqrt{\alpha^2 \gamma^{ll}(1+4\kappa_1)})$$

The hyperbolicity of (5): $\begin{cases} \text{symmetric hyperbolic} & \text{when } \kappa_1 = 3/2 \\ \text{strongly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) > 0 \\ \text{weakly hyperbolic} & \text{when } \alpha^2\gamma^{ll}(1+4\kappa_1) \geq 0 \end{cases}$

- On the Minkowskii background metric, the linear order terms of the Fourier-transformed constraint propagation equations gives the eigenvalues

$$\Lambda^l = (0, 0, \pm\sqrt{-k^2(1+4\kappa_1)}).$$

That is, $\begin{cases} \text{(two 0s, two pure imaginary)} & \text{for the standard ADM} \\ \text{(four 0s)} & \text{for the original ADM} \end{cases}$ BETTER STABILITY

Comparisons of Adjusted ADM systems (Teukolsky wave)

3-dim, harmonic slice, periodic BC

HS original Cactus/GR code

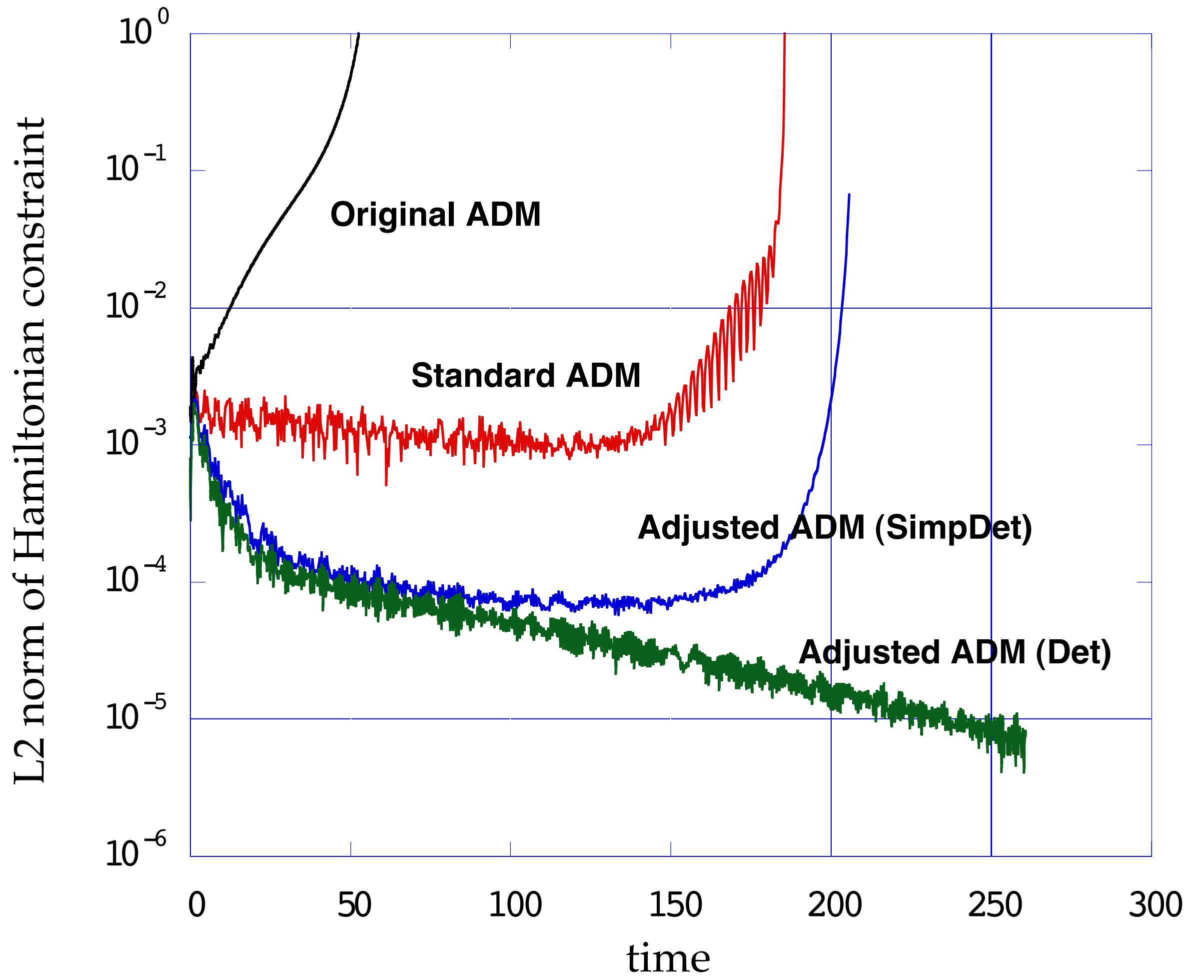


Figure 1: Violation of Hamiltonian constraints versus time: Adjusted ADM systems applied for Teukolsky wave initial data evolution with harmonic slicing, and with periodic boundary condition. Cactus/GR/evolveADMeq code was used. Grid $= 24^3$, $\Delta x = 0.25$, iterative Crank-Nicholson method.

4 Constraint propagations in spherically symmetric spacetime

4.1 The procedure

The discussion becomes clear if we expand the constraint $C_\mu := (\mathcal{H}, \mathcal{M}_i)^T$ using vector harmonics.

$$C_\mu = \sum_{l,m} \left(A^{lm}(t,r) a_{lm}(\theta,\varphi) + B^{lm} b_{lm} + C^{lm} c_{lm} + D^{lm} d_{lm} \right), \tag{1}$$

where we choose the basis of the vector harmonics as

$$a_{lm} = \begin{pmatrix} Y_{lm} \\ 0 \\ 0 \\ 0 \end{pmatrix}, b_{lm} = \begin{pmatrix} 0 \\ Y_{lm} \\ 0 \\ 0 \end{pmatrix}, c_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ \partial_\theta Y_{lm} \\ \partial_\varphi Y_{lm} \end{pmatrix}, d_{lm} = \frac{r}{\sqrt{l(l+1)}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\sin\theta}\partial_\varphi Y_{lm} \\ \sin\theta\, \partial_\theta Y_{lm} \end{pmatrix}.$$

The basis are normalized so that they satisfy

$$\langle C_\mu, C_\nu \rangle = \int_0^{2\pi} d\varphi \int_0^{\pi} C_\mu^* C_\rho \, \eta^{\nu\rho} \sin\theta d\theta,$$

where $\eta^{\nu\rho}$ is Minkowskii metric and the asterisk denotes the complex conjugate. Therefore

$$A^{lm} = \langle a^{lm}_{(\nu)}, C_\nu \rangle, \quad \partial_t A^{lm} = \langle a^{lm}_{(\nu)}, \partial_t C_\nu \rangle, \quad \text{etc.}$$

We also express these evolution equations using the Fourier expansion on the radial coordinate,

$$A^{lm} = \sum_k \hat{A}^{lm}_{(k)}(t)\, e^{ikr} \quad \text{etc.} \tag{2}$$

So that we will be able to obtain the RHS of the evolution equations for $(\hat{A}^{lm}_{(k)}(t), \cdots, \hat{D}^{lm}_{(k)}(t))^T$ in a homogeneous form.

4.2 Constraint propagations in Schwarzschild spacetime

1. the standard Schwarzschild coordinate

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2, \qquad \text{(the standard expression)}$$

2. the isotropic coordinate, which is given by, $r = (1 + M/2r_{iso})^2 r_{iso}$:

$$ds^2 = -(\frac{1 - M/2r_{iso}}{1 + M/2r_{iso}})^2 dt^2 + (1 + \frac{M}{2r_{iso}})^4 [dr_{iso}^2 + r_{iso}^2 d\Omega^2], \qquad \text{(the isotropic expression)}$$

3. the ingoing Eddington-Finkelstein (iEF) coordinate, by $t_{iEF} = t + 2M \log(r - 2M)$:

$$ds^2 = -(1 - \frac{2M}{r})dt_{iEF}^2 + \frac{4M}{r} dt_{iEF} dr + (1 + \frac{2M}{r})dr^2 + r^2 d\Omega^2 \qquad \text{(the iEF expression)}$$

4. the Painlevé-Gullstrand (PG) coordinates,

$$ds^2 = -\left(1 - \frac{2\,M}{r}\right) dt_{PG}^2 + 2\sqrt{\frac{2\,M}{r}} dt_{PG}\, dr + dr^2 + r^2 d\Omega^2 \;, \qquad \text{(the PG expression)}$$

which is given by $t_{PG} = t + \sqrt{8Mr} - 2M \log\{(\sqrt{r/2M} + 1)/(\sqrt{r/2M} - 1)\}$

Example 1: standard ADM vs original ADM (in Schwarzschild coordinate)

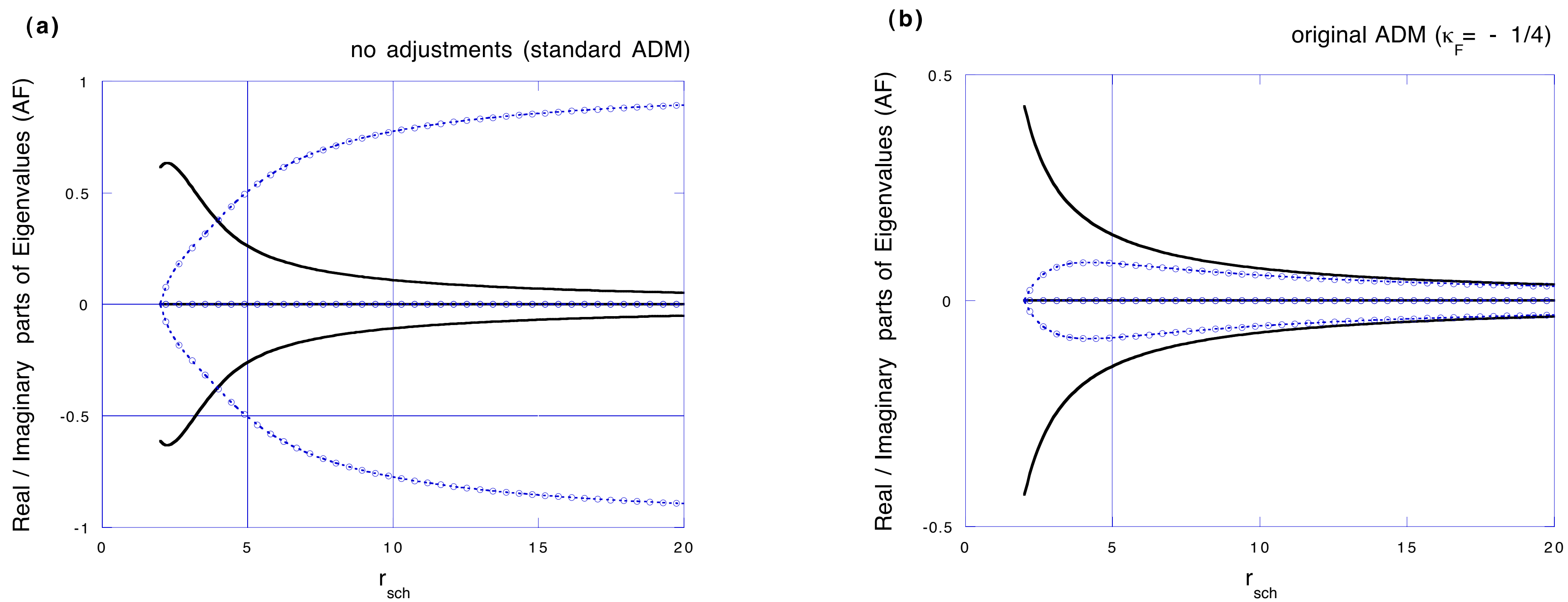


Figure 1: Amplification factors (AFs, eigenvalues of homogenized constraint propagation equations) are shown for the standard Schwarzschild coordinate, with (a) no adjustments, i.e., standard ADM, (b) original ADM ($\kappa_F = -1/4$). The solid lines and the dotted lines with circles are real parts and imaginary parts, respectively. They are four lines each, but actually the two eigenvalues are zero for all cases. Plotting range is $2 < r \leq 20$ using Schwarzschild radial coordinate. We set $k = 1, l = 2$, and $m = 2$ throughout the article.

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} + \kappa_F \alpha \gamma_{ij} \mathcal{H},
\end{aligned}
$$

Example 2: Detweiler-type adjusted (in Schwarzschild coord.)

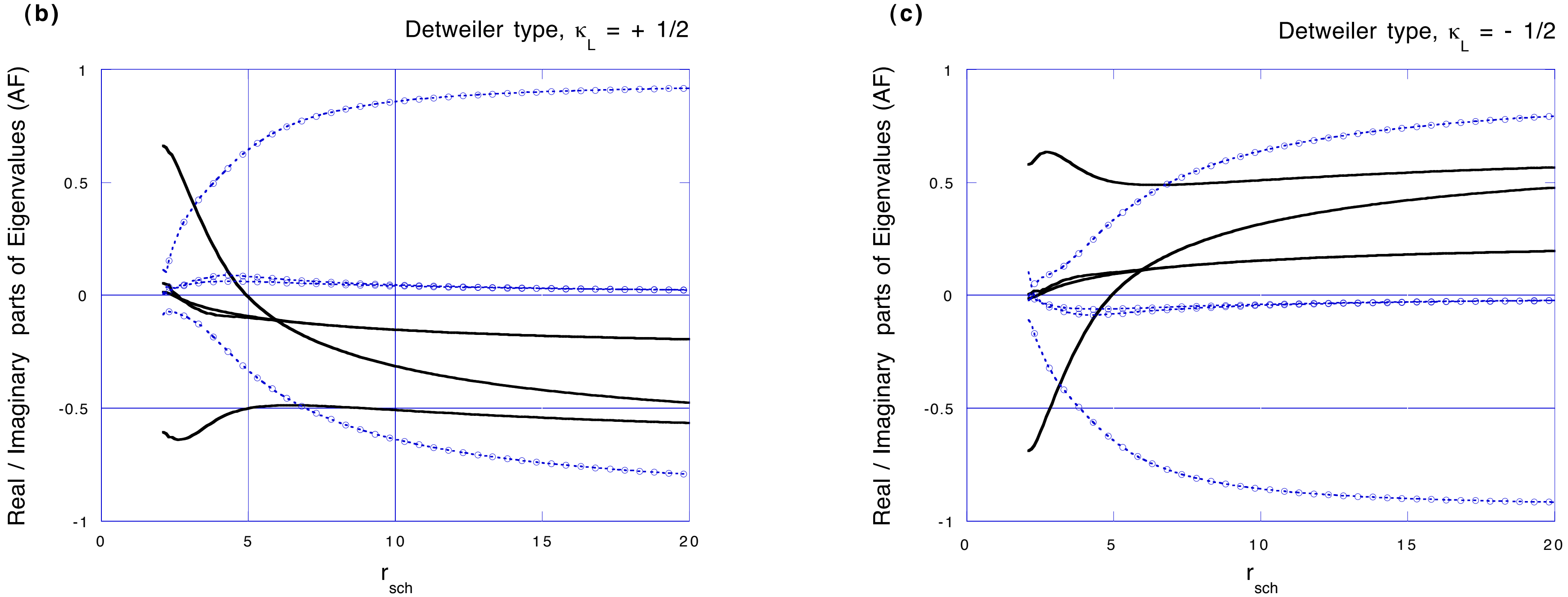


Figure 2: Amplification factors of the standard Schwarzschild coordinate, with Detweiler type adjustments. Multipliers used in the plot are (b) $\kappa_L = +1/2$, and (c) $\kappa_L = -1/2$.

$$\begin{aligned}
\partial_t \gamma_{ij} &= \text{(original terms)} + P_{ij}\mathcal{H}, \\
\partial_t K_{ij} &= \text{(original terms)} + R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), \\
& \text{where} \quad P_{ij} = -\kappa_L \alpha^3 \gamma_{ij}, \quad R_{ij} = \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}), \\
& \qquad S^k{}_{ij} = \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}], \quad s^{kl}{}_{ij} = \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}],
\end{aligned}$$

Detweiler's criteria vs Our criteria

- Detweiler calculated the L2 norm of the constraints, C_α, over the 3-hypersurface and imposed its negative definiteness of its evolution,

$$\text{Detweiler's criteria} \;\Leftrightarrow\; \partial_t \int \sum_\alpha C_\alpha^2 \; dV < 0,$$

  This is rewritten by supposing the constraint propagation to be $\partial_t \hat{C}_\alpha = A_\alpha{}^\beta \hat{C}_\beta$ in the Fourier components,

$$\begin{aligned} &\Leftrightarrow \quad \partial_t \int \sum_\alpha \hat{C}_\alpha \bar{\hat{C}}_\alpha \; dV = \int \sum_\alpha A_\alpha{}^\beta \hat{C}_\beta \bar{\hat{C}}_\alpha + \hat{C}_\alpha \bar{A}_\alpha{}^\beta \bar{\hat{C}}_\beta \, dV < 0, \; \forall \text{ non zero } \hat{C}_\alpha \\ &\Leftrightarrow \quad \underline{\text{eigenvalues of } (A + A^\dagger) \text{ are all negative for } \forall k}. \end{aligned}$$

- Our criteria is that <u>the eigenvalues of A are all negative</u>. Therefore,

$$\boxed{\text{Our criteria} \ni \text{Detweiler's criteria}}$$

- We remark that Detweiler's truncations on higher order terms in C-norm corresponds our perturbative analysis, both based on the idea that the deviations from constraint surface (the errors expressed non-zero constraint value) are initially small.

Constraint propagation of ADM systems

(2) Detweiler's system

Detweiler's modification to ADM [PRD35(87)1095] can be realized in our notation as:

$$\begin{aligned} P_{ij} &= -L\alpha^3\gamma_{ij}, \\ R_{ij} &= L\alpha^3(K_{ij} - (1/3)K\gamma_{ij}), \\ S^k_{ij} &= L\alpha^2[3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l\alpha)\gamma_{ij}\gamma^{kl}], \\ s^{kl}_{ij} &= L\alpha^3[2\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}], \end{aligned}$$

and else zero, where L is a constant.

- This adjustment does not make constraint propagation equation in the first order form, so that we can not discuss the hyperbolicity nor the characteristic speed of the constraints.
- For the Minkowskii background spacetime, the adjusted constraint propagation equations with above choice of multiplier become

$$\begin{aligned} \partial_t\mathcal{H} &= -2(\partial_j\mathcal{M}_j) + 4L(\partial_j\partial_j\mathcal{H}), \\ \partial_t\mathcal{M}_i &= -(1/2)(\partial_i\mathcal{H}) + (L/2)(\partial_k\partial_k\mathcal{M}_i) + (L/6)(\partial_i\partial_k\mathcal{M}_k). \end{aligned}$$

Constraint Amp. Factors (the eigenvalues of their Fourier expression) are

$$\Lambda^l = (-(L/2)k^2 (\text{multiplicity } 2), -(7L/3)k^2 \pm (1/3)\sqrt{k^2(-9+25L^2k^2)}.)$$

This indicates negative real eigenvalues if we chose small positive L.

Example 3: standard ADM (in isotropic/iEF coord.)

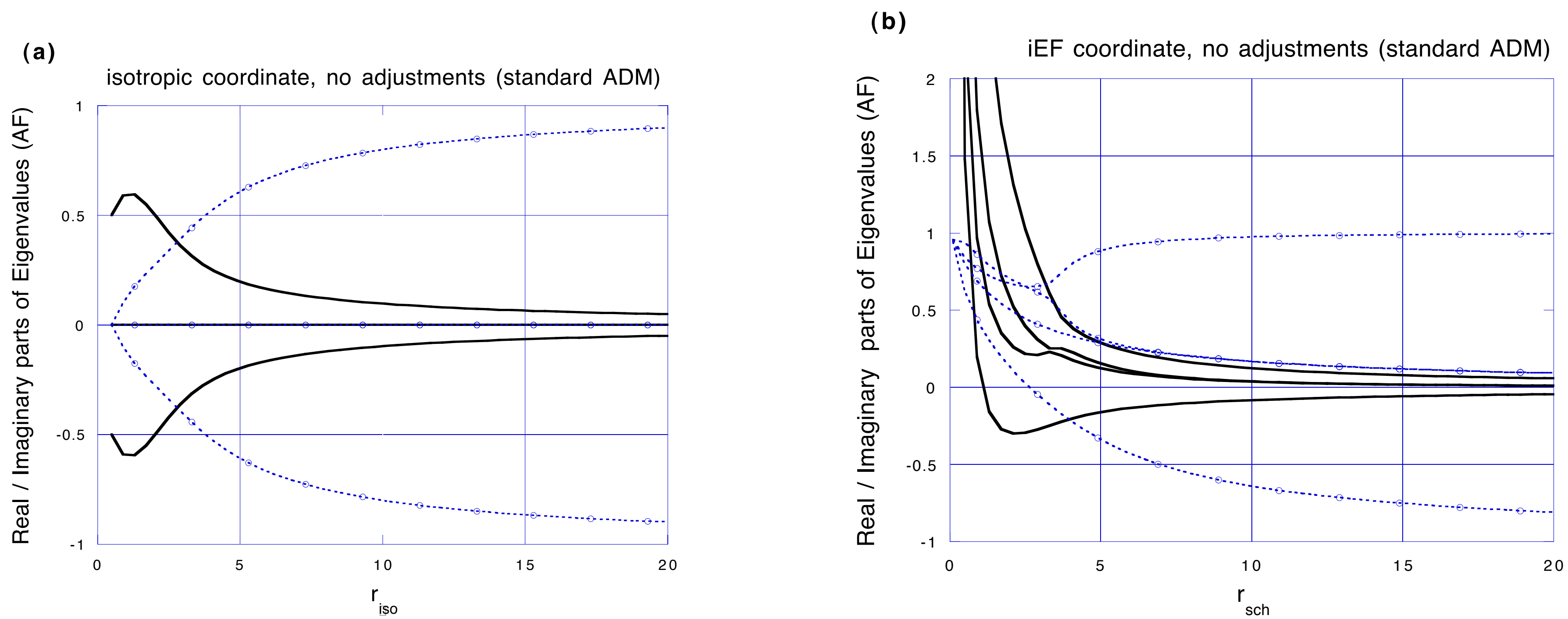


Figure 3: Comparison of amplification factors between different coordinate expressions for the standard ADM formulation (i.e. no adjustments). Fig. (a) is for the isotropic coordinate (1), and the plotting range is $1/2 \leq r_{iso}$. Fig. (b) is for the iEF coordinate (1) and we plot lines on the $t = 0$ slice for each expression. The solid four lines and the dotted four lines with circles are real parts and imaginary parts, respectively.

Example 4: Detweiler-type adjusted (in iEF/PG coord.)

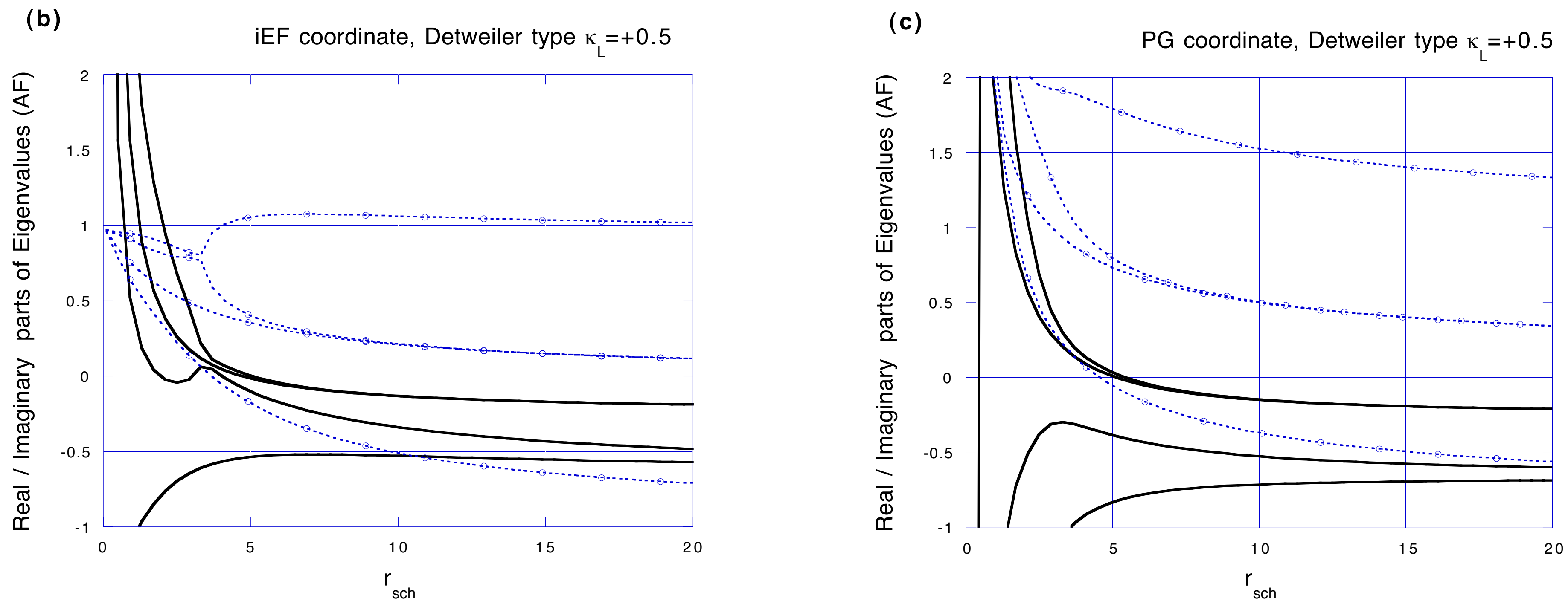


Figure 4: Similar comparison for Detweiler adjustments. $\kappa_L = +1/2$ for all plots.

"Einstein equations" are time-reversal invariant. So ...

Why all negative amplification factors (AFs) are available?

Explanation by the time-reversal invariance (TRI)

- the adjustment of the system I,

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{K_{ij}}_{(-)} = \kappa_1 \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

preserves TRI. ... so the AFs remain zero (unchange).

- the adjustment by (a part of) Detweiler

$$\text{adjust term to } \underbrace{\partial_t}_{(-)} \underbrace{\gamma_{ij}}_{(+)} = -L \underbrace{\alpha}_{(+)} \underbrace{\gamma_{ij}}_{(+)} \underbrace{\mathcal{H}}_{(+)}$$

violates TRI. ... so the AFs can become negative.

Therefore

$$\boxed{\text{We can break the time-reversal invariant feature of the ``ADM equations''.}}$$

Adjusted ADM systems

PRD 63 (2001) 120419, CQG 19 (2002) 1027

We adjust the standard ADM system using constraints as:

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i, && (1)\\
&\quad +P_{ij}\mathcal{H} + Q^k{}_{ij}\mathcal{M}_k + p^k{}_{ij}(\nabla_k \mathcal{H}) + q^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (2)\\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}, && (3)\\
&\quad +R_{ij}\mathcal{H} + S^k{}_{ij}\mathcal{M}_k + r^k{}_{ij}(\nabla_k \mathcal{H}) + s^{kl}{}_{ij}(\nabla_k \mathcal{M}_l), && (4)
\end{aligned}
$$

with constraint equations

$$
\begin{aligned}
\mathcal{H} &:= R^{(3)} + K^2 - K_{ij}K^{ij}, && (5)\\
\mathcal{M}_i &:= \nabla_j K^j{}_i - \nabla_i K. && (6)
\end{aligned}
$$

We can write the adjusted constraint propagation equations as

$$
\begin{aligned}
\partial_t \mathcal{H} &= \text{(original terms)} + H_1^{mn}[(2)] + H_2^{imn}\partial_i[(2)] + H_3^{ijmn}\partial_i\partial_j[(2)] + H_4^{mn}[(4)], && (7)\\
\partial_t \mathcal{M}_i &= \text{(original terms)} + M_{1i}{}^{mn}[(2)] + M_{2i}{}^{jmn}\partial_j[(2)] + M_{3i}{}^{mn}[(4)] + M_{4i}{}^{jmn}\partial_j[(4)]. && (8)
\end{aligned}
$$

Table 3. List of adjustments we tested in the Schwarzschild spacetime. The column of adjustments are nonzero multipliers in terms of (13) and (14). The column '1st?' and 'TRS' are the same as in table 1. The effects to amplification factors (when $\kappa > 0$) are commented for each coordinate system and for real/imaginary parts of AFs, respectively. The 'N/A' means that there is no effect due to the coordinate properties; 'not apparent' means the adjustment does not change the AFs effectively according to our conjecture; 'enl./red./min.' means enlarge/reduce/minimize, and 'Pos./Neg.' means positive/negative, respectively. These judgements are made at the $r \sim O(10M)$ region on their $t = 0$ slice.

					Schwarzschild/isotropic coordinates			iEF/PG coordinates	
No	No in table 1		Adjustment	1st?	TRS	Real	Imaginary	Real	Imaginary
0	0	–	no adjustments	yes	–	–	–	–	–
P-1	2-P	P_{ij}	$-\kappa_L \alpha^3 \gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-2	3	P_{ij}	$-\kappa_L \alpha \gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-3	–	P_{ij}	$P_{rr} = -\kappa$ or $P_{rr} = -\kappa\alpha$	no	no	slightly enl.Neg.	not apparent	slightly enl.Neg.	not apparent
P-4	–	P_{ij}	$-\kappa \gamma_{ij}$	no	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
P-5	–	P_{ij}	$-\kappa \gamma_{rr}$	no	no	red. Pos./enl.Neg.	not apparent	red.Pos./enl.Neg.	not apparent
Q-1	–	$Q^k{}_{ij}$	$\kappa \alpha \beta^k \gamma_{ij}$	no	no	N/A	N/A	$\kappa \sim 1.35$ min. vals.	not apparent
Q-2	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
Q-3	–	$Q^k{}_{ij}$	$Q^r{}_{ij} = \kappa \gamma_{ij}$ or $Q^r{}_{ij} = \kappa \alpha \gamma_{ij}$	no	yes	red. abs vals.	not apparent	enl.Neg.	enl. vals.
Q-4	–	$Q^k{}_{ij}$	$Q^r{}_{rr} = \kappa \gamma_{rr}$	no	yes	red. abs vals.	not apparent	red. abs vals.	not apparent
R-1	1	R_{ij}	$\kappa_F \alpha \gamma_{ij}$	yes	yes	$\kappa_F = -1/4$ min. abs vals.		$\kappa_F = -1/4$ min. vals.	
R-2	4	R_{ij}	$R_{rr} = -\kappa_\mu \alpha$ or $R_{rr} = -\kappa_\mu$	yes	no	not apparent	not apparent	red.Pos./enl.Neg.	enl. vals.
R-3	–	R_{ij}	$R_{rr} = -\kappa \gamma_{rr}$	yes	no	enl. vals.	not apparent	red.Pos./enl.Neg.	enl. vals.
S-1	2-S	$S^k{}_{ij}$	$\kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}]$	yes	no	not apparent	not apparent	not apparent	not apparent
S-2	–	$S^k{}_{ij}$	$\kappa \alpha \gamma^{lk} (\partial_l \gamma_{ij})$	yes	no	makes 2 Neg.	not apparent	makes 2 Neg.	not apparent
p-1	–	$p^k{}_{ij}$	$p^r{}_{ij} = -\kappa \alpha \gamma_{ij}$	no	no	red. Pos.	red. vals.	red. Pos.	enl. vals.
p-2	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa \alpha$	no	no	red. Pos.	red. vals.	red.Pos/enl.Neg.	enl. vals.
p-3	–	$p^k{}_{ij}$	$p^r{}_{rr} = \kappa \alpha \gamma_{rr}$	no	no	makes 2 Neg.	enl. vals.	red. Pos. vals.	red. vals.
q-1	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{ij} = \kappa \alpha \gamma_{ij}$	no	no	$\kappa = 1/2$ min. vals.	red. vals.	not apparent	enl. vals.
q-2	–	$q^{kl}{}_{ij}$	$q^{rr}{}_{rr} = -\kappa \alpha \gamma_{rr}$	no	yes	red. abs vals.	not apparent	not apparent	not apparent
r-1	–	$r^k{}_{ij}$	$r^r{}_{ij} = \kappa \alpha \gamma_{ij}$	no	yes	not apparent	not apparent	not apparent	enl. vals.
r-2	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa \alpha$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
r-3	–	$r^k{}_{ij}$	$r^r{}_{rr} = -\kappa \alpha \gamma_{rr}$	no	yes	red. abs vals.	enl. vals.	red. abs vals.	enl. vals.
s-1	2-s	$s^{kl}{}_{ij}$	$\kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}]$	no	no	makes 4 Neg.	not apparent	makes 4 Neg.	not apparent
s-2	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{ij} = -\kappa \alpha \gamma_{ij}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.
s-3	–	$s^{kl}{}_{ij}$	$s^{rr}{}_{rr} = -\kappa \alpha \gamma_{rr}$	no	no	makes 2 Neg.	red. vals.	makes 2 Neg.	red. vals.

Numerical Tests (method)

- Cactus-based original “GR” code
  http://www.cactuscode.org/
  [CactusBase+CactusPUGH+GR]
- 3+1dim, linear wave evolution
  (Teukolsky wave)
- harmonic slice
- periodic boundary, [-3,+3]
- iterative Crank-Nicholson method
- 12^3, 24^3, 48^3, 96^3

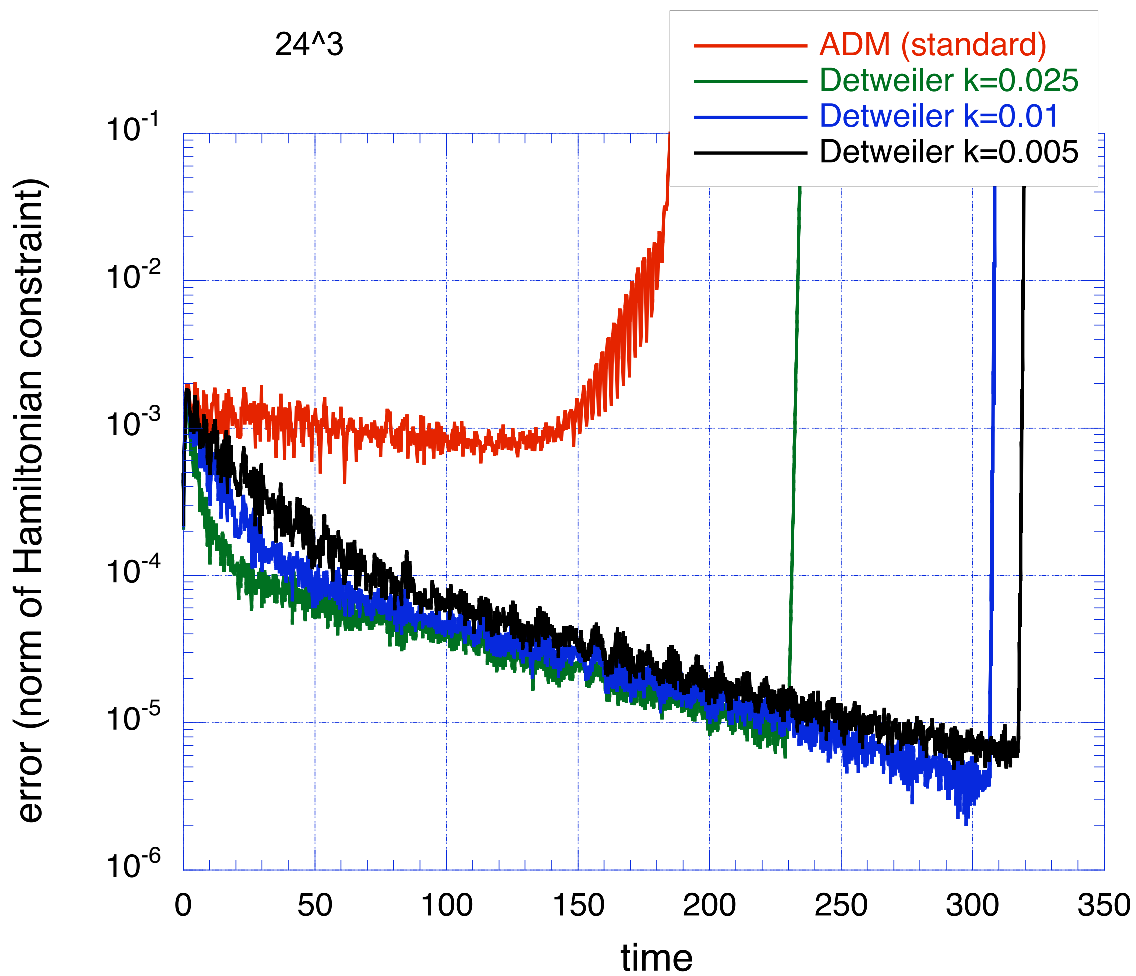
Towards standard testbeds for numerical relativity
Mexico Numerical Relativity Workshop 2002 Participants
CQG 21 (2004) 589-613

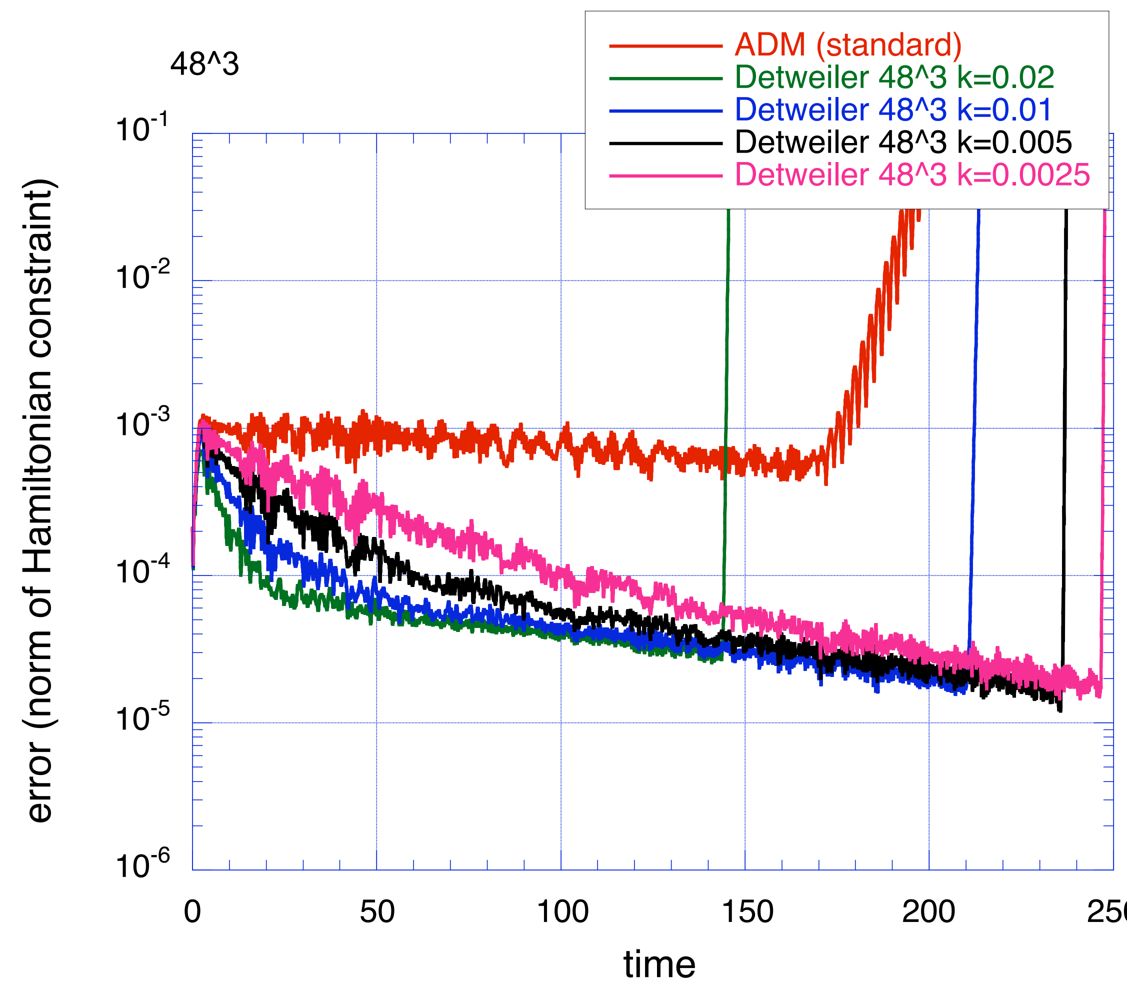
Numerical Tests (Detweiler-type)

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H}$$

PRD35(1987)1095

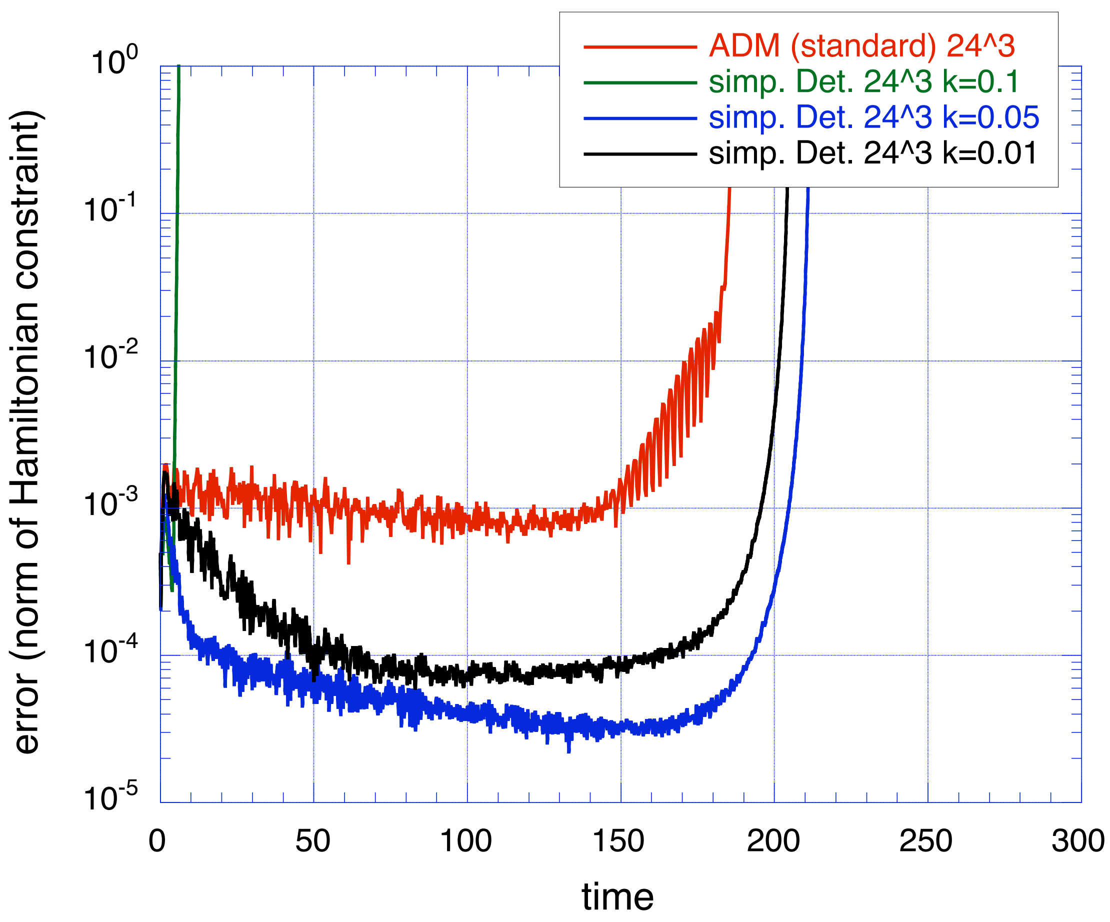
$$\begin{aligned}\partial_t K_{ij} = {}& \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\ & + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}] \mathcal{M}_k \\ & + \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}] (\nabla_k \mathcal{M}_l)\end{aligned}$$

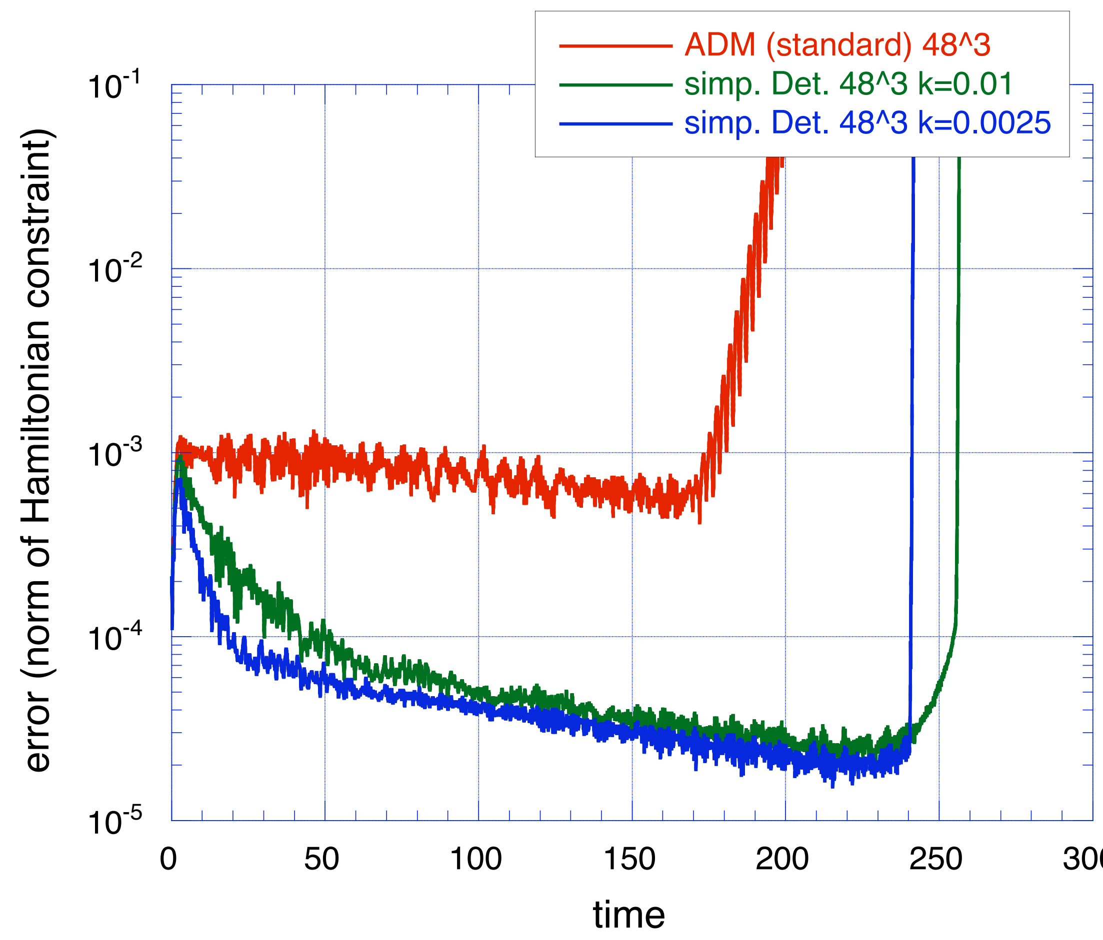




Numerical Tests (Simplified Detweiler)

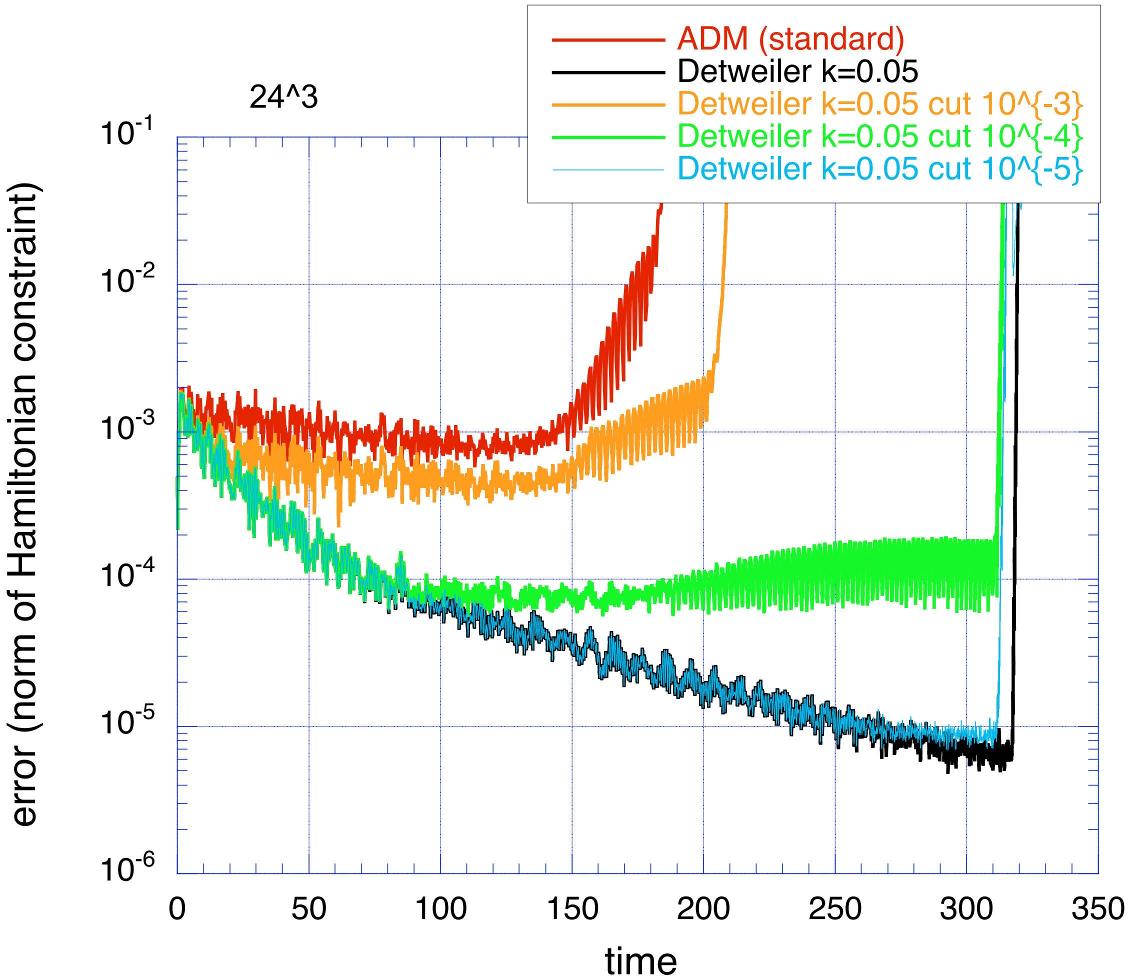
$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha \gamma_{ij} \mathcal{H}$$
$$\partial_t K_{ij} = \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij}$$

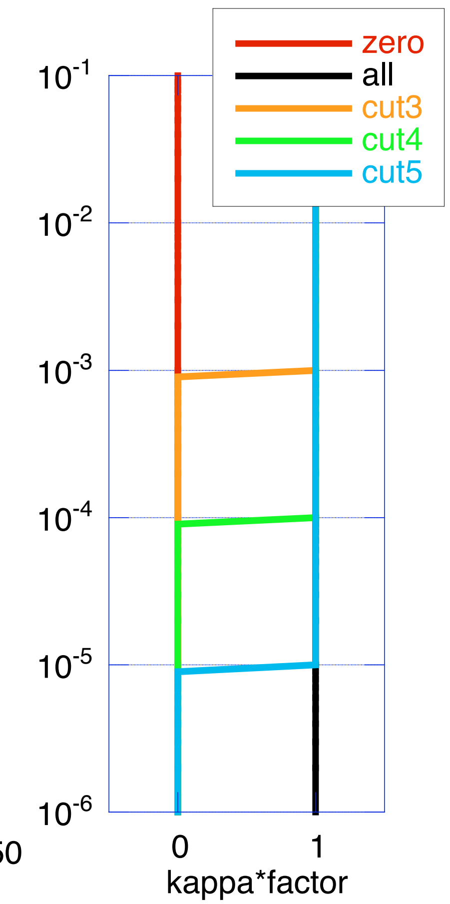




Numerical Tests (Detweiler, k-adjust)

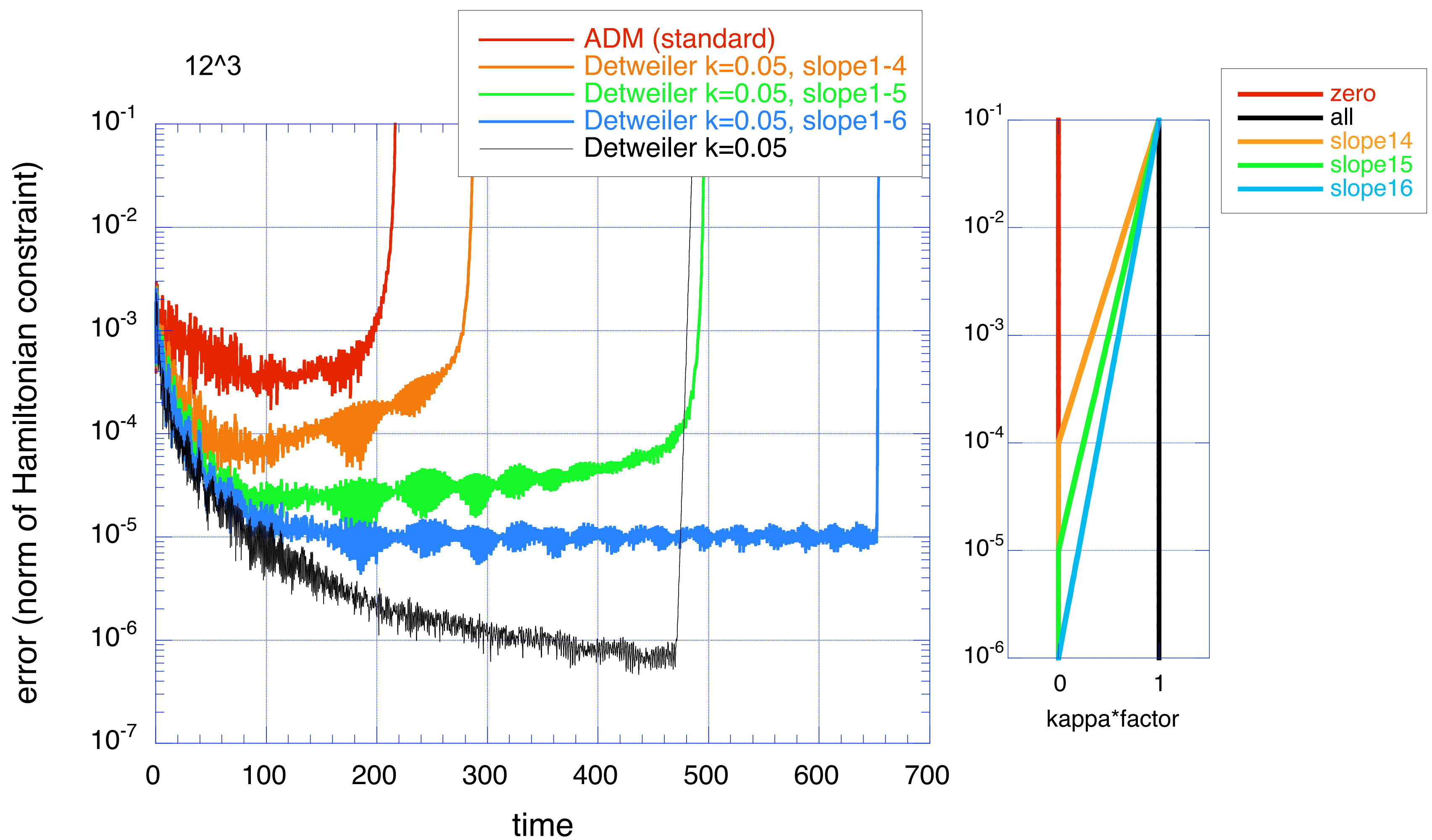
$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H} \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}] \mathcal{M}_k \\
&\quad + \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}] (\nabla_k \mathcal{M}_l)
\end{aligned}
$$





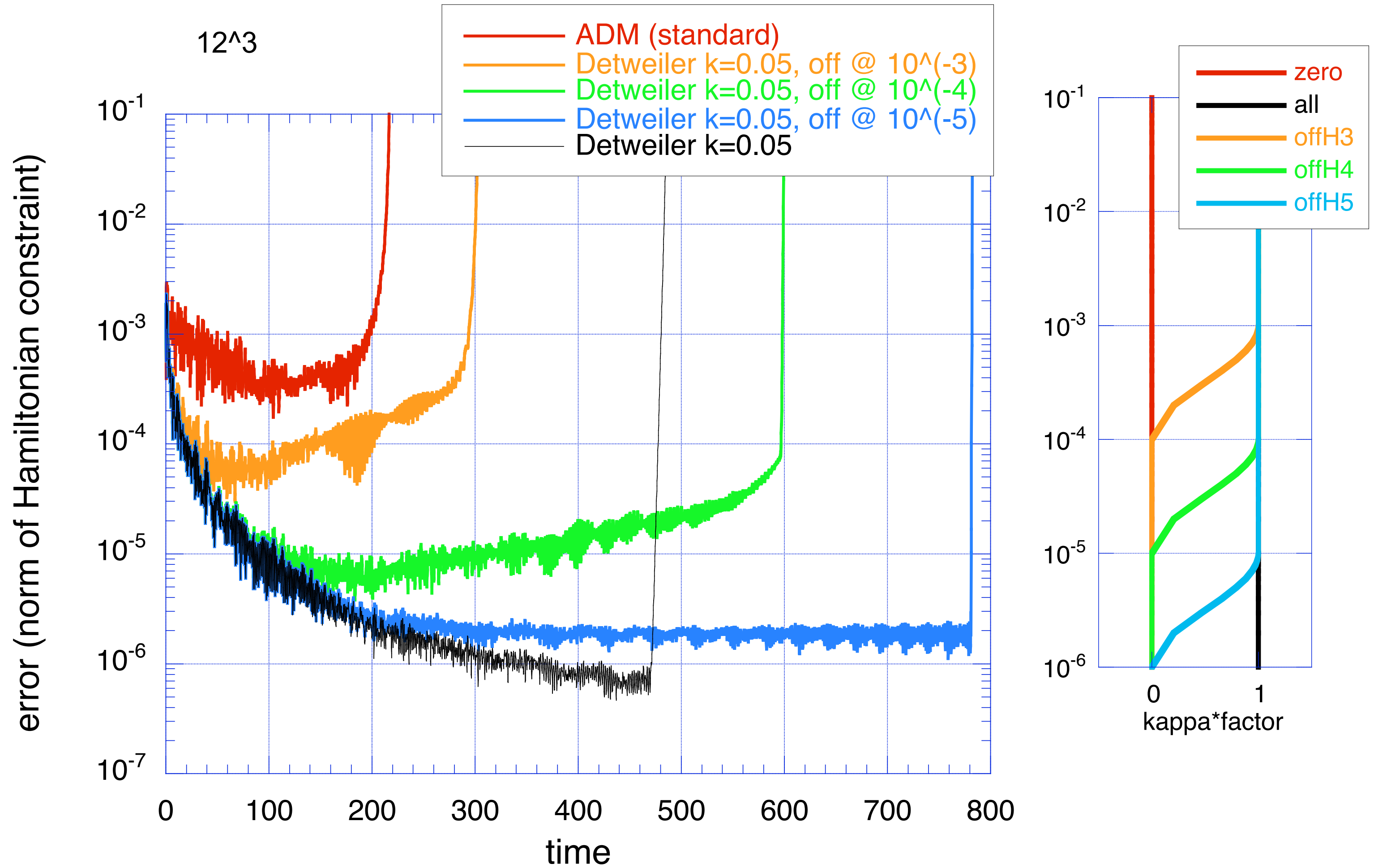
Numerical Tests (Detweiler, k-adjust)

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H} \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}] \mathcal{M}_k \\
&\quad + \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}] (\nabla_k \mathcal{M}_l)
\end{aligned}
$$



Numerical Tests (Detweiler, k-adjust)

$$
\begin{aligned}
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + \nabla_i \beta_j + \nabla_j \beta_i - \kappa_L \alpha^3 \gamma_{ij} \mathcal{H} \\
\partial_t K_{ij} &= \alpha R^{(3)}_{ij} + \alpha K K_{ij} - 2\alpha K_{ik} K^k{}_j - \nabla_i \nabla_j \alpha + (\nabla_i \beta^k) K_{kj} + (\nabla_j \beta^k) K_{ki} + \beta^k \nabla_k K_{ij} \\
&\quad + \kappa_L \alpha^3 (K_{ij} - (1/3) K \gamma_{ij}) \mathcal{H} + \kappa_L \alpha^2 [3(\partial_{(i}\alpha)\delta^k_{j)} - (\partial_l \alpha)\gamma_{ij}\gamma^{kl}] \mathcal{M}_k \\
&\quad + \kappa_L \alpha^3 [\delta^k_{(i}\delta^l_{j)} - (1/3)\gamma_{ij}\gamma^{kl}] (\nabla_k \mathcal{M}_l)
\end{aligned}
$$



APCTP Winter School, January 25-26, 2008

Formulation Problem in Numerical Relativity

Hisaaki Shinkai (Osaka Institute of Technology, Japan)

신 카 이 히 사 아 키

1. Introduction
2. The Standard Approach to Numerical Relativity
   ADM/BSSN/hyperbolic formulations
3. Robust system for Constraint Violation
   Adjusted systems
   Adjusted ADM system -- why the standard ADM brows up?
   Adjusted BSSN system -- should be better than BSSN
4. Outlook

strategy 1 Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation

T. Nakamura, K. Oohara and Y. Kojima, Prog. Theor. Phys. Suppl. **90**, 1 (1987)
M. Shibata and T. Nakamura, Phys. Rev. D **52**, 5428 (1995)
T.W. Baumgarte and S.L. Shapiro, Phys. Rev. D **59**, 024007 (1999)

The popular approach. Nakamura's idea in 1980s.
BSSN is a tricky nickname. BS (1999) introduced a paper of SN (1995).

- define new set of variables $(\phi, \tilde{\gamma}_{ij}, K, \tilde{A}_{ij}, \tilde{\Gamma}^i)$, instead of the ADM's (γ_{ij}, K_{ij}) where

$$\boxed{\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij}, \qquad \tilde{A}_{ij} \equiv e^{-4\phi}(K_{ij} - (1/3)\gamma_{ij}K), \qquad \tilde{\Gamma}^i \equiv \tilde{\Gamma}^i_{jk}\tilde{\gamma}^{jk},}$$

  and impose $\det\tilde{\gamma}_{ij} = 1$ during the evolutions.

- The set of evolution equations become

$$\boxed{\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\phi &= -(1/6)\alpha K, \\
(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij}, \\
(\partial_t - \mathcal{L}_\beta)K &= \alpha\tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - \gamma^{ij}(\nabla_i\nabla_j\alpha), \\
(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} &= -e^{-4\phi}(\nabla_i\nabla_j\alpha)^{TF} + e^{-4\phi}\alpha R^{(3)}_{ij} - e^{-4\phi}\alpha(1/3)\gamma_{ij}R^{(3)} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j) \\
\partial_t\tilde{\Gamma}^i &= -2(\partial_j\alpha)\tilde{A}^{ij} - (4/3)\alpha(\partial_j K)\tilde{\gamma}^{ij} + 12\alpha\tilde{A}^{ji}(\partial_j\phi) - 2\alpha\tilde{A}_k{}^j(\partial_j\tilde{\gamma}^{ik}) - 2\alpha\tilde{\Gamma}^k{}_{lj}\tilde{A}^j{}_k\tilde{\gamma}^{il} \\
&\quad -\partial_j\left(\beta^k\partial_k\tilde{\gamma}^{ij} - \tilde{\gamma}^{kj}(\partial_k\beta^i) - \tilde{\gamma}^{ki}(\partial_k\beta^j) + (2/3)\tilde{\gamma}^{ij}(\partial_k\beta^k)\right)
\end{aligned}}$$

Momentum constraint was used in Γ^i-eq.

- Calculate Riemann tensor as

$$\begin{aligned} R_{ij} &= \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^m_{ij}\Gamma^k_{mk} - \Gamma^m_{kj}\Gamma^k_{mi} =: \tilde{R}_{ij} + R^\phi_{ij} \\ & R^\phi_{ij} = -2\tilde{D}_i\tilde{D}_j\phi - 2\tilde{g}_{ij}\tilde{D}^l\tilde{D}_l\phi + 4(\tilde{D}_i\phi)(\tilde{D}_j\phi) - 4\tilde{g}_{ij}(\tilde{D}^l\phi)(\tilde{D}_l\phi) \\ & \tilde{R}_{ij} = -(1/2)\tilde{g}^{lm}\partial_{lm}\tilde{g}_{ij} + \tilde{g}_{k(i}\partial_{j)}\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} + 2\tilde{g}^{lm}\tilde{\Gamma}^k_{l(i}\tilde{\Gamma}_{j)km} + \tilde{g}^{lm}\tilde{\Gamma}^k_{im}\tilde{\Gamma}_{klj} \end{aligned}$$

- Constraints are $\mathcal{H}, \mathcal{M}_i$.
  But thre are additional ones, $\mathcal{G}^i, \mathcal{A}, \mathcal{S}$.

Hamiltonian and the momentum constraint equations

$$\mathcal{H}^{BSSN} = R^{BSSN} + K^2 - K_{ij}K^{ij}, \tag{1}$$

$$\mathcal{M}_i^{BSSN} = \mathcal{M}_i^{ADM}, \tag{2}$$

Additionally, we regard the following three as the constraints:

$$\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \tag{3}$$

$$\mathcal{A} = \tilde{A}_{ij}\tilde{\gamma}^{ij}, \tag{4}$$

$$\mathcal{S} = \tilde{\gamma} - 1, \tag{5}$$

Why BSSN better than ADM?
Is the BSSN best? Are there any alternatives?

Constraints in BSSN system

The normal Hamiltonian and momentum constraints

$$\mathcal{H}^{BSSN} = R^{BSSN} + K^2 - K_{ij}K^{ij}, \quad (1)$$

$$\mathcal{M}_i^{BSSN} = \mathcal{M}_i^{ADM}, \quad (2)$$

Additionally, we regard the following three as the constraints:

$$\mathcal{G}^i = \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}, \quad (3)$$

$$\mathcal{A} = \tilde{A}_{ij}\tilde{\gamma}^{ij}, \quad (4)$$

$$\mathcal{S} = \tilde{\gamma} - 1, \quad (5)$$

Adjustments in evolution equations

$$\begin{aligned}
\partial_t^B \varphi &= \partial_t^A \varphi + (1/6)\alpha\mathcal{A} - (1/12)\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\beta^j, & (6)\\
\partial_t^B \tilde{\gamma}_{ij} &= \partial_t^A \tilde{\gamma}_{ij} - (2/3)\alpha\tilde{\gamma}_{ij}\mathcal{A} + (1/3)\tilde{\gamma}^{-1}(\partial_k\mathcal{S})\beta^k\tilde{\gamma}_{ij}, & (7)\\
\partial_t^B K &= \partial_t^A K - (2/3)\alpha K\mathcal{A} - \alpha\mathcal{H}^{BSSN} + \alpha e^{-4\varphi}(\tilde{D}_j\mathcal{G}^j), & (8)\\
\partial_t^B \tilde{A}_{ij} &= \partial_t^A \tilde{A}_{ij} + ((1/3)\alpha\tilde{\gamma}_{ij}K - (2/3)\alpha\tilde{A}_{ij})\mathcal{A} + \alpha e^{-4\varphi}((1/2)(\partial_k\tilde{\gamma}_{ij}) - (1/6)\tilde{\gamma}_{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}))\mathcal{G}^k & \\
&\quad +\alpha e^{-4\varphi}\tilde{\gamma}_{k(i}(\partial_{j)}\mathcal{G}^k) - (1/3)\alpha e^{-4\varphi}\tilde{\gamma}_{ij}(\partial_k\mathcal{G}^k) & (9)\\
\partial_t^B \tilde{\Gamma}^i &= \partial_t^A \tilde{\Gamma}^i - ((2/3)(\partial_j\alpha)\tilde{\gamma}^{ji} + (2/3)\alpha(\partial_j\tilde{\gamma}^{ji}) + (1/3)\alpha\tilde{\gamma}^{ji}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) - 4\alpha\tilde{\gamma}^{ij}(\partial_j\varphi))\mathcal{A} & \\
&\quad -(2/3)\alpha\tilde{\gamma}^{ji}(\partial_j\mathcal{A}) + 2\alpha\tilde{\gamma}^{ij}\mathcal{M}_j - (1/2)(\partial_k\beta^i)\tilde{\gamma}^{kj}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (1/6)(\partial_j\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_k\mathcal{S}) & \\
&\quad +(1/3)(\partial_k\beta^k)\tilde{\gamma}^{ij}\tilde{\gamma}^{-1}(\partial_j\mathcal{S}) + (5/6)\beta^k\tilde{\gamma}^{-2}\tilde{\gamma}^{ij}(\partial_k\mathcal{S})(\partial_j\mathcal{S}) + (1/2)\beta^k\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\mathcal{S}) & \\
&\quad +(1/3)\beta^k\tilde{\gamma}^{-1}(\partial_j\tilde{\gamma}^{ji})(\partial_k\mathcal{S}). & (10)
\end{aligned}$$

A Full set of BSSN constraint propagation eqs.

$$\partial_t^{BS}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_i\\ \mathcal{G}^i\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ -(1/3)(\partial_i\alpha)+(1/6)\partial_i & \alpha K & A_{23} & 0 & A_{25} \\ 0 & \alpha\tilde{\gamma}^{ij} & 0 & A_{34} & A_{35} \\ 0 & 0 & 0 & \beta^k(\partial_k\mathcal{S}) & -2\alpha\tilde{\gamma} \\ 0 & 0 & 0 & 0 & \alpha K+\beta^k\partial_k \end{pmatrix}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_j\\ \mathcal{G}^j\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix}$$

$$\begin{aligned}
A_{11} &= +(2/3)\alpha K+(2/3)\alpha\mathcal{A}+\beta^k\partial_k\\
A_{12} &= -4e^{-4\varphi}\alpha(\partial_k\varphi)\tilde{\gamma}^{kj}-2e^{-4\varphi}(\partial_k\alpha)\tilde{\gamma}^{jk}\\
A_{13} &= -2\alpha e^{-4\varphi}\tilde{A}^k{}_j\partial_k-\alpha e^{-4\varphi}(\partial_j\tilde{A}_{kl})\tilde{\gamma}^{kl}-e^{-4\varphi}(\partial_j\alpha)\mathcal{A}-e^{-4\varphi}\beta^k\partial_k\partial_j-(1/2)e^{-4\varphi}\beta^k\tilde{\gamma}^{-1}(\partial_j\mathcal{S})\partial_k\\
&\quad +(1/6)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_j\beta^k)(\partial_k\mathcal{S})-(2/3)e^{-4\varphi}(\partial_k\beta^k)\partial_j\\
A_{14} &= 2\alpha e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{lk}(\partial_l\varphi)\mathcal{A}\partial_k+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l\mathcal{A})\tilde{\gamma}^{lk}\partial_k+(1/2)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_l\alpha)\tilde{\gamma}^{lk}\mathcal{A}\partial_k+(1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m\tilde{\gamma}^{lk}\partial_m\partial_l\partial_k\\
&\quad -(5/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^m\tilde{\gamma}^{lk}(\partial_m\mathcal{S})\partial_l\partial_k+e^{-4\varphi}\tilde{\gamma}^{-1}\beta^m(\partial_m\tilde{\gamma}^{lk})\partial_l\partial_k+(1/2)e^{-4\varphi}\tilde{\gamma}^{-1}\beta^i(\partial_j\partial_i\tilde{\gamma}^{jk})\partial_k\\
&\quad +(3/4)e^{-4\varphi}\tilde{\gamma}^{-3}\beta^i\tilde{\gamma}^{jk}(\partial_i\mathcal{S})(\partial_j\mathcal{S})\partial_k-(3/4)e^{-4\varphi}\tilde{\gamma}^{-2}\beta^i(\partial_i\tilde{\gamma}^{jk})(\partial_j\mathcal{S})\partial_k+(1/3)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{pj}(\partial_j\beta^k)\partial_p\partial_k\\
&\quad -(5/12)e^{-4\varphi}\tilde{\gamma}^{-2}\tilde{\gamma}^{jk}(\partial_k\beta^i)(\partial_i\mathcal{S})\partial_j+(1/3)e^{-4\varphi}\tilde{\gamma}^{-1}(\partial_k\tilde{\gamma}^{ij})(\partial_j\beta^k)\partial_i-(1/6)e^{-4\varphi}\tilde{\gamma}^{-1}\tilde{\gamma}^{mk}(\partial_k\partial_l\beta^l)\partial_m\\
A_{15} &= (4/9)\alpha K\mathcal{A}-(8/9)\alpha K^2+(4/3)\alpha e^{-4\varphi}(\partial_i\partial_j\varphi)\tilde{\gamma}^{ij}+(8/3)\alpha e^{-4\varphi}(\partial_k\varphi)(\partial_l\tilde{\gamma}^{lk})+\alpha e^{-4\varphi}(\partial_j\tilde{\gamma}^{jk})\partial_k\\
&\quad +8\alpha e^{-4\varphi}\tilde{\gamma}^{jk}(\partial_j\varphi)\partial_k+\alpha e^{-4\varphi}\tilde{\gamma}^{jk}\partial_j\partial_k+8e^{-4\varphi}(\partial_l\alpha)(\partial_k\varphi)\tilde{\gamma}^{lk}+e^{-4\varphi}(\partial_l\alpha)(\partial_k\tilde{\gamma}^{lk})+2e^{-4\varphi}(\partial_l\alpha)\tilde{\gamma}^{lk}\partial_k\\
&\quad +e^{-4\varphi}\tilde{\gamma}^{lk}(\partial_l\partial_k\alpha)\\
A_{23} &= \alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k\varphi)(\partial_j\tilde{\gamma}_{mi})-(1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}(\partial_j\tilde{\gamma}_{mi})\\
&\quad +(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_k\partial_j\tilde{\gamma}_{mi})+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{-2}(\partial_i\mathcal{S})(\partial_j\mathcal{S})-(1/4)\alpha e^{-4\varphi}(\partial_i\tilde{\gamma}_{kl})(\partial_j\tilde{\gamma}^{kl})+\alpha e^{-4\varphi}\tilde{\gamma}^{km}(\partial_k\varphi)\tilde{\gamma}_{ji}\partial_m\\
&\quad +\alpha e^{-4\varphi}(\partial_j\varphi)\partial_i-(1/2)\alpha e^{-4\varphi}\tilde{\Gamma}^m_{kl}\tilde{\gamma}^{kl}\tilde{\gamma}_{ji}\partial_m+\alpha e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\Gamma}_{ijk}\partial_m+(1/2)\alpha e^{-4\varphi}\tilde{\gamma}^{lk}\tilde{\gamma}_{ji}\partial_k\partial_l\\
&\quad +(1/2)e^{-4\varphi}\tilde{\gamma}^{mk}(\partial_j\tilde{\gamma}_{im})(\partial_k\alpha)+(1/2)e^{-4\varphi}(\partial_j\alpha)\partial_i+(1/2)e^{-4\varphi}\tilde{\gamma}^{mk}\tilde{\gamma}_{ji}(\partial_k\alpha)\partial_m\\
A_{25} &= -\tilde{A}^k{}_i(\partial_k\alpha)+(1/9)(\partial_i\alpha)K+(4/9)\alpha(\partial_iK)+(1/9)\alpha K\partial_i-\alpha\tilde{A}^k{}_i\partial_k\\
A_{34} &= -(1/2)\beta^k\tilde{\gamma}^{il}\tilde{\gamma}^{-2}(\partial_l\mathcal{S})\partial_k-(1/2)(\partial_l\beta^i)\tilde{\gamma}^{lk}\tilde{\gamma}^{-1}\partial_k+(1/3)(\partial_l\beta^l)\tilde{\gamma}^{ik}\tilde{\gamma}^{-1}\partial_k-(1/2)\beta^l\tilde{\gamma}^{in}(\partial_l\tilde{\gamma}_{mn})\tilde{\gamma}^{mk}\tilde{\gamma}^{-1}\partial_k\\
&\quad +(1/2)\beta^k\tilde{\gamma}^{il}\tilde{\gamma}^{-1}\partial_l\partial_k\\
A_{35} &= -(\partial_k\alpha)\tilde{\gamma}^{ik}+4\alpha\tilde{\gamma}^{ik}(\partial_k\varphi)-\alpha\tilde{\gamma}^{ik}\partial_k
\end{aligned}$$

BSSN Constraint propagation analysis in flat spacetime

- The set of the constraint propagation equations, $\partial_t(\mathcal{H}^{BSSN}, \mathcal{M}_i, \mathcal{G}^i, \mathcal{A}, \mathcal{S})^T$?
- For the flat background metric $g_{\mu\nu} = \eta_{\mu\nu}$, the first order perturbation equations of (6)-(10):

$$\begin{aligned}
\partial_t {}^{(1)}\varphi &= -(1/6){}^{(1)}K + (1/6)\kappa_\varphi {}^{(1)}\mathcal{A} && (11)\\
\partial_t {}^{(1)}\tilde{\gamma}_{ij} &= -2{}^{(1)}\tilde{A}_{ij} - (2/3)\kappa_{\tilde{\gamma}}\delta_{ij}{}^{(1)}\mathcal{A} && (12)\\
\partial_t {}^{(1)}K &= -(\partial_j\partial_j{}^{(1)}\alpha) + \kappa_{K1}\partial_j{}^{(1)}\mathcal{G}^j - \kappa_{K2}{}^{(1)}\mathcal{H}^{BSSN} && (13)\\
\partial_t {}^{(1)}\tilde{A}_{ij} &= {}^{(1)}(R_{ij}^{BSSN})^{TF} - {}^{(1)}(\tilde{D}_i\tilde{D}_j\alpha)^{TF} + \kappa_{A1}\delta_{k(i}(\partial_{j)}{}^{(1)}\mathcal{G}^k) - (1/3)\kappa_{A2}\delta_{ij}(\partial_k{}^{(1)}\mathcal{G}^k) && (14)\\
\partial_t {}^{(1)}\tilde{\Gamma}^i &= -(4/3)(\partial_i{}^{(1)}K) - (2/3)\kappa_{\tilde{\Gamma}1}(\partial_i{}^{(1)}\mathcal{A}) + 2\kappa_{\tilde{\Gamma}2}{}^{(1)}\mathcal{M}_i && (15)
\end{aligned}$$

We express the adjustements as

$$\kappa_{adj} := (\kappa_\varphi, \kappa_{\tilde{\gamma}}, \kappa_{K1}, \kappa_{K2}, \kappa_{A1}, \kappa_{A2}, \kappa_{\tilde{\Gamma}1}, \kappa_{\tilde{\Gamma}2}). \qquad (16)$$

- Constraint propagation equations at the first order in the flat spacetime:

$$\begin{aligned}
\partial_t {}^{(1)}\mathcal{H}^{BSSN} &= (\kappa_{\tilde{\gamma}} - (2/3)\kappa_{\tilde{\Gamma}1} - (4/3)\kappa_\varphi + 2)\,\partial_j\partial_j{}^{(1)}\mathcal{A} + 2(\kappa_{\tilde{\Gamma}2} - 1)(\partial_j{}^{(1)}\mathcal{M}_j), && (17)\\
\partial_t {}^{(1)}\mathcal{M}_i &= (-(2/3)\kappa_{K1} + (1/2)\kappa_{A1} - (1/3)\kappa_{A2} + (1/2))\,\partial_i\partial_j{}^{(1)}\mathcal{G}^j\\
&\quad +(1/2)\kappa_{A1}\partial_j\partial_j{}^{(1)}\mathcal{G}^i + ((2/3)\kappa_{K2} - (1/2))\,\partial_i{}^{(1)}\mathcal{H}^{BSSN}, && (18)\\
\partial_t {}^{(1)}\mathcal{G}^i &= 2\kappa_{\tilde{\Gamma}2}{}^{(1)}\mathcal{M}_i + (-(2/3)\kappa_{\tilde{\Gamma}1} - (1/3)\kappa_{\tilde{\gamma}})(\partial_i{}^{(1)}\mathcal{A}), && (19)\\
\partial_t {}^{(1)}\mathcal{S} &= -2\kappa_{\tilde{\gamma}}{}^{(1)}\mathcal{A}, && (20)\\
\partial_t {}^{(1)}\mathcal{A} &= (\kappa_{A1} - \kappa_{A2})(\partial_j{}^{(1)}\mathcal{G}^j). && (21)
\end{aligned}$$

Effect of adjustments

No.		Constraints (number of components) $\mathcal{H}$ (1)	$\mathcal{M}_i$ (3)	$\mathcal{G}^i$ (3)	$\mathcal{A}$ (1)	$\mathcal{S}$ (1)	Amplification Factors (AFs) in Minkowskii background
0.	standard ADM	use	use	-	-	-	$(0, 0, \Im, \Im)$
1.	BSSN no adjustment	use	use	use	use	use	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
2.	the BSSN	use+adj	use+adj	use+adj	use+adj	use+adj	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
3.	no $\mathcal{S}$ adjustment	use+adj	use+adj	use+adj	use+adj	use	no difference in flat background
4.	no $\mathcal{A}$ adjustment	use+adj	use+adj	use+adj	use	use+adj	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
5.	no $\mathcal{G}^i$ adjustment	use+adj	use+adj	use	use+adj	use+adj	$(0, 0, 0, 0, 0, 0, 0, \Im, \Im)$
6.	no $\mathcal{M}_i$ adjustment	use+adj	use	use+adj	use+adj	use+adj	$(0, 0, 0, 0, 0, 0, 0, \Re, \Re)$ Growing modes!
7.	no $\mathcal{H}$ adjustment	use	use+adj	use+adj	use+adj	use+adj	$(0, 0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
8.	ignore $\mathcal{G}^i$, $\mathcal{A}$, $\mathcal{S}$	use+adj	use+adj	-	-	-	$(0, 0, 0, 0)$
9.	ignore $\mathcal{G}^i$, $\mathcal{A}$	use+adj	use+adj	use+adj	-	-	$(0, \Im, \Im, \Im, \Im, \Im, \Im)$
10.	ignore $\mathcal{G}^i$	use+adj	use+adj	-	use+adj	use+adj	$(0, 0, 0, 0, 0, 0)$
11.	ignore $\mathcal{A}$	use+adj	use+adj	use+adj	-	use+adj	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$
12.	ignore $\mathcal{S}$	use+adj	use+adj	use+adj	use+adj	-	$(0, 0, \Im, \Im, \Im, \Im, \Im, \Im)$

New Proposals :: Improved (adjusted) BSSN systems

TRS breaking adjustments

In order to break time reversal symmetry (TRS) of the evolution eqs, to adjust $\partial_t\phi$,$\partial_t\tilde{\gamma}_{ij}$,$\partial_t\tilde{\Gamma}^i$ using $\mathcal{S}, \mathcal{G}^i$, or to adjust $\partial_t K, \partial_t \tilde{A}_{ij}$ using $\tilde{\mathcal{A}}$.

$$\begin{aligned}
\partial_t\phi &= \partial_t^{BS}\phi + \kappa_{\phi\mathcal{H}}\alpha\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\alpha\tilde{D}_k\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}\alpha\mathcal{S} + \kappa_{\phi\mathcal{S}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{S} \\
\partial_t\tilde{\gamma}_{ij} &= \partial_t^{BS}\tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}\mathcal{H}}\alpha\tilde{\gamma}_{ij}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{G}2}\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k + \kappa_{\tilde{\gamma}\mathcal{S}1}\alpha\tilde{\gamma}_{ij}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\alpha\tilde{D}_i\tilde{D}_j\mathcal{S} \\
\partial_t K &= \partial_t^{BS}K + \kappa_{K\mathcal{M}}\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k) + \kappa_{K\tilde{\mathcal{A}}1}\alpha\tilde{\mathcal{A}} + \kappa_{K\tilde{\mathcal{A}}2}\alpha\tilde{D}^j\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{A}_{ij} &= \partial_t^{BS}\tilde{A}_{ij} + \kappa_{A\mathcal{M}1}\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k) + \kappa_{A\mathcal{M}2}\alpha(\tilde{D}_{(i}\mathcal{M}_{j)}) + \kappa_{A\tilde{\mathcal{A}}1}\alpha\tilde{\gamma}_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{\mathcal{A}}2}\alpha\tilde{D}_i\tilde{D}_j\tilde{\mathcal{A}} \\
\partial_t\tilde{\Gamma}^i &= \partial_t^{BS}\tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}\mathcal{H}}\alpha\tilde{D}^i\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}\alpha\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\alpha\tilde{D}^i\mathcal{H}^{BS}
\end{aligned}$$

or in the flat background

$$\begin{aligned}
\partial_t^{ADJ(1)}\phi &= +\kappa_{\phi\mathcal{H}}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\phi\mathcal{G}}\partial_k{}^{(1)}\mathcal{G}^k + \kappa_{\phi\mathcal{S}1}{}^{(1)}\mathcal{S} + \kappa_{\phi\mathcal{S}2}\partial_j\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}\tilde{\gamma}_{ij} &= +\kappa_{\tilde{\gamma}\mathcal{H}}\delta_{ij}{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\gamma}\mathcal{G}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{G}^k + (1/2)\kappa_{\tilde{\gamma}\mathcal{G}2}(\partial_j{}^{(1)}\mathcal{G}^i + \partial_i{}^{(1)}\mathcal{G}^j) + \kappa_{\tilde{\gamma}\mathcal{S}1}\delta_{ij}{}^{(1)}\mathcal{S} + \kappa_{\tilde{\gamma}\mathcal{S}2}\partial_i\partial_j{}^{(1)}\mathcal{S} \\
\partial_t^{ADJ(1)}K &= +\kappa_{K\mathcal{M}}\partial_j{}^{(1)}\mathcal{M}_j + \kappa_{K\tilde{\mathcal{A}}1}{}^{(1)}\tilde{\mathcal{A}} + \kappa_{K\tilde{\mathcal{A}}2}\partial_j\partial_j{}^{(1)}\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{A}_{ij} &= +\kappa_{A\mathcal{M}1}\delta_{ij}\partial_k{}^{(1)}\mathcal{M}_k + (1/2)\kappa_{A\mathcal{M}2}(\partial_i\mathcal{M}_j + \partial_j\mathcal{M}_i) + \kappa_{A\tilde{\mathcal{A}}1}\delta_{ij}\tilde{\mathcal{A}} + \kappa_{A\tilde{\mathcal{A}}2}\partial_i\partial_j\tilde{\mathcal{A}} \\
\partial_t^{ADJ(1)}\tilde{\Gamma}^i &= +\kappa_{\tilde{\Gamma}\mathcal{H}}\partial_i{}^{(1)}\mathcal{H}^{BS} + \kappa_{\tilde{\Gamma}\mathcal{G}1}{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}2}\partial_j\partial_j{}^{(1)}\mathcal{G}^i + \kappa_{\tilde{\Gamma}\mathcal{G}3}\partial_i\partial_j{}^{(1)}\mathcal{G}^j + \kappa_{\tilde{\Gamma}\mathcal{S}}\partial_i{}^{(1)}\mathcal{S}
\end{aligned}$$

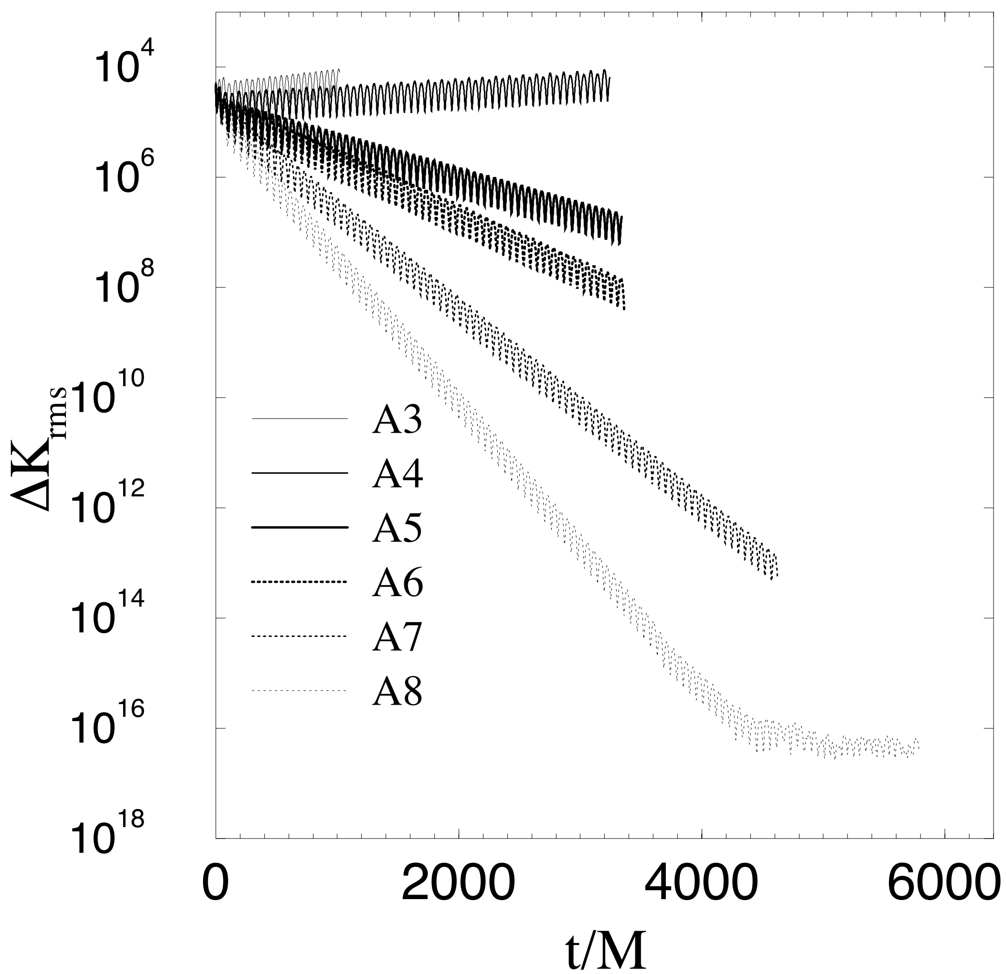
Constraint Amplification Factors with each adjustment

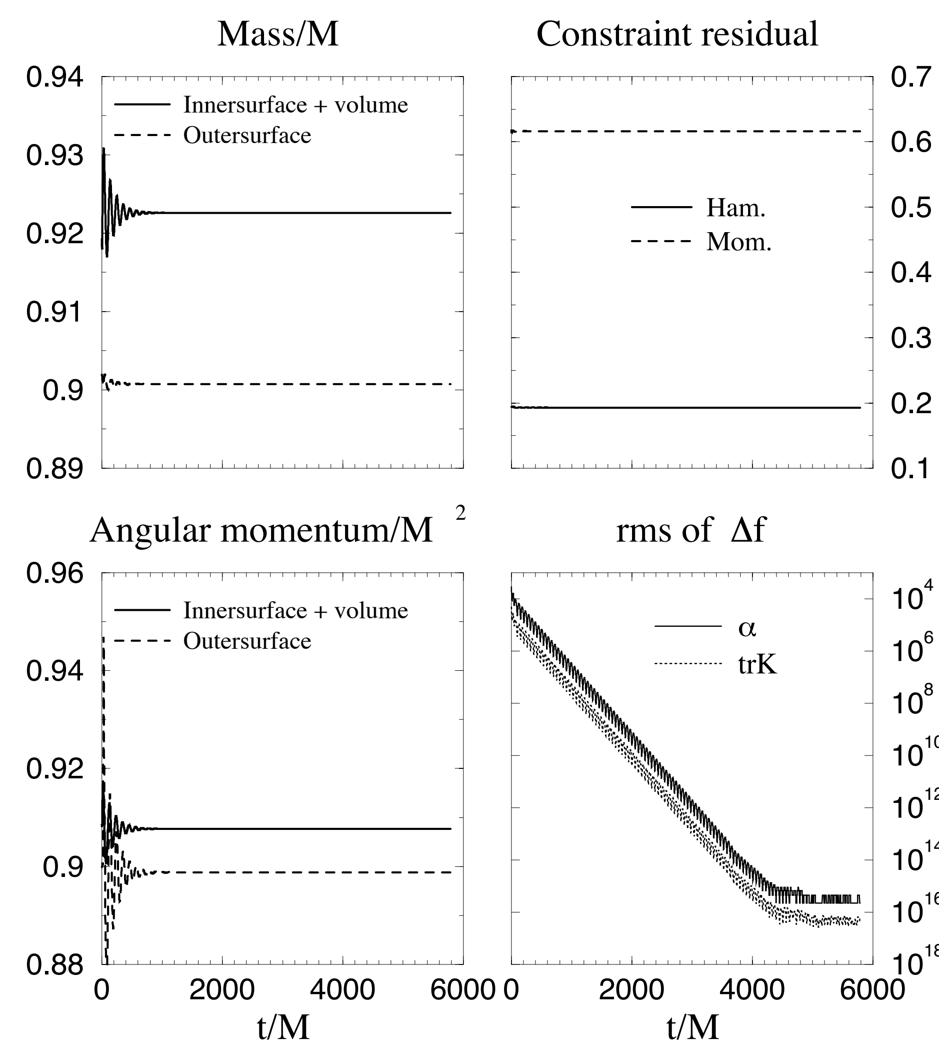
adjustment		CAFs	diag?	effect of the adjustment	
$\partial_t\phi$	$\kappa_{\phi\mathcal{H}}\,\alpha\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3), 8\kappa_{\phi\mathcal{H}}k^2)$	no	$\kappa_{\phi\mathcal{H}}<0$ makes 1 Neg.	
$\partial_t\phi$	$\kappa_{\phi\mathcal{G}}\,\alpha\tilde{D}_k\mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\phi\mathcal{G}}<0$ makes 2 Neg. 1 Pos.	
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{SD}\,\alpha\tilde{\gamma}_{ij}\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3),(3/2)\kappa_{SD}k^2)$	yes	$\kappa_{SD}<0$ makes 1 Neg.	Case (B)
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}1}\,\alpha\tilde{\gamma}_{ij}\tilde{D}_k\mathcal{G}^k$	$(0,0,\pm\sqrt{-k^2}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}1}>0$ makes 1 Neg.	
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{G}2}\,\alpha\tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k$	(0,0, $(1/4)k^2\kappa_{\tilde{\gamma}\mathcal{G}2}\pm\sqrt{k^2(-1+k^2\kappa_{\tilde{\gamma}\mathcal{G}2}/16)}(*2)$, long expressions)	yes	$\kappa_{\tilde{\gamma}\mathcal{G}2}<0$ makes 6 Neg. 1 Pos.	Case (E1)
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3),3\kappa_{\tilde{\gamma}\mathcal{S}1})$	no	$\kappa_{\tilde{\gamma}\mathcal{S}1}<0$ makes 1 Neg.	
$\partial_t\tilde{\gamma}_{ij}$	$\kappa_{\tilde{\gamma}\mathcal{S}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{S}$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{\tilde{\gamma}\mathcal{S}2}k^2)$	no	$\kappa_{\tilde{\gamma}\mathcal{S}2}>0$ makes 1 Neg.	
$\partial_t K$	$\kappa_{K\mathcal{M}}\,\alpha\tilde{\gamma}^{jk}(\tilde{D}_j\mathcal{M}_k)$	$(0,0,0,\pm\sqrt{-k^2}(*2),$ $(1/3)\kappa_{K\mathcal{M}}k^2\pm(1/3)\sqrt{k^2(-9+k^2\kappa_{K\mathcal{M}}^2)})$	no	$\kappa_{K\mathcal{M}}<0$ makes 2 Neg.	
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{M}1}\,\alpha\tilde{\gamma}_{ij}(\tilde{D}^k\mathcal{M}_k)$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{A\mathcal{M}1}k^2)$	yes	$\kappa_{A\mathcal{M}1}>0$ makes 1 Neg.	
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{M}2}\,\alpha(\tilde{D}_{(i}\mathcal{M}_{j)})$	(0,0, $-k^2\kappa_{A\mathcal{M}2}/4\pm\sqrt{k^2(-1+k^2\kappa_{A\mathcal{M}2}/16)}(*2)$, long expressions)	yes	$\kappa_{A\mathcal{M}2}>0$ makes 7 Neg	Case (D)
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{A}1}\,\alpha\tilde{\gamma}_{ij}\mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3),3\kappa_{A\mathcal{A}1})$	yes	$\kappa_{A\mathcal{A}1}<0$ makes 1 Neg.	
$\partial_t\tilde{A}_{ij}$	$\kappa_{A\mathcal{A}2}\,\alpha\tilde{D}_i\tilde{D}_j\mathcal{A}$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{A\mathcal{A}2}k^2)$	yes	$\kappa_{A\mathcal{A}2}>0$ makes 1 Neg.	
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{H}}\,\alpha\tilde{D}^i\mathcal{H}$	$(0,0,\pm\sqrt{-k^2}(*3),-\kappa_{A\mathcal{A}2}k^2)$	no	$\kappa_{\tilde{\Gamma}\mathcal{H}}>0$ makes 1 Neg.	
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}1}\,\alpha\mathcal{G}^i$	$(0,0,(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}1}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}1}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}1}<0$ makes 6 Neg. 1 Pos.	Case (E2)
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}2}\,\alpha\tilde{D}^j\tilde{D}_j\mathcal{G}^i$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}2}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}2}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}2}>0$ makes 2 Neg. 1 Pos.	
$\partial_t\tilde{\Gamma}^i$	$\kappa_{\tilde{\Gamma}\mathcal{G}3}\,\alpha\tilde{D}^i\tilde{D}_j\mathcal{G}^j$	$(0,0,-(1/2)\kappa_{\tilde{\Gamma}\mathcal{G}3}\pm\sqrt{-k^2+\kappa_{\tilde{\Gamma}\mathcal{G}3}^2}(*2)$, long.)	yes	$\kappa_{\tilde{\Gamma}\mathcal{G}3}>0$ makes 2 Neg. 1 Pos.	

Yoneda-HS, PRD66 (2002) 124003

An Evolution of Adjusted BSSN Formulation

by Yo-Baumgarte-Shapiro, PRD 66 (2002) 084026

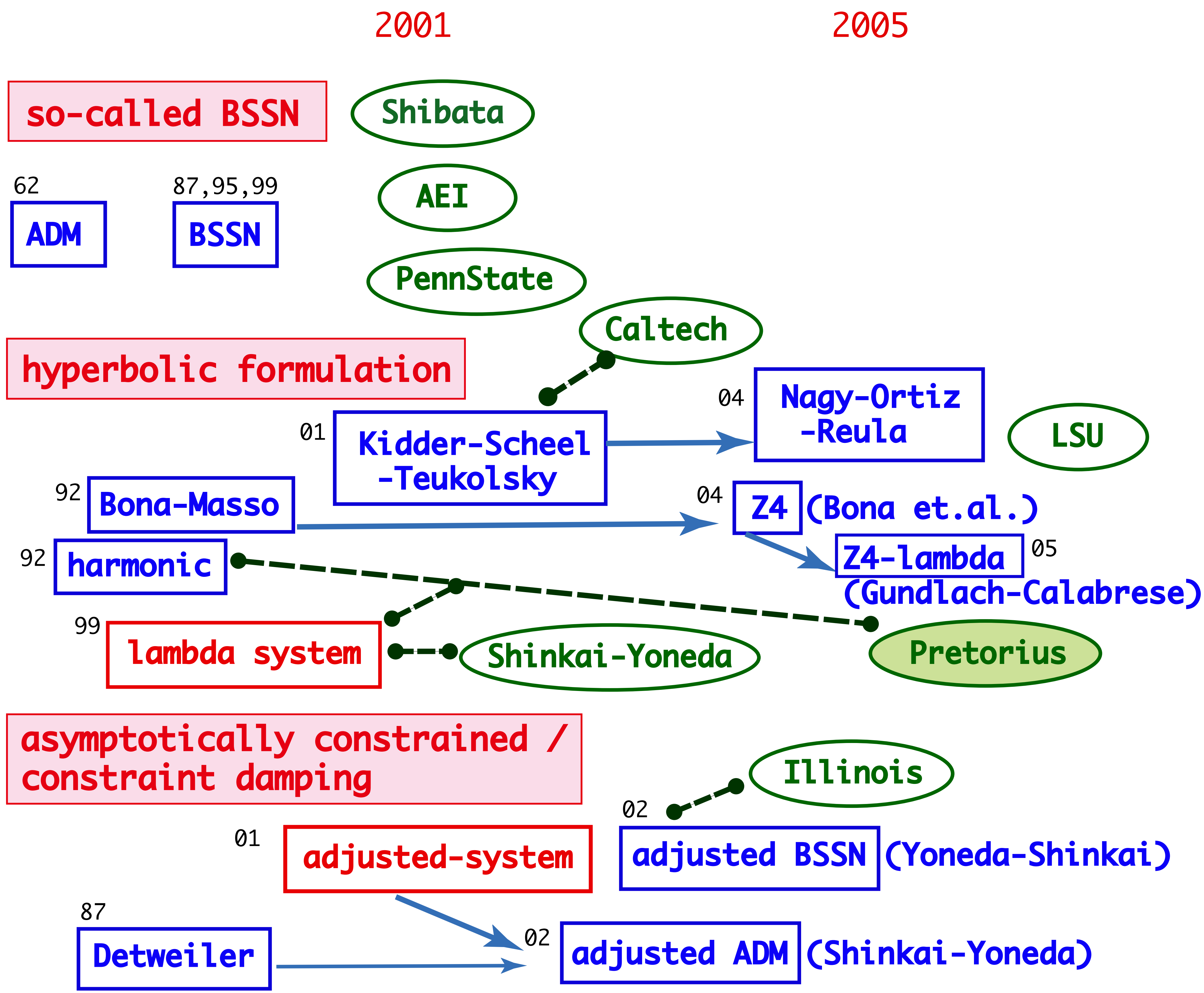


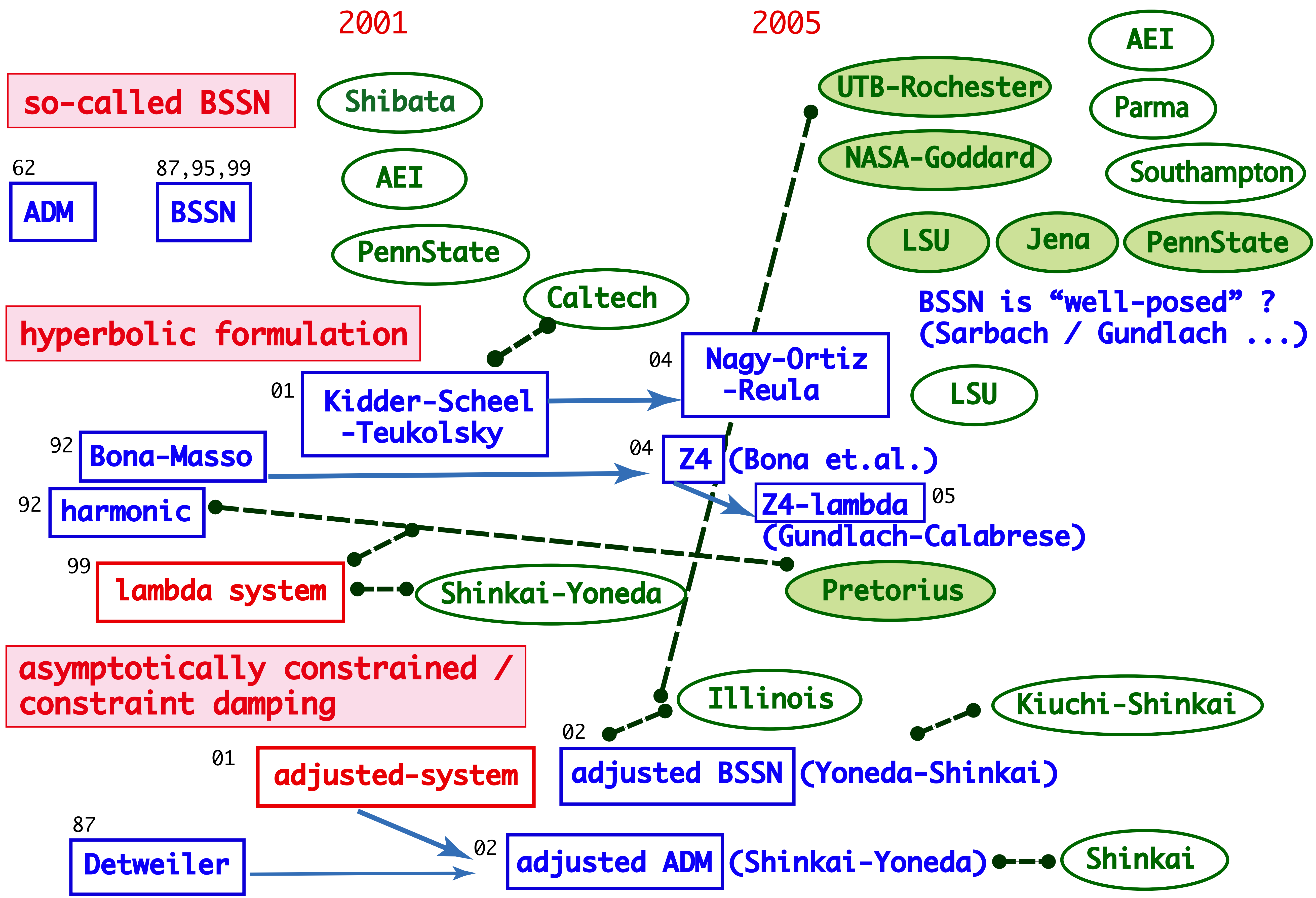


Kerr-Schild BH (0.9 J/M), excision with cube, $1+\log$-lapse, Γ-driver shift.

$$\partial_t \tilde{\Gamma}^i = (\cdots) + \frac{2}{3}\tilde{\Gamma}^i \beta^i{}_{,j} - (\chi + \frac{2}{3})\mathcal{G}^i \beta^j{}_{,j} \qquad \chi = 2/3 \quad \textbf{for (A4)-(A8)}$$

$$\partial_t \tilde{\gamma}_{ij} = (\cdots) - \kappa \alpha \tilde{\gamma}_{ij} \mathcal{H} \qquad \kappa = 0.1 \sim 0.2 \quad \textbf{for (A5), (A6) and (A8)}$$


2001
2005
so-called BSSN
Shibata
62
ADM
87,95,99
BSSN
AEI
PennState
Caltech
hyperbolic formulation
04
Nagy-Ortiz -Reula
01
Kidder-Scheel -Teukolsky
LSU
92
Bona-Masso
04
Z4 (Bona et.al.)
Z4-lambda
05
(Gundlach-Calabrese)
92
harmonic
99
lambda system
Shinkai-Yoneda
Pretorius
asymptotically constrained / constraint damping
Illinois
02
01
adjusted-system
adjusted BSSN (Yoneda-Shinkai)
87
Detweiler
02
adjusted ADM (Shinkai-Yoneda)


2001
2005
so-called BSSN
62
ADM
87,95,99
BSSN
Shibata
AEI
PennState
UTB-Rochester
NASA-Goddard
AEI
Parma
Southampton
LSU
Jena
PennState
BSSN is “well-posed” ?
(Sarbach / Gundlach ...)
hyperbolic formulation
Caltech
01
Kidder-Scheel
-Teukolsky
04
Nagy-Ortiz
-Reula
LSU
92
Bona-Masso
04
Z4
(Bona et.al.)
05
Z4-lambda
(Gundlach-Calabrese)
92
harmonic
99
lambda system
Shinkai-Yoneda
Pretorius
asymptotically constrained /
constraint damping
Illinois
Kiuchi-Shinkai
02
adjusted BSSN
(Yoneda-Shinkai)
01
adjusted-system
87
Detweiler
02
adjusted ADM
(Shinkai-Yoneda)
Shinkai

Some known fact (technical):

- Trace-out A_{ij} at every time step helps the stability.

  Alcubierre, et al, [PRD 62 (2000) 044034]

- "The essential improvement is in the process of replacing terms by the momentum constraints",

  Alcubierre, et al, [PRD 62 (2000) 124011]

- $\tilde{\Gamma}^i$ is replaced by $-\partial_j \tilde{\gamma}^{ij}$ where it is not differentiated,

  Campanelli, et al, [PRL96 (2006) 111101; PRD 73 (2006) 061501R]

- $\tilde{\Gamma}^i$-equation has been modified as suggested in Yo-Baumgarte-Shapiro [PRD 66 (2002) 084026]

  Baker et al, [PRL96 (2006) 111102; PRD73 (2006) 104002]

Some known fact (technical):

- Trace-out A_{ij} at every time step helps the stability.

  Alcubierre, et al, [PRD 62 (2000) 044034]

  This is because $\mathcal{A}$-violation affects to all other constraint violations.

- "The essential improvement is in the process of replacing terms by the momentum constraints",

  Alcubierre, et al, [PRD 62 (2000) 124011]

  This is because $\mathcal{M}$-replacement in Γ^i equation kills the positive real eigenvalues of CAFs. eigenvalues

- $\tilde{\Gamma}^i$ is replaced by $-\partial_j \tilde{\gamma}^{ij}$ where it is not differentiated,

  Campanelli, et al, [PRL96 (2006) 111101; PRD 73 (2006) 061501R]

  This is because $\mathcal{G}$-violation affects to $\mathcal{H}, \mathcal{M}_i$-violation constraint violations.

- $\tilde{\Gamma}^i$-equation has been modified as suggested in Yo-Baumgarte-Shapiro [PRD 66 (2002) 084026]

  Baker et al, [PRL96 (2006) 111102; PRD73 (2006) 104002]

  No doubt about this.

Numerical Experiments of Adjusted BSSN Systems

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- BSSN vs adjusted BSSN Numerical tests
- gauge-wave, linear wave, and Gowdy-wave tests, proposed by the Mexico workshop 2002
- 3 adjusted BSSN systems.
- Work as Expected
  - When the original BSSN system already shows satisfactory good evolutions (e.g., linear wave test), the adjusted versions also coincide with those evolutions.
  - For some cases (e.g., gauge-wave or Gowdy-wave tests) the simulations using the adjusted systems last 10 times longer than the standard BSSN.

arXiv:0711.3575, to be published in Phys. Rev. D. (2008)

Adjusted BSSN systems; we tested

from the proposals in Yoneda & HS, Phys. Rev. D66 (2002) 124003

1. $\tilde{A}$-equation with the momentum constraint:

$$\partial_t \tilde{A}_{ij} = \partial_t^B \tilde{A}_{ij} + \kappa_A \alpha \tilde{D}_{(i} \mathcal{M}_{j)}, \tag{1}$$

with $\kappa_{\mathcal{A}} > 0$ (predicted from the eigenvalue analysis).

2. $\tilde{\gamma}$-equation with $\mathcal{G}$ constraint:

$$\partial_t \tilde{\gamma}_{ij} = \partial_t^B \tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}} \alpha \tilde{\gamma}_{k(i} \tilde{D}_{j)} \mathcal{G}^k, \tag{2}$$

with $\kappa_{\tilde{\gamma}} < 0$.

3. $\tilde{\Gamma}$-equation with $\mathcal{G}$ constraint:

$$\partial_t \tilde{\Gamma}^i = \partial_t^B \tilde{\Gamma}^i + \kappa_{\tilde{\Gamma}} \alpha \mathcal{G}^i. \tag{3}$$

with $\kappa_{\tilde{\Gamma}} < 0$.

Numerical Testbed Models | A: Gauge-wave testbed

from the proposals in Mexico Workshop 2002, Class. Quant. Gravity 21 (2004) 589

The trivial Minkowski space-time, but time-dependent tilded slice.

$$ds^2 = -H dt^2 + H dx^2 + dy^2 + dz^2,$$
$$H = H(x-t) = 1 - A \sin\left(\frac{2\pi(x-t)}{d}\right),$$

Parameters:

- Gauge-wave parameters: $d = 1$ and $A = 10^{-2}$
- Simulation domain: $x \in [-0.5, 0.5]$, $y = z = 0$
- Grid: $x^i = -0.5 + (n - \frac{1}{2})dx$ with $n = 1, \cdots 50\rho$, where $dx = 1/(50\rho)$ with $\rho = 2, 4, 8$
- Time step: $dt = 0.25 dx$
- Periodic boundary condition in x direction
- Gauge conditions: $\partial_t \alpha = -\alpha^2 K, \quad \beta^i = 0.$

The 1D simulation is carried out for a $T = 1000$ crossing-time or until the code crashes, where one crossing-time is defined by the length of the simulation domain.

Error evaluation methods

It should be emphasized that the adjustment effect has two meanings, improvement of stability and of accuracy. Even if a simulation is stable, it does not imply that the result is accurate.

- We judge the **stability of the evolution** by monitoring the L2 norm of each constraint,

$$||\delta \mathcal{C}||_2(t) \equiv \sqrt{\frac{1}{N} \sum_{x,y,z} \left(\mathcal{C}(t;x,y,z)\right)^2},$$

  where N is the total number of grid points,

- We judge the **accuracy** by the difference of the metric components $g_{ij}(t;x,y,z)$ from the exact solution $g_{ij}^{(\mathrm{exact})}(t;x,y,z)$,

$$||\delta g_{ij}||_2(t) \equiv \sqrt{\frac{1}{N} \sum_{x,y,z} \left(g_{ij} - g_{ij}^{(\mathrm{exact})}\right)^2}.$$

A.1 The plain BSSN system

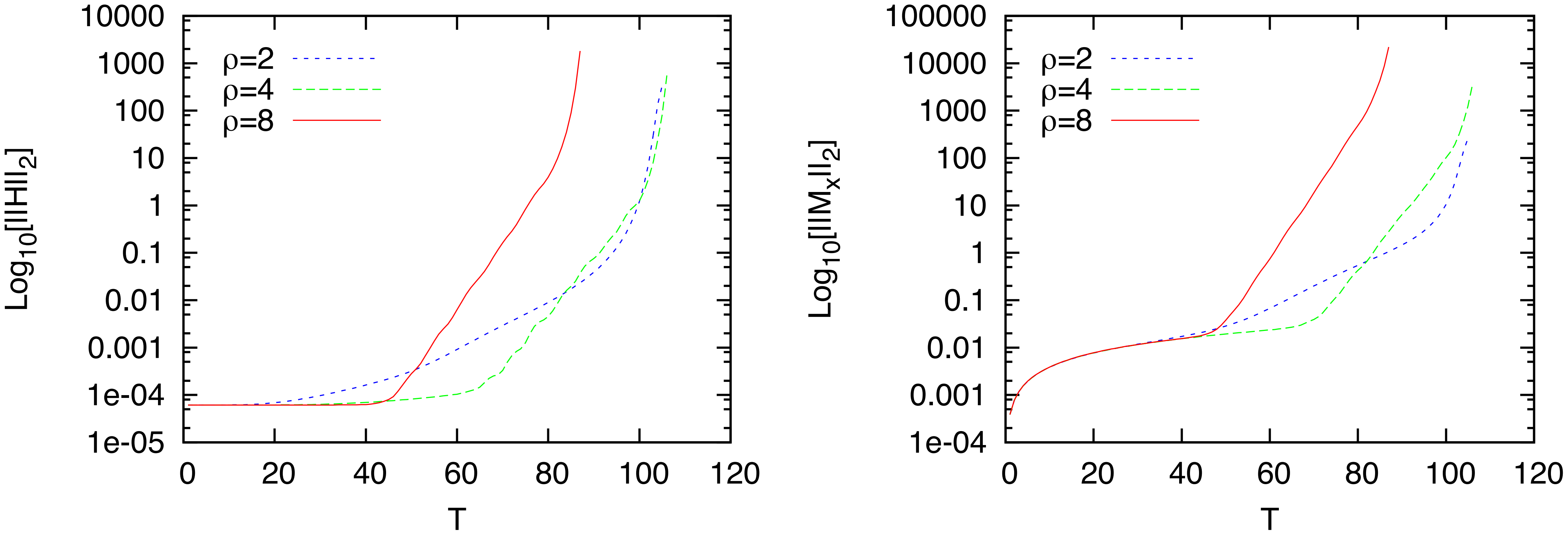


FIG. 1: The one-dimensional gauge-wave test with the plain BSSN system. The L2 norm of $\mathcal{H}$ and $\mathcal{M}_x$, rescaled by $\rho^2/4$, are plotted with a function of the crossing-time. The amplitude of the wave is $A = 0.01$. The loss of convergence at the early time, near the 20 crossing-time, can be seen, and it will produce the blow-ups of the calculation in the end.

- The poor performance of the plain BSSN system has been reported.
  Jansen, Bruegmann, & Tichy, PRD 74 (2006) 084022.
- The 4th-order finite differencing scheme improves the results.
  Zlochower, Baker, Campanelli, & Lousto, PRD 72 (2005) 024021.

A.2 Adjusted BSSN with $\tilde{A}$-equation

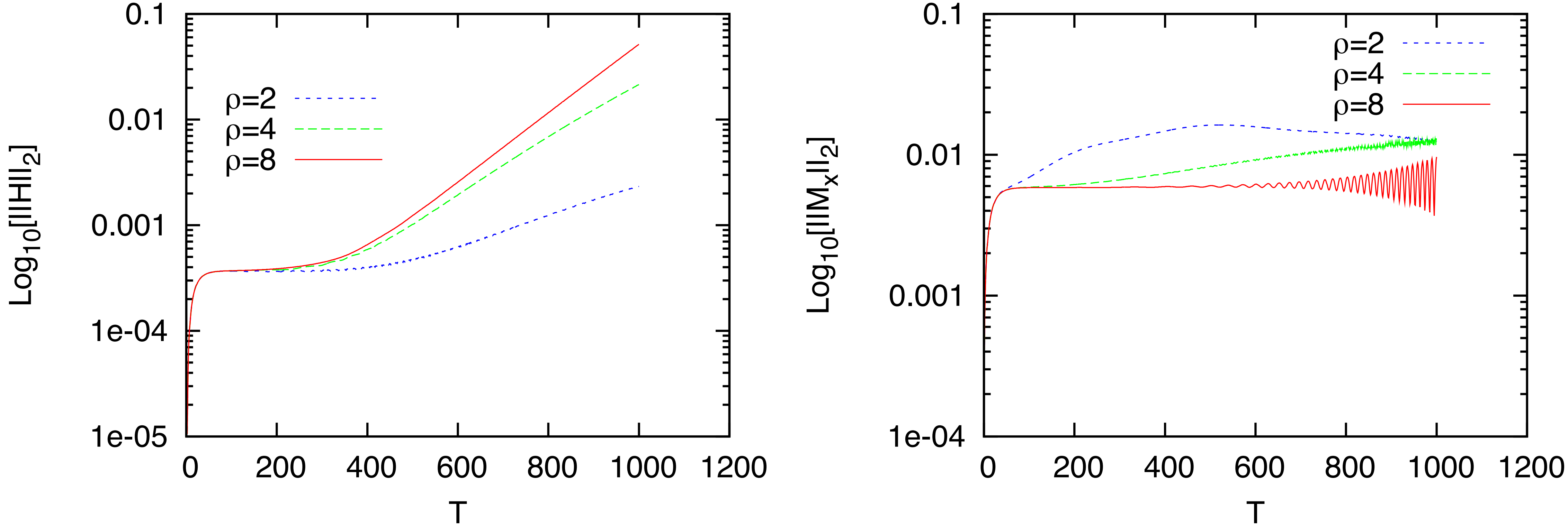


FIG. 2: The one-dimensional gauge-wave test with the adjusted BSSN system in the $\tilde{A}$-equation (1). The L2 norm of $\mathcal{H}$ and $\mathcal{M}_x$, rescaled by $\rho^2/4$, are plotted with a function of the crossing-time. The wave parameter is the same as with Fig. 1, and the adjustment parameter κ_A is set to $\kappa_A = 0.005$. We see the higher resolution runs show convergence longer, i.e., the 300 crossing-time in $\mathcal{H}$ and the 200 crossing-time in $\mathcal{M}_x$ with $\rho = 4$ and 8 runs. All runs can stably evolve up to the 1000 crossing-time.

- We found that the simulation continues 10 times longer.
- Convergence behaviors are apparently improved than those of the plain BSSN.
- However, growth of the error in later time at higher resolution.

$$\partial_t \tilde{A}_{ij} = -e^{-4\phi}\left[D_i D_j \alpha + \alpha R_{ij}\right]^{\mathsf{TF}} + \alpha K \tilde{A}_{ij} - 2\alpha \tilde{A}_{ik}\tilde{A}^k{}_j + \partial_i \beta^k \tilde{A}_{kj} + \partial_j \beta^k \tilde{A}_{ki} - \frac{2}{3}\partial_k \beta^k \tilde{A}_{ij} + \beta^k \partial_k \tilde{A}_{ij} + \kappa_A \alpha \tilde{D}_{(i}\mathcal{M}_{j)}$$

A.4 Evaluation of Accuracy

- L2 norm of the error in γ_{xx}, (4), with the function of time.
- The error is induced by distortion of the wave; the both phase and amplitude errors.

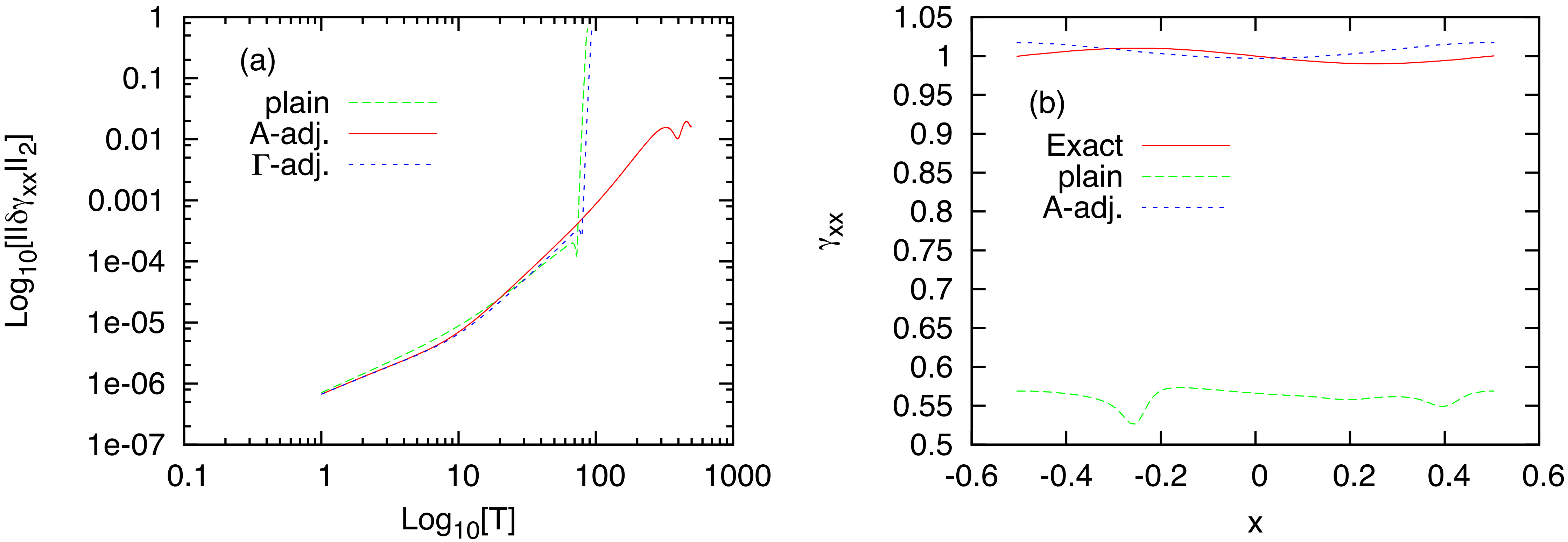


FIG. 4: Evaluation of the accuracy of the one-dimensional gauge-wave testbed. Lines show the plain BSSN, the adjusted BSSN with $\mathcal{A}$-equation, and with $\tilde{\Gamma}$-equation. (a) The L2 norm of the error in γ_{xx}, using (4). (b) A snapshot of the exact and numerical solution at $T = 100$.

Numerical Testbed Models | B: Linear wave testbed

from the proposals in Mexico Workshop 2002, Class. Quant. Gravity 21 (2004) 589

Check the ability of handling a travelling gravitational wave.

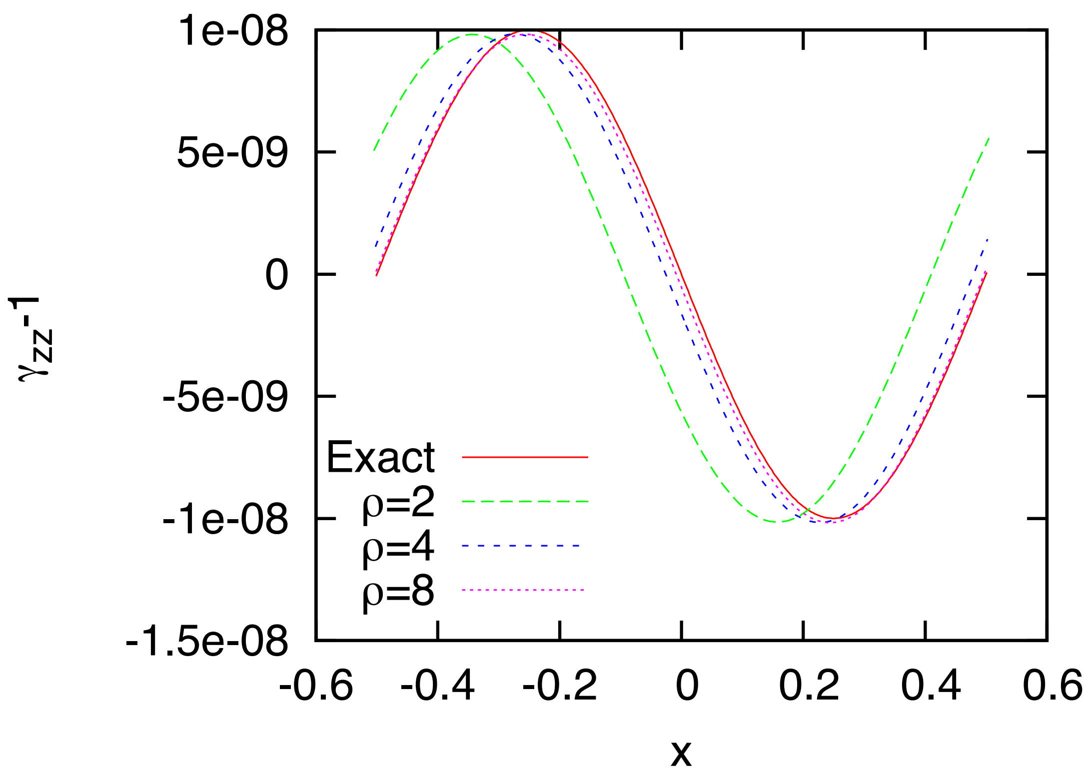
$$ds^2 = -dt^2 + dx^2 + (1+b)dy^2 + (1-b)dz^2,$$
$$b = A \sin\left(\frac{2\pi(x-t)}{d}\right)$$

Parameters:

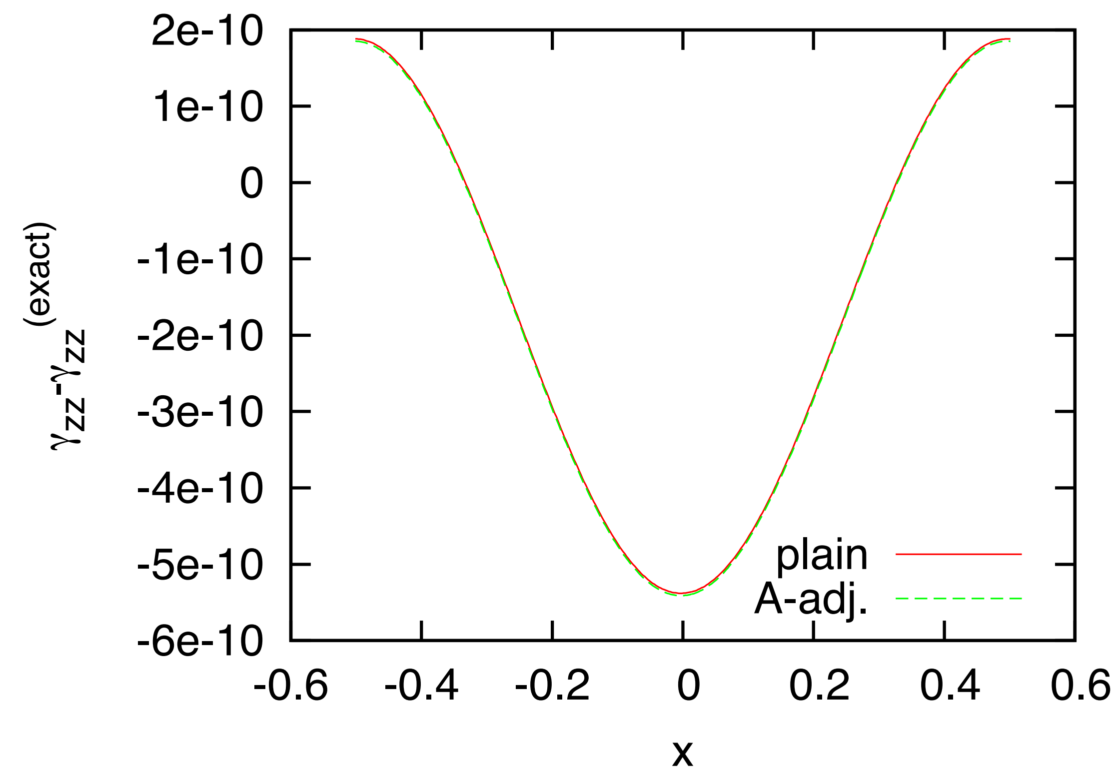
- Linear wave parameters: $d = 1$ and $A = 10^{-8}$
- Simulation domain: $x \in [-0.5, 0.5]$, $y = 0$, $z = 0$
- Grid: $x^i = -0.5 + (n - \frac{1}{2})dx$ with $n = 1, \cdots 50\rho$, where $dx = 1/(50\rho)$ with $\rho = 2, 4, 8$
- Time step: $dt = 0.25dx$
- Periodic boundary condition in x direction
- Gauge conditions: $\alpha = 1$ and $\beta^i = 0$

The 1D simulation is carried out for a $T = 1000$ crossing-time or until the code crashes.

Numerical Results B: Linear Wave Test



Snapshots of the one-dimensional linear wave at different resolutions with the plain BSSN system at the simulation time 500 crossing-time. We see there exists phase error, but they are convergent away at higher resolution runs.



Snapshot of errors with the exact solution for the Linear Wave testbed with the plain BSSN system and the adjusted BSSN system with the $\tilde{A}$ equation at $T = 500$. The highest resolution $\rho = 8$ is used in both runs. The difference between the plain and the adjusted BSSN system with the $\tilde{A}$ equation is indistinguishable. Note that the maximum amplitude is set to be 10^{-8} in this problem.

- The linear wave testbed does not produce a significant constraint violation.
- The plain BSSN and adjusted BSSN results are indistinguishable.
  This is because the adjusted terms of the equations are small due to the small violations of constraints.

Numerical Testbed Models — C: Collapsing polarized Gowdy-wave testbed

from the proposals in Mexico Workshop 2002, Class. Quant. Gravity 21 (2004) 589

Check the formulation in a strong field context using the polarized Gowdy metric.

$$
\begin{aligned}
ds^2 &= t^{-1/2}e^{\lambda/2}(-dt^2 + dz^2) + t(e^P dx^2 + e^{-P}dy^2). \\
P &= J_0(2\pi t)\cos(2\pi z), \\
\lambda &= -2\pi t J_0(2\pi t)J_1(2\pi t)\cos^2(2\pi z) + 2\pi^2 t^2[J_0^2(2\pi t) + J_1^2(2\pi t)] \\
&\quad -\frac{1}{2}[(2\pi)^2[J_0^2(2\pi) + J_1^2(2\pi)] - 2\pi J_0(2\pi)J_1(2\pi)],
\end{aligned}
$$

where J_n is the Bessel function.

Parameters:

- Perform the evolution in the collapsing (i.e. backward in time) direction.
- Simulation domain: $z \in [-0.5, 0.5]$, $x = y = 0$
- Grid: $z = -0.5 + (n - \frac{1}{2})dz$ with $n = 1, \cdots 50\rho$, where $dz = 1/(50\rho)$ with $\rho = 2, 4, 8$
- Time step: $dt = 0.25dz$
- Periodic boundary condition in z-direction
- Gauge conditions: the harmonic slicing $\partial_t\alpha = -\alpha^2 K, \quad \beta^i = 0.$ and $\beta^i = 0$
- Set the initial lapse function is 1, using coordinate transformation.

The 1D simulation is carried out for a $T = 1000$ crossing-time or until the code crashes.

Numerical Results | C: Collapsing polarized Gowdy-wave testbed (1)

C.1 The plain BSSN

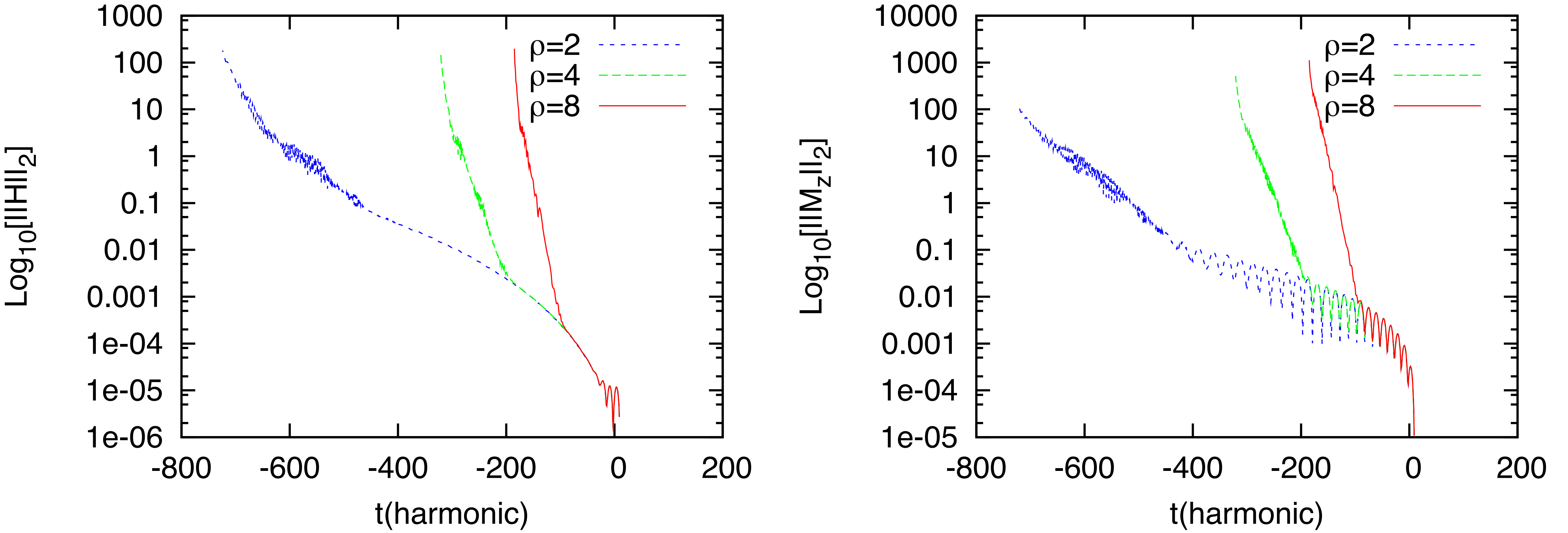


FIG. 5: Collapsing polarized Gowdy-wave test with the plain BSSN system. The L2 norm of $\mathcal{H}$ and $\mathcal{M}_z$, rescaled by $\rho^2/4$, are plotted with a function of the crossing-time. (Simulation proceeds backwards from $t = 0$.) We see almost perfect overlap for the initial 100 crossing-time, and the higher resolution runs crash earlier. This result is quite similar to those achieved with the Cactus BSSN code, reported by [**?**].

- Our result shows similar crashing time with that of *Cactus* BSSN code.
  Alcubierre et al. CQG **21**, 589 (2004)
- Higher order differencing scheme with Kreiss-Oliger dissipation term improves the results.
  Zlochower, Baker, Campanelli & Lousto, PRD **72**, 024021 (2005)

C.2 Adjusted BSSN with $\tilde{A}$-equation

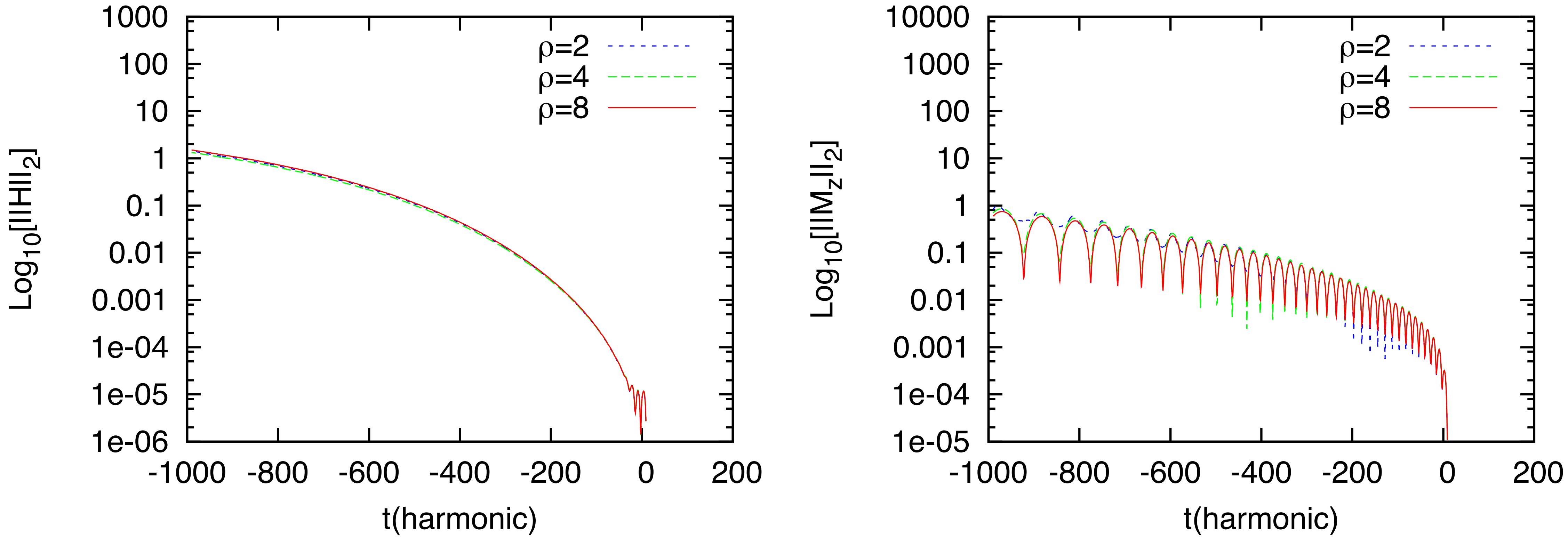
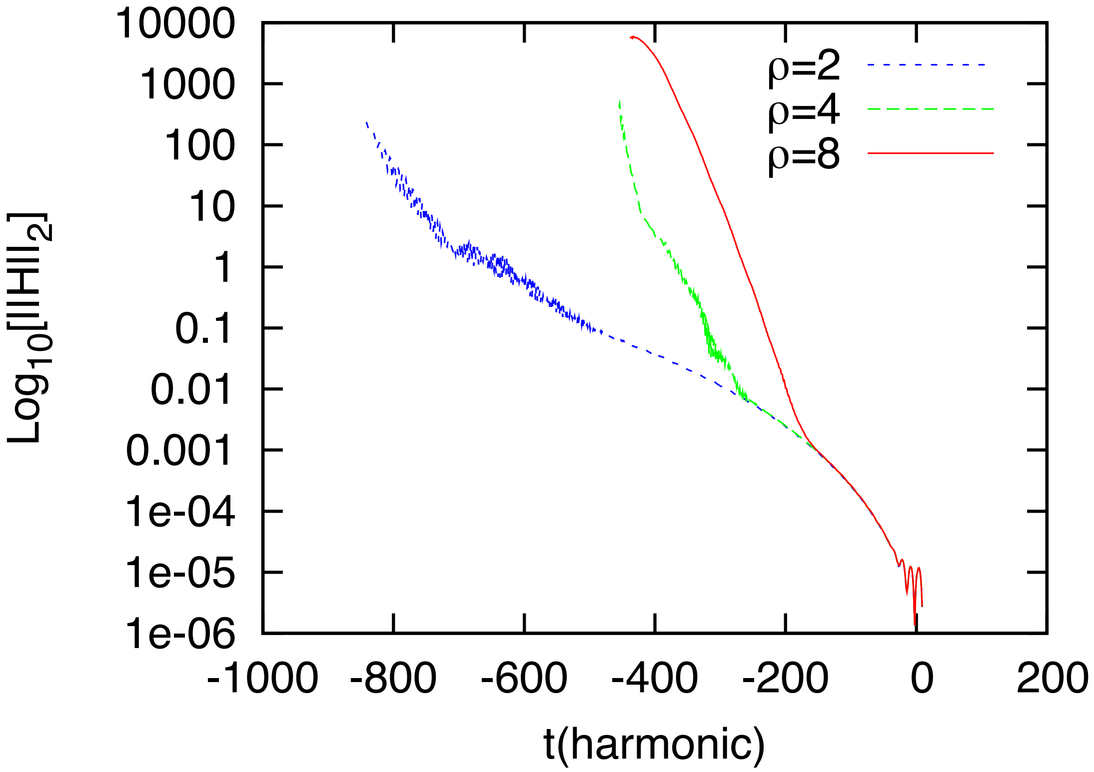


FIG. 6: Collapsing polarized Gowdy-wave test with the adjusted BSSN system in the $\tilde{A}$-equation (1), with $\kappa_{\mathcal{A}} = -0.001$. The style is the same as in Fig. 5 and note that both constraints are normalized by $\rho^2/4$. We see almost perfect overlap for the initial 1000 crossing-time in both constraint equations, $\mathcal{H}$ and $\mathcal{M}_z$, even for the highest resolution run.

- Adjustment extends the life-time of the simulation 10 times longer.
- Almost perfect convergence upto $t = 1000t_{cross}$ for both $\mathcal{H}$ and $\mathcal{M}_z$, while we find oscillations in $\mathcal{M}_z$ later time.

$$\partial_t \tilde{A}_{ij} = -e^{-4\phi}\left[D_i D_j \alpha + \alpha R_{ij}\right]^{\mathsf{TF}} + \alpha K \tilde{A}_{ij} - 2\alpha \tilde{A}_{ik}\tilde{A}^k{}_j + \partial_i \beta^k \tilde{A}_{kj} + \partial_j \beta^k \tilde{A}_{ki} - \frac{2}{3}\partial_k \beta^k \tilde{A}_{ij} + \beta^k \partial_k \tilde{A}_{ij} + \kappa_A \alpha \tilde{D}_{(i}\mathcal{M}_{j)}$$

C.3 Adjusted BSSN with $\tilde{\gamma}$-equation



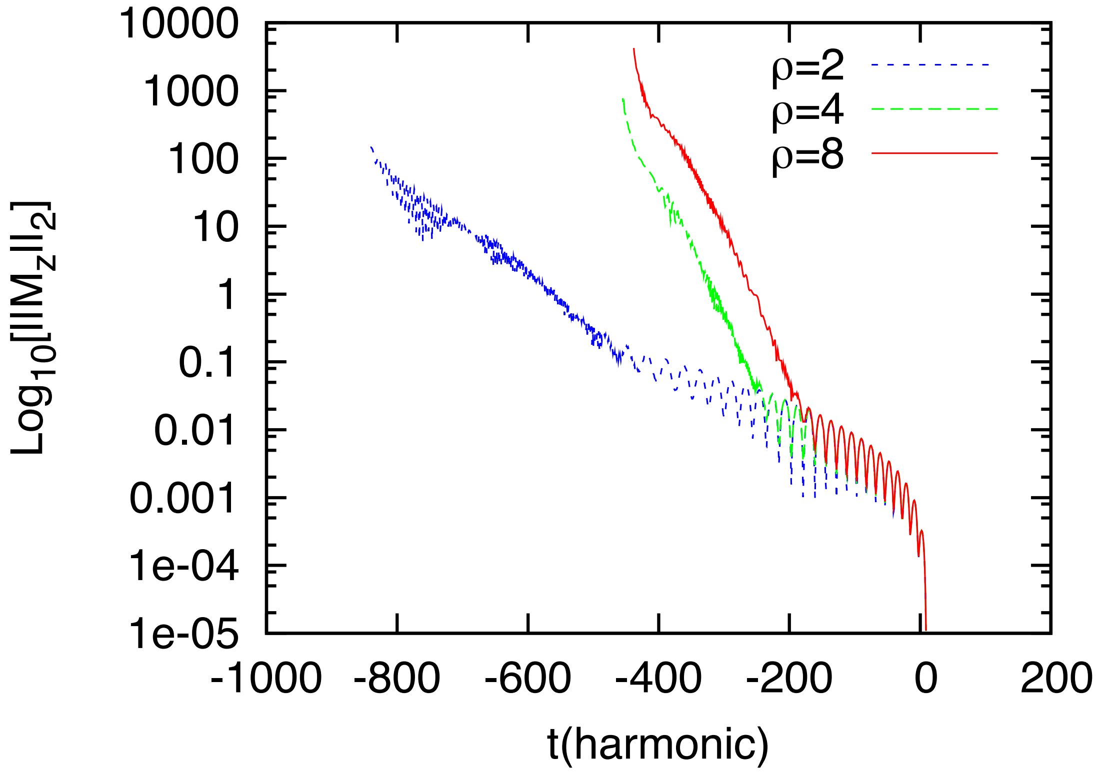


FIG. 7: Collapsing polarized Gowdy-wave test with the adjusted BSSN system in the $\tilde{\gamma}$-equation (2), with $\kappa_{\tilde{\gamma}} = 0.000025$. The figure style is the same as Figure 5. Note the almost perfect overlap for 200 crossing-time in the both the Hamiltonian and Momentum constraint and the $\rho = 2$ run can evolve stably for 1000 crossing-time.

- Almost perfect convergence up to $t = 200 t_{cross}$ in both $\mathcal{H}$ and $\mathcal{M}_z$.

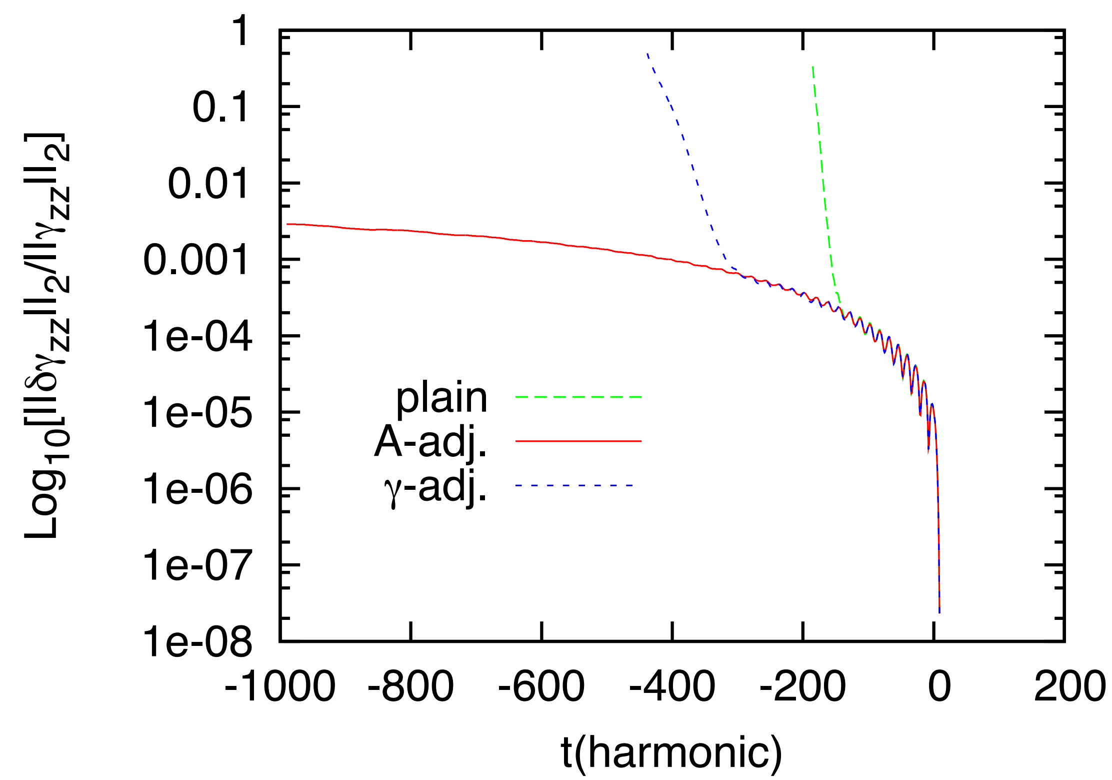
$$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta^k + \tilde{\gamma}_{jk}\partial_i \beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k + \beta^k \partial_k \tilde{\gamma}_{ij} + \kappa_{\tilde{\gamma}} \alpha \tilde{\gamma}_{k(i}\tilde{D}_{j)}\mathcal{G}^k$$

C.4 Adjustment works for Accuracy

Error of γ_{zz} to the exact solution normalized by γ_{zz}.

- *Accurate Evolution* $\Leftrightarrow$ Error < 1 %.
  (Zlochower, et al., PRD72 (2005) 024021)

the Plain BSSN $\approx t = 200 t_{cross}$
adjusted BSSN $\tilde{\mathcal{A}}$-eq $\approx t = 1000 t_{cross}$
adjusted BSSN $\tilde{\gamma}$-eq $\approx t = 400 t_{cross}$



Comparisons of systems in the collapsing polarized Gowdy-wave test. The L2 norm of the error in γ_{zz}, rescaled by the L2 norm of γ_{zz}, for the plain BSSN, adjusted BSSN with $\tilde{\mathcal{A}}$-equation, and with $\tilde{\gamma}$-equation are shown. The highest resolution run, $\rho = 8$, is depicted for the plots. We can conclude that the adjustments make longer accurate runs available. Note that the evolution is backwards in time.

A Full set of BSSN constraint propagation eqs.

$$\partial_t^{BS}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_i\\ \mathcal{G}^i\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12} & A_{13} & A_{14} & A_{15}\\ -(1/3)(\partial_i\alpha)+(1/6)\partial_i & \alpha K & A_{23} & 0 & A_{25}\\ 0 & \alpha\tilde\gamma^{ij} & 0 & A_{34} & A_{35}\\ 0 & 0 & 0 & \beta^k(\partial_k\mathcal{S}) & -2\alpha\tilde\gamma\\ 0 & 0 & 0 & 0 & \alpha K+\beta^k\partial_k\end{pmatrix}\begin{pmatrix}\mathcal{H}^{BS}\\ \mathcal{M}_j\\ \mathcal{G}^j\\ \mathcal{S}\\ \mathcal{A}\end{pmatrix}$$

$$\begin{aligned}
A_{11} &= +(2/3)\alpha K+(2/3)\alpha\mathcal{A}+\beta^k\partial_k\\
A_{12} &= -4e^{-4\varphi}\alpha(\partial_k\varphi)\tilde\gamma^{kj}-2e^{-4\varphi}(\partial_k\alpha)\tilde\gamma^{jk}\\
A_{13} &= -2\alpha e^{-4\varphi}\tilde A^k{}_j\partial_k-\alpha e^{-4\varphi}(\partial_j\tilde A_{kl})\tilde\gamma^{kl}-e^{-4\varphi}(\partial_j\alpha)\mathcal{A}-e^{-4\varphi}\beta^k\partial_k\partial_j-(1/2)e^{-4\varphi}\beta^k\tilde\gamma^{-1}(\partial_j\mathcal{S})\partial_k\\
&\quad +(1/6)e^{-4\varphi}\tilde\gamma^{-1}(\partial_j\beta^k)(\partial_k\mathcal{S})-(2/3)e^{-4\varphi}(\partial_k\beta^k)\partial_j\\
A_{14} &= 2\alpha e^{-4\varphi}\tilde\gamma^{-1}\tilde\gamma^{lk}(\partial_l\varphi)\mathcal{A}\partial_k+(1/2)\alpha e^{-4\varphi}\tilde\gamma^{-1}(\partial_l\mathcal{A})\tilde\gamma^{lk}\partial_k+(1/2)e^{-4\varphi}\tilde\gamma^{-1}(\partial_l\alpha)\tilde\gamma^{lk}\mathcal{A}\partial_k+(1/2)e^{-4\varphi}\tilde\gamma^{-1}\beta^m\tilde\gamma^{lk}\partial_m\partial_l\partial_k\\
&\quad -(5/4)e^{-4\varphi}\tilde\gamma^{-2}\beta^m\tilde\gamma^{lk}(\partial_m\mathcal{S})\partial_l\partial_k+e^{-4\varphi}\tilde\gamma^{-1}\beta^m(\partial_m\tilde\gamma^{lk})\partial_l\partial_k+(1/2)e^{-4\varphi}\tilde\gamma^{-1}\beta^i(\partial_j\partial_i\tilde\gamma^{jk})\partial_k\\
&\quad +(3/4)e^{-4\varphi}\tilde\gamma^{-3}\beta^i\tilde\gamma^{jk}(\partial_i\mathcal{S})(\partial_j\mathcal{S})\partial_k-(3/4)e^{-4\varphi}\tilde\gamma^{-2}\beta^i(\partial_i\tilde\gamma^{jk})(\partial_j\mathcal{S})\partial_k+(1/3)e^{-4\varphi}\tilde\gamma^{-1}\tilde\gamma^{pj}(\partial_j\beta^k)\partial_p\partial_k\\
&\quad -(5/12)e^{-4\varphi}\tilde\gamma^{-2}\tilde\gamma^{jk}(\partial_k\beta^i)(\partial_i\mathcal{S})\partial_j+(1/3)e^{-4\varphi}\tilde\gamma^{-1}(\partial_k\tilde\gamma^{ij})(\partial_j\beta^k)\partial_i-(1/6)e^{-4\varphi}\tilde\gamma^{-1}\tilde\gamma^{mk}(\partial_k\partial_l\beta^l)\partial_m\\
A_{15} &= (4/9)\alpha K\mathcal{A}-(8/9)\alpha K^2+(4/3)\alpha e^{-4\varphi}(\partial_i\partial_j\varphi)\tilde\gamma^{ij}+(8/3)\alpha e^{-4\varphi}(\partial_k\varphi)(\partial_l\tilde\gamma^{lk})+\alpha e^{-4\varphi}(\partial_j\tilde\gamma^{jk})\partial_k\\
&\quad +8\alpha e^{-4\varphi}\tilde\gamma^{jk}(\partial_j\varphi)\partial_k+\alpha e^{-4\varphi}\tilde\gamma^{jk}\partial_j\partial_k+8e^{-4\varphi}(\partial_l\alpha)(\partial_k\varphi)\tilde\gamma^{lk}+e^{-4\varphi}(\partial_l\alpha)(\partial_k\tilde\gamma^{lk})+2e^{-4\varphi}(\partial_l\alpha)\tilde\gamma^{lk}\partial_k\\
&\quad +e^{-4\varphi}\tilde\gamma^{lk}(\partial_l\partial_k\alpha)\\
A_{23} &= \alpha e^{-4\varphi}\tilde\gamma^{km}(\partial_k\varphi)(\partial_j\tilde\gamma_{mi})-(1/2)\alpha e^{-4\varphi}\tilde\Gamma^m_{kl}\tilde\gamma^{kl}(\partial_j\tilde\gamma_{mi})\\
&\quad +(1/2)\alpha e^{-4\varphi}\tilde\gamma^{mk}(\partial_k\partial_j\tilde\gamma_{mi})+(1/2)\alpha e^{-4\varphi}\tilde\gamma^{-2}(\partial_i\mathcal{S})(\partial_j\mathcal{S})-(1/4)\alpha e^{-4\varphi}(\partial_i\tilde\gamma_{kl})(\partial_j\tilde\gamma^{kl})+\alpha e^{-4\varphi}\tilde\gamma^{km}(\partial_k\varphi)\tilde\gamma_{ji}\partial_m\\
&\quad +\alpha e^{-4\varphi}(\partial_j\varphi)\partial_i-(1/2)\alpha e^{-4\varphi}\tilde\Gamma^m_{kl}\tilde\gamma^{kl}\tilde\gamma_{ji}\partial_m+\alpha e^{-4\varphi}\tilde\gamma^{mk}\tilde\Gamma_{ijk}\partial_m+(1/2)\alpha e^{-4\varphi}\tilde\gamma^{lk}\tilde\gamma_{ji}\partial_k\partial_l\\
&\quad +(1/2)e^{-4\varphi}\tilde\gamma^{mk}(\partial_j\tilde\gamma_{im})(\partial_k\alpha)+(1/2)e^{-4\varphi}(\partial_j\alpha)\partial_i+(1/2)e^{-4\varphi}\tilde\gamma^{mk}\tilde\gamma_{ji}(\partial_k\alpha)\partial_m\\
A_{25} &= -\tilde A^k{}_i(\partial_k\alpha)+(1/9)(\partial_i\alpha)K+(4/9)\alpha(\partial_i K)+(1/9)\alpha K\partial_i-\alpha\tilde A^k{}_i\partial_k\\
A_{34} &= -(1/2)\beta^k\tilde\gamma^{il}\tilde\gamma^{-2}(\partial_l\mathcal{S})\partial_k-(1/2)(\partial_l\beta^i)\tilde\gamma^{lk}\tilde\gamma^{-1}\partial_k+(1/3)(\partial_l\beta^l)\tilde\gamma^{ik}\tilde\gamma^{-1}\partial_k-(1/2)\beta^l\tilde\gamma^{in}(\partial_l\tilde\gamma_{mn})\tilde\gamma^{mk}\tilde\gamma^{-1}\partial_k\\
&\quad +(1/2)\beta^k\tilde\gamma^{il}\tilde\gamma^{-1}\partial_l\partial_k\\
A_{35} &= -(\partial_k\alpha)\tilde\gamma^{ik}+4\alpha\tilde\gamma^{ik}(\partial_k\varphi)-\alpha\tilde\gamma^{ik}\partial_k
\end{aligned}$$

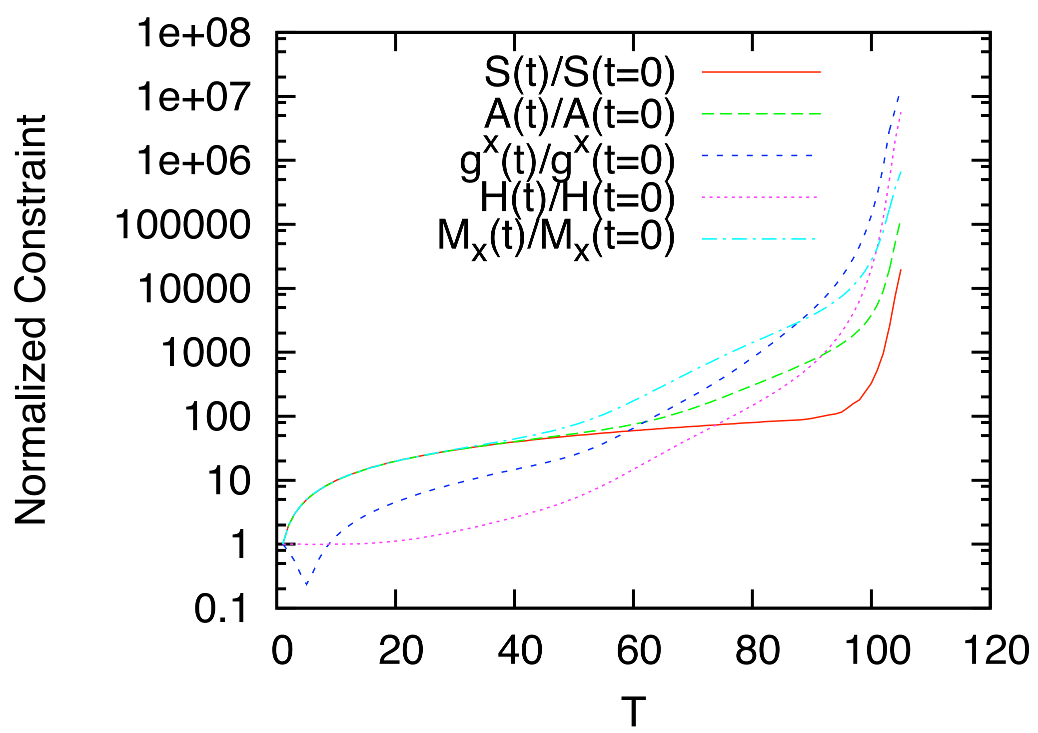
Which constraint should be monitored?

Yoneda & HS, PRD 66 (2002) 124003

Kiuchi & HS, arXiv:0711.3575, PRD (2008)

Order of constraint violation?

- $\mathcal{A}$ and $\mathcal{S}$ constraints propagate independently of the other constraints.
- $\mathcal{G}$-constraint is triggered by the violation of the momentum constraint.
- $\mathcal{H}$ and $\mathcal{M}$ constraints are affected by all the other constraints.



The violation of all constraints normalized with their initial values, $||\delta\mathcal{C}||_2(t)/||\delta\mathcal{C}||_2(0)$, are plotted with a function of time. The evolutions of the gauge-wave testbeds with the plain BSSN system are shown.

By observing which constraint triggers the other constraint's violation from the constraint propagation equations, we may guess the mechanism by which the entire system is violating accuracy and stability.

Summary up to here (2nd half)

[Keyword 1] Adjusted Systems
Adjusting the EoM with constraints is common to all previous approaches.
Just add constraints to evolution eqs, while lambda-system requires symmetric hyperbolicity.

[Keyword 2] Constraint Propagation Analysis -> Constraint Damping System
By evaluating the propagation eqs of constraints, we can predict the suitable adjustments to the EoM in advance.

(Step 1) Fourier mode expression of all terms of constraint propagation eqs.

(Step 2) Eigenvalues and Diagonalizability of constraint propagation matrix.
Eigenvalues = Constraint Amplification Factors

(Step 3) If CAF=negatives -> Constraint surface becomes the attractor.

[Keyword 3] Adjusted ADM systems
We show the standard ADM has constraint violating mode.
We predict several adjustments, which give better stability.

[Keyword 3] Adjusted BSSN systems
We show the advantage of BSSN is the adjustment using M.
We predict several adjustments, which give better stability.

ADMに代わる発展方程式の模索：まとめ

	アプローチ	利点	課題
1	●日本型 (BSSN) Nakamura-Oohara-Shibata, Baumgarte-Shapiro	何故か上手くいく	何故か? ⇒拘束発展方程式の固有値解析
2	●双曲形式の発展方程式 Bona-Masso, Frittelli-Reula Anderson-York, Kidder-Scheel-Teukolsky Ashtekar変数版(Yoneda-HS)	対称双曲型ならば, 「wellposed発展が期待される」 伝播固有値を使って境界条件改善 IBVP問題への手がかり	非線形な項の振る舞い不明 ⇒非特性項が予測を裏切る ⇒非特性項をなくす工夫 汎用性不明
3	●漸近的拘束型 (λ-system) Brodbeck-Frittelli-Hübner-Reula Ashtekar変数版(HS-Yoneda)	拘束条件の破れを強制収束	対称双曲形式が必要 変数多くて非現実的
4	●漸近的拘束型 (adjusted system) Ashtekar変数版(HS-Yoneda) ADM変数版(Yoneda-HS)	拘束条件の発展方程式を固有値解析 双曲形式の議論を必要としない	他のアプローチも説明できるか? 3次元数値計算でも予言通りか? 乗数決定プロセスの汎用化

Discussion
Application 1 : Constraint Propagation in $N+1$ dim. space-time

HS-Yoneda, GRG 36 (2004) 1931

Dynamical equation has N-dependency

Only the matter term in $\partial_t K_{ij}$ has N-dependency.

$$
\begin{aligned}
0 \approx \mathcal{C}_H &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu n^\nu = \frac{1}{2}({}^{(N)}R + K^2 - K^{ij}K_{ij}) - 8\pi\rho_H - \Lambda, \\
0 \approx \mathcal{C}_{Mi} &\equiv (G_{\mu\nu} - 8\pi T_{\mu\nu}) n^\mu \perp^\nu_i = D_j K^j_i - D_i K - 8\pi J_i, \\
\partial_t \gamma_{ij} &= -2\alpha K_{ij} + D_j\beta_i + D_i\beta_j, \\
\partial_t K_{ij} &= \alpha {}^{(N)}R_{ij} + \alpha K K_{ij} - 2\alpha K^\ell{}_j K_{i\ell} - D_i D_j \alpha \\
&\quad + \beta^k (D_k K_{ij}) + (D_j\beta^k) K_{ik} + (D_i\beta^k) K_{kj} - 8\pi\alpha \left(S_{ij} - \frac{1}{N-1}\gamma_{ij} T \right) - \frac{2\alpha}{N-1}\gamma_{ij}\Lambda,
\end{aligned}
$$

Constraint Propagations remain the same

From the Bianchi identity, $\nabla^\nu \mathcal{S}_{\mu\nu} = 0$ with $\mathcal{S}_{\mu\nu} = X n_\mu n_\nu + Y_\mu n_\nu + Y_\nu n_\mu + Z_{\mu\nu}$, we get

$$
\begin{aligned}
0 = n^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= -Z_{\mu\nu}(\nabla^\mu n^\nu) - \nabla^\mu Y_\mu + Y_\nu n^\mu \nabla_\mu n^\nu - 2Y_\mu n_\nu (\nabla^\nu n^\mu) - X(\nabla^\mu n_\mu) - n_\mu(\nabla^\mu X), \\
0 = h_i{}^\mu \nabla^\nu \mathcal{S}_{\mu\nu} &= \nabla^\mu Z_{i\mu} + Y_i(\nabla^\mu n_\mu) + Y_\mu(\nabla^\mu n_i) + X(\nabla^\mu n_i) n_\mu + n_\mu(\nabla^\mu Y_i).
\end{aligned}
$$

- $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (T_{\mu\nu}, \rho_H, J_i, S_{ij})$ with $\nabla^\mu T_{\mu\nu} = 0 \Rightarrow$ matter eq.
- $(\mathcal{S}_{\mu\nu}, X, Y_i, Z_{ij}) = (G_{\mu\nu} - 8\pi T_{\mu\nu}, \mathcal{C}_H, \mathcal{C}_{Mi}, \kappa\gamma_{ij}\mathcal{C}_H)$ with $\nabla^\mu (G_{\mu\nu} - 8\pi T_{\mu\nu}) = 0 \Rightarrow$ CP eq.

Discussion
Future : Construct a robust adjusted system

HS-Yoneda, in preparation

(1) dynamic & automatic determination of κ under a suitable principle.

e.g.) Efforts in Multi-body Constrained Dynamics simulations

$$\frac{\partial}{\partial t} p_i = F_i + \lambda_a \frac{\partial C^a}{\partial x^i}, \qquad \text{with} \quad C^a(x_i, t) \approx 0$$

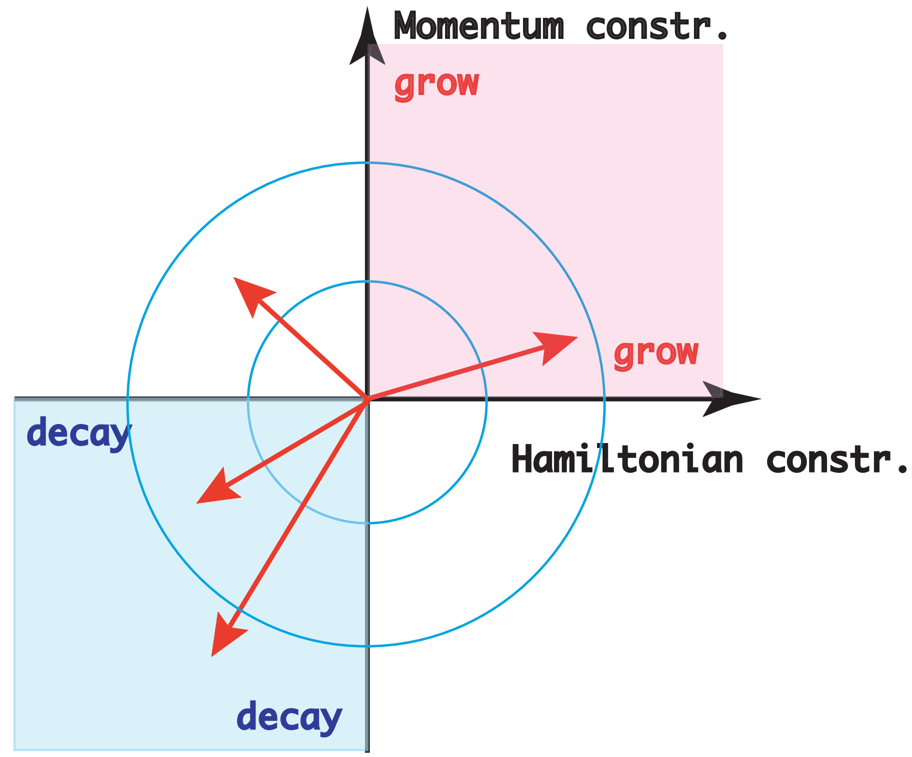
- J. Baumgarte (1972, Comp. Methods in Appl. Mech. Eng.)
  Replace a holonomic constraint $\partial_t^2 C = 0$ as $\partial_t^2 C + \alpha \partial_t C + \beta^2 C = 0$.
- Park-Chiou (1988, J. Guidance), "penalty method"
  Derive "stabilization eq." for Lagrange multiplier $\lambda(t)$.
- Nagata (2002, Multibody Dyn.)
  Introduce a scaled norm, $J = C^T S C$, apply $\partial_t J + w^2 J = 0$, and adjust $\lambda(t)$.

e.g.) Efforts in Molecular Dynamics simulations

- Constant pressure $\cdots\cdots$ potential piston!
- Constant temperature $\cdots\cdots$ potential thermostat!! (Nosé, 1991, PTP)

(2) target to control each constraint violation by adjusting multipliers.

CP-eigenvectors indicate directions of constraint grow/decay, if CP-matrix is diagonalizable.



(3) clarify the reasons of non-linear violation in the last stage of current test evolutions.

(4) Alternative new ideas?

– control theories, optimization methods (convex functional theories), mathematical programming methods, or ....

(5) Numerical comparisons of formulations, links to other systems, ...

– "Comparisons of Formulations" (e.g. Mexico NR workshop, 2002-2003); more formulations to be tested, ...

Find a RECIPE for all. Avoid un-essential techniques.

Goals of the Lecture

What is the guiding principle for selecting evolution equations for simulations in GR?

Why many groups use the BSSN equations?

Are there an alternative formulation better than the BSSN?

Goals of the Lecture

What is the guiding principle for selecting evolution equations for simulations in GR?

--- Constraint Propagation eqs.

Why many groups use the BSSN equations?

Are there an alternative formulation better than the BSSN?

Goals of the Lecture

What is the guiding principle for selecting evolution equations for simulations in GR?

--- Constraint Propagation eqs.

Why many groups use the BSSN equations?

--- Just rush, not to be late.

Are there an alternative formulation better than the BSSN?

Goals of the Lecture

What is the guiding principle for selecting evolution equations for simulations in GR?

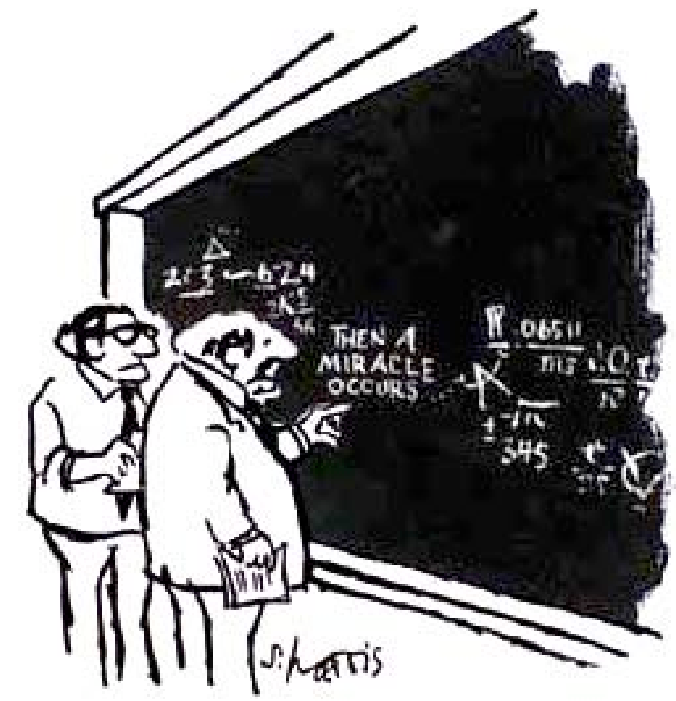
--- Constraint Propagation eqs.

Why many groups use the BSSN equations?

--- Just rush, not to be late.

Are there an alternative formulation better than the BSSN?

--- Yes, there are. But we do not the best one.



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Discussion
Application 2 : Constraint Propagation of Maxwell field in Curved space

HS-Yoneda, in preparation

Towards a robust GR-MHD system:

- Maxwell eqs in curved space-time

$$\begin{aligned}
\partial_t E^i &= \epsilon^{ijk} D_j(\alpha B_k) - 4\pi\alpha J^i + \alpha K E^i + \pounds_\beta E^i \\
\partial_t B^i &= -\epsilon^{ijk} D_j(\alpha E_k) + \alpha K B^i + \pounds_\beta B^i \\
\mathcal{C}_E &:= D_i E^i - 4\pi\rho_e \\
\mathcal{C}_B &:= D_i B^i
\end{aligned}$$

- CP of Maxwell system in curved space-time

$$\begin{aligned}
\partial_t C_E &= \alpha K C_E + \beta^j D_j C_E \\
\partial_t C_B &= \alpha K C_B + \beta^j D_j C_B
\end{aligned}$$

- CP of ADM+Maxwell

$$\partial_t \begin{pmatrix} \mathcal{C}_E \\ \mathcal{C}_B \\ \mathcal{H} \\ \mathcal{M}_i \end{pmatrix} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \begin{pmatrix} \mathcal{C}_E \\ \mathcal{C}_B \\ \mathcal{H} \\ \mathcal{M}_i \end{pmatrix}$$

- CP of ADM+Maxwell+Hydro
  in progress.

Constraint propagation and constraint-damping for C^2-adjusted formulations

土屋 拓也 [1]
共同研究者: 米田 元 [1], 真貝 寿明 [2]

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[2] 大阪工業大学 情報科学部 情報システム学科

2011 年 11 月 7 日

ADM case: Phys. Rev. D **83**, 064032 (2011)
BSSN case: [gr-qc/1109.5782], submitted

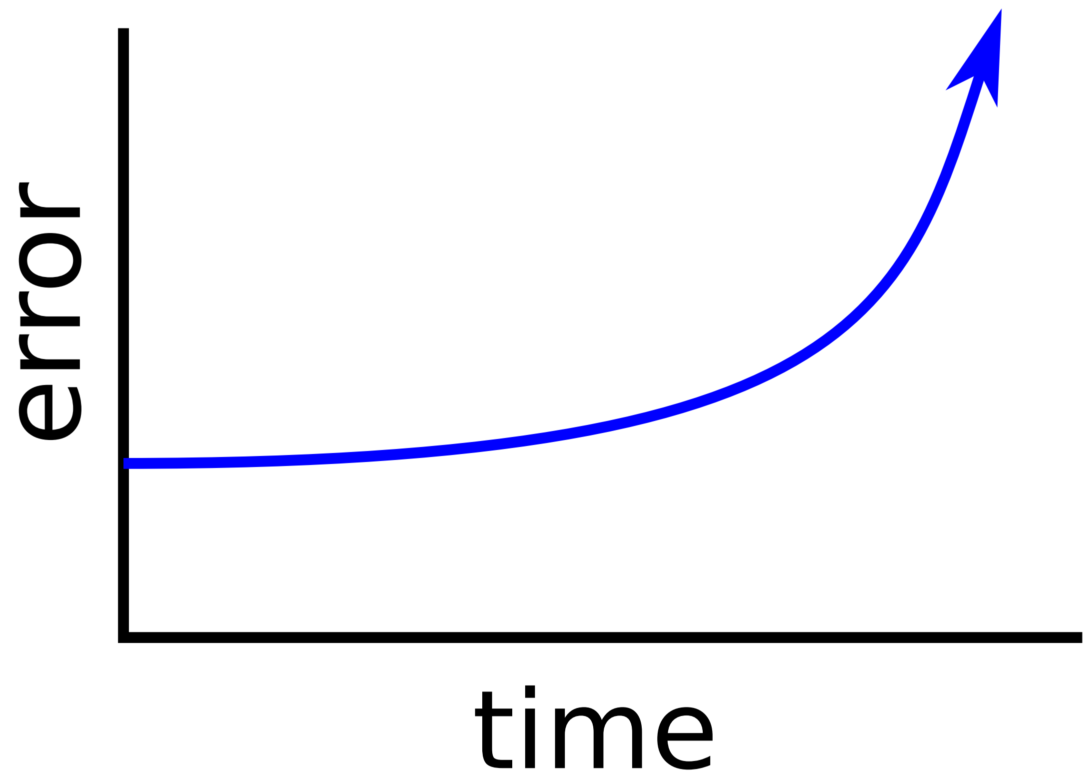
動機と研究背景

数値相対論で実際に数値計算を行うには, おもに以下の設定をする必要がある:

- 初期条件
- 境界条件
- gauge 条件の設定
- formulation の選択
- スキームの設定

今回の話は formulation に関するもの.

動機と研究背景



数値相対論では, formulation 以外の条件をうまく設定したとしても, 拘束値の破れが増大し, 計算が止まってしまう.

これを数値相対論における formulation 問題と呼ぶ.

最近では

- ADM formulation は用いられない
- Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation が広く用いられるようになっている

目的は, さらなる数値安定な formulation を構築することである.

動機と研究背景

formulation を改善する 1 つの方法として, 発展方程式に拘束方程式を付加する方法がある. (これを constraint damping technique と呼ぶ.)

補正システム[1]

ある発展方程式に拘束方程式を付加する:

$$\partial_t u^i = [\textbf{Original Terms}] + f(C^i, \partial_j C^i, \cdots) \tag{1}$$

このとき, 拘束方程式の発展方程式 (拘束伝播方程式) は

$$\partial_t C^i = [\textbf{Original Terms}] + g(C^i, \partial_j C^i, \cdots) \tag{2}$$

と変化する. 一般的には, 背景時空を固定して, (2) の係数行列の固有値解析を行うことで, 新しい方程式系の安定性を調べることができる.

- どのように付加項 $f(C^i, \partial_j C^i, \cdots)$ を加えればよいか?
- 計算とともに背景時空が変化していく場合にこの解析は正しいのか?

⇒ 背景時空に依存しない付加項を設定する 1 つの方法を紹介する.
それが, C^2-adjusted system である.

[1] G. Yoneda and H. Shinkai in PRD **63**, 124019 and PRD **66**, 124003

C^2-adjusted System の説明

ある拘束条件付き発展方程式

$$\begin{cases} \partial_t u^i = f^i(u^i, \partial_j u^i, \dots) \\ C^i = g^i(u^i, \partial_j u^i, \dots) \approx 0 \end{cases} \tag{3}$$

に対して, 以下のように発展方程式を修正する:

$$\partial_t u^i = f^i(u^i, \partial_j u^i, \dots) - \kappa^{ij} \frac{\delta C^2}{\delta u^j} \tag{4}$$

$$\text{where,} \quad C^2 = \int C^i C_i dx^3, \quad \kappa^{ij} : \text{Positive definite} \tag{5}$$

このとき, C^2 の拘束伝播方程式は以下のようになる:

$$\partial_t C^2 = [\textbf{Original terms}] - \kappa^{ij} \left(\frac{\delta C^2}{\delta u^i} \right) \left(\frac{\delta C^2}{\delta u^j} \right) \tag{6}$$

(この考えは D. R. Fiske (Phys. Rev. D **69**, 047501 (2004)) によって提案された)

C^2-adjusted System の説明

ある拘束条件付き発展方程式

$$\begin{cases} \partial_t u^i = f^i(u^i, \partial_j u^i, \dots) \\ C^i = g^i(u^i, \partial_j u^i, \dots) \approx 0 \end{cases} \tag{3}$$

に対して, 以下のように発展方程式を修正する:

$$\partial_t u^i = f^i(u^i, \partial_j u^i, \dots) - \kappa^{ij} \frac{\delta C^2}{\delta u^j} \tag{4}$$

$$\text{where,} \quad C^2 = \int C^i C_i dx^3, \quad \kappa^{ij} : \text{Positive definite} \tag{5}$$

このとき, C^2 の拘束伝播方程式は以下のようになる:

$$\partial_t C^2 = [\textbf{Original terms}] - \kappa^{ij} \left(\frac{\delta C^2}{\delta u^i} \right) \left(\frac{\delta C^2}{\delta u^j} \right) < 0 \tag{6}$$

(この考えは D. R. Fiske (Phys. Rev. D **69**, 047501 (2004)) によって提案された)

Outline

Standard ADM Formulation

Einstein 方程式 ($G_{\mu\nu} = 8\pi T_{\mu\nu}$) の時空分解[2].

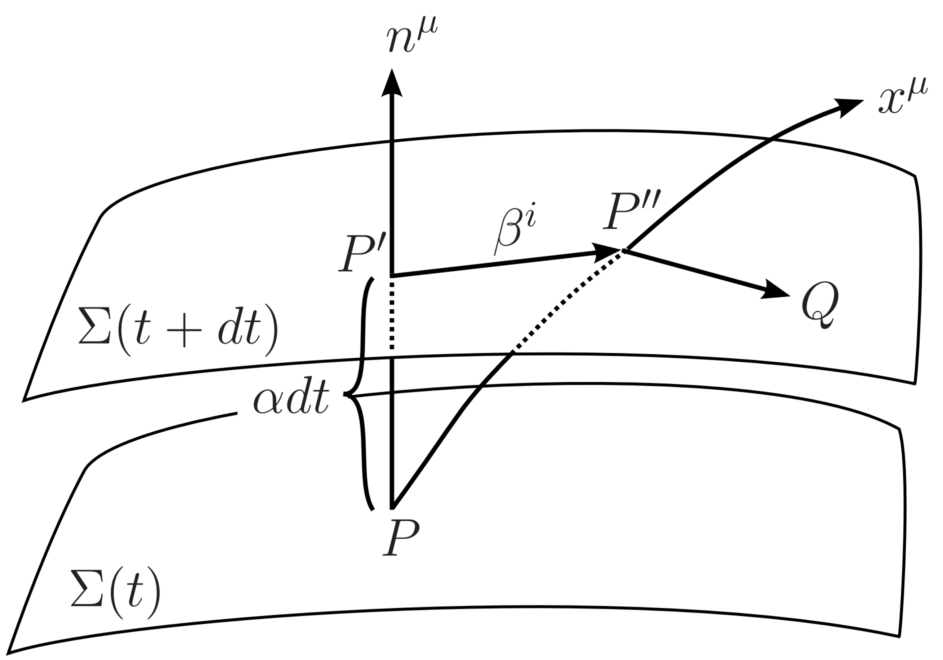


Figure: 時空分解の概念図

$$n^\mu n^\nu G_{\mu\nu} = 8\pi\rho_H. \tag{7}$$

$$P^\mu{}_i n^\nu G_{\mu\nu} = -8\pi J_i. \tag{8}$$

$$P^\mu{}_i P^\nu{}_j G_{\mu\nu} = 8\pi S_{ij}. \tag{9}$$

ただし, $P_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$,
n^μ は超曲面上の単位法線ベクトル.

[2]J. W. York, Jr., in Sources of Gravitational Radiation, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979);
L. Smarr and J. W. York, Jr., Phys. Rev. D **17**, 2529 (1978).

Standard ADM Formulation

発展方程式:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i \tag{10}$$

$$\begin{aligned} \partial_t K_{ij} &= \alpha (R_{ij} + K K_{ij} - 2 K_i{}^{\ell} K_{\ell j}) - D_i D_j \alpha \\ &\quad + K_{i\ell} D_j \beta^{\ell} + K_{j\ell} D_i \beta^{\ell} + \beta^{\ell} D_{\ell} K_{ij} \end{aligned} \tag{11}$$

拘束方程式:

$$\mathcal{H}^{ADM} = R + K^2 - K_{ij} K^{ij} \approx 0 \tag{12}$$

$$\mathcal{M}_i^{ADM} = D_j K^j{}_i - D_i K \approx 0 \tag{13}$$

C^2-adjusted ADM Formulation

C^2-adjusted ADM formulation の発展方程式:

$$\partial_t \gamma_{ij} = [\textbf{Original Terms}] - \kappa_{\gamma ijmn} \frac{\delta (C^{ADM})^2}{\delta \gamma_{mn}} \tag{14}$$

$$\partial_t K_{ij} = [\textbf{Original Terms}] - \kappa_{Kijmn} \frac{\delta (C^{ADM})^2}{\delta K_{mn}} \tag{15}$$

where

$$(C^{ADM})^2 = \int \left\{ (\mathcal{H}^{ADM})^2 + \gamma^{ij} (\mathcal{M}_i^{ADM})(\mathcal{M}_j^{ADM}) \right\} dx^3 \tag{16}$$

Constraint Propagation Equations

背景時空を Minkowskii, Lagrange 乗数係数を $\kappa_{\gamma ijmn} = \kappa_\gamma \delta_{im}\delta_{jn}, \kappa_{Kijmn} = \kappa_K \delta_{im}\delta_{jn}$ としたとき, 各拘束伝播方程式は以下のようになる:

$$\partial_t \mathcal{H} = [\textbf{Original Terms}] - 2\kappa_\gamma \Delta^2 \mathcal{H} \quad (17)$$

$$\partial_t \mathcal{M}_i = [\textbf{Original Terms}] + \kappa_K \Delta \mathcal{M}_i + 3\kappa_K \partial_j \partial_i \mathcal{M}^j \quad (18)$$

補正項の部分に拡散項が現れる. これが拘束値の破れ減少に大きな影響を与えると考えられる.

Outline

Standard BSSN Formulation

BSSN formulation の発展変数:

$$\varphi = \frac{1}{12} \log(\det(\gamma_{ij})) \tag{19}$$

$$\widetilde{\gamma}_{ij} = e^{-4\varphi} \gamma_{ij} \tag{20}$$

$$K = \gamma^{ij} K_{ij} \tag{21}$$

$$\widetilde{A}_{ij} = e^{-4\varphi} \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right) \tag{22}$$

$$\widetilde{\Gamma}^i = \widetilde{\gamma}^{ab} \widetilde{\Gamma}^i{}_{ab} \tag{23}$$

発展方程式

$$\partial_t \varphi = -(1/6)\alpha K + (1/6)(\partial_i \beta^i) + \beta^i(\partial_i \varphi) \tag{24}$$

$$\partial_t K = \alpha \tilde{A}_{ij}\tilde{A}^{ij} + (1/3)\alpha K^2 - D_i D^i \alpha + \beta^i(\partial_i K) \tag{25}$$

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} - (2/3)\tilde{\gamma}_{ij}(\partial_\ell \beta^\ell) + \tilde{\gamma}_{j\ell}(\partial_i \beta^\ell) + \tilde{\gamma}_{i\ell}(\partial_j \beta^\ell) + \beta^\ell(\partial_\ell \tilde{\gamma}_{ij}) \tag{26}$$

$$\begin{aligned}\partial_t \tilde{A}_{ij} &= \alpha K \tilde{A}_{ij} - 2\alpha \tilde{A}_{i\ell}\tilde{A}^\ell{}_j + \alpha e^{-4\varphi} R_{ij}{}^{TF} - e^{-4\varphi}(D_i D_j \alpha)^{TF} \\ &\quad - (2/3)\tilde{A}_{ij}(\partial_\ell \beta^\ell) + (\partial_i \beta^\ell)\tilde{A}_{j\ell} + (\partial_j \beta^\ell)\tilde{A}_{i\ell} + \beta^\ell(\partial_\ell \tilde{A}_{ij})\end{aligned} \tag{27}$$

$$\begin{aligned}\partial_t \tilde{\Gamma}^i &= 2\alpha\{6(\partial_j \varphi)\tilde{A}^{ij} + \tilde{\Gamma}^i{}_{j\ell}\tilde{A}^{j\ell} - (2/3)\tilde{\gamma}^{ij}(\partial_j K)\} - 2(\partial_j \alpha)\tilde{A}^{ij} \\ &\quad + (2/3)\tilde{\Gamma}^i(\partial_j \beta^j) + (1/3)\tilde{\gamma}^{ij}(\partial_\ell \partial_j \beta^\ell) + \beta^\ell(\partial_\ell \tilde{\Gamma}^i) - \tilde{\Gamma}^j(\partial_j \beta^i) \\ &\quad + \tilde{\gamma}^{j\ell}(\partial_j \partial_\ell \beta^i)\end{aligned} \tag{28}$$

Standard BSSN formulation

拘束方程式:

“運動学的” 拘束方程式:

$$\mathcal{H}^{BSSN} \equiv e^{-4\varphi}\tilde{R} - 8e^{-4\varphi}(\tilde{D}_i\tilde{D}^i\varphi + (\tilde{D}^m\varphi)(\tilde{D}_m\varphi)) + (2/3)K^2 - \tilde{A}_{ij}\tilde{A}^{ij} - (2/3)\mathcal{A}K \approx 0 \quad (29)$$

$$\mathcal{M}_i^{BSSN} \equiv -(2/3)\tilde{D}_iK + 6(\tilde{D}_j\varphi)\tilde{A}^j{}_i + \tilde{D}_j\tilde{A}^j{}_i - 2(\tilde{D}_i\varphi)\mathcal{A} \approx 0 \quad (30)$$

“代数的” 拘束方程式:

$$\mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{j\ell}\tilde{\Gamma}^i{}_{j\ell} \approx 0 \quad (31)$$

$$\mathcal{A} \equiv \tilde{A}^{ij}\tilde{\gamma}_{ij} \approx 0 \quad (32)$$

$$\mathcal{S} \equiv \det(\tilde{\gamma}_{ij}) - 1 \approx 0 \quad (33)$$

もし, 代数的拘束方程式 (31)-(33) が満たされない場合, BSSN formulation は数学的に ADM formulation に一致しない.

C^2-adjusted BSSN formulation の発展方程式:

$$\partial_t \varphi = [\textbf{Original Terms}] - \lambda_\varphi \left(\frac{\delta (C^{BSSN})^2}{\delta \varphi} \right) \tag{34}$$

$$\partial_t K = [\textbf{Original Terms}] - \lambda_K \left(\frac{\delta (C^{BSSN})^2}{\delta K} \right) \tag{35}$$

$$\partial_t \tilde{\gamma}_{ij} = [\textbf{Original Terms}] - \lambda_{\tilde{\gamma} ijmn} \left(\frac{\delta (C^{BSSN})^2}{\delta \tilde{\gamma}_{mn}} \right) \tag{36}$$

$$\partial_t \tilde{A}_{ij} = [\textbf{Original Terms}] - \lambda_{\tilde{A} ijmn} \left(\frac{\delta (C^{BSSN})^2}{\delta \tilde{A}_{mn}} \right) \tag{37}$$

$$\partial_t \tilde{\Gamma}^i = [\textbf{Original Terms}] - \lambda_{\tilde{\Gamma}}^{ij} \left(\frac{\delta (C^{BSSN})^2}{\delta \tilde{\Gamma}^j} \right) \tag{38}$$

where

$$(C^{BSSN})^2 = \int \{ (\mathcal{H}^{BSSN})^2 + \gamma^{ij} (\mathcal{M}^{BSSN})_i (\mathcal{M}^{BSSN})_j + \gamma_{ij} \mathcal{G}^i \mathcal{G}^j + \mathcal{A}^2 + \mathcal{S}^2 \} dx^3$$

拘束伝播方程式

背景時空を Minkowskii, Lagrange 乗数係数を $\lambda_{\tilde{\gamma}ijmn} = \lambda_{\tilde{\gamma}}\delta_{im}\delta_{jn}$, $\lambda_{\tilde{A}ijmn} = \lambda_{\tilde{A}}\delta_{im}\delta_{jn}$, $\lambda_{\tilde{\Gamma}}^{ij} = \lambda_{\tilde{\Gamma}}\delta^{ij}$ としたとき, 各拘束伝播方程式は以下のようになる:

$$\begin{aligned}
\partial_t\mathcal{H} &= [\textbf{Original Terms}] + \{-128\lambda_{\varphi}\Delta^2 - (3/2)\lambda_{\tilde{\gamma}}\Delta^2 + 2\lambda_{\tilde{\Gamma}}\Delta\}\mathcal{H} \\
&\quad + \{-(1/2)\lambda_{\tilde{\gamma}}\Delta\partial_m - 2\lambda_{\tilde{\Gamma}}\partial_m\}\mathcal{G}^m + 3\lambda_{\tilde{\gamma}}\Delta\mathcal{S} && (39) \\
\partial_t\mathcal{M}_a &= [\textbf{Original Terms}] - 2\lambda_{\tilde{A}}\partial_a\mathcal{A} \\
&\quad + \{(8/9)\lambda_K\delta^{bc}\partial_a\partial_b + \lambda_{\tilde{A}}\Delta\delta_a{}^c + \lambda_{\tilde{A}}\delta^{bc}\partial_a\partial_b\}\mathcal{M}_c && (40) \\
\partial_t\mathcal{G}^a &= [\textbf{Original Terms}] + \delta^{ab}\{(1/2)\lambda_{\tilde{\gamma}}\partial_b\Delta + 2\lambda_{\tilde{\Gamma}}\partial_b\}\mathcal{H} \\
&\quad - \lambda_{\tilde{\gamma}}\delta^{ab}\partial_b\mathcal{S} + \left(\lambda_{\tilde{\gamma}}\Delta\delta^a{}_b + (1/2)\lambda_{\tilde{\gamma}}\delta^{ac}\partial_c\partial_b - 2\lambda_{\tilde{\Gamma}}\delta^a{}_b\right)\mathcal{G}^b && (41) \\
\partial_t\mathcal{A} &= [\textbf{Original Terms}] + 2\lambda_{\tilde{A}}\delta^{ij}(\partial_i\mathcal{M}_j) - 6\lambda_{\tilde{A}}\mathcal{A} && (42) \\
\partial_t\mathcal{S} &= [\textbf{Original Terms}] + 3\lambda_{\tilde{\gamma}}\Delta\mathcal{H} + \lambda_{\tilde{\gamma}}\partial_\ell\mathcal{G}^\ell - 6\lambda_{\tilde{\gamma}}\mathcal{S} && (43)
\end{aligned}$$

拡散項が現れ, これが拘束値の破れの減少に大きな影響を与えると考えられる.

拘束伝播方程式

もし, $(C^{BSSN})^2$ が代数学的拘束方程式 $(\mathcal{G}^i, \mathcal{A}, \mathcal{S})$ を 含まない場合:

$$(C^{BSSN})^2 = \int \left\{ (\mathcal{H}^{BSSN})^2 + \gamma^{ij}(\mathcal{M}^{BSSN})_i(\mathcal{M}^{BSSN})_j \right\} dx^3,$$

拘束伝播方程式は以下のようになる:

$$\partial_t \mathcal{H} = [\textbf{Original Terms}] + \{-128\lambda_\varphi \Delta^2 - (3/2)\lambda_{\tilde{\gamma}}\Delta^2 + 2\lambda_{\tilde{\Gamma}}\Delta\}\mathcal{H} \quad (44)$$

$$\begin{aligned}\partial_t \mathcal{M}_a =& [\textbf{Original Terms}] \\ &+ \{(8/9)\lambda_K \delta^{bc}\partial_a\partial_b + \lambda_{\tilde{A}}\Delta\delta_a{}^c + \lambda_{\tilde{A}}\delta^{bc}\partial_a\partial_b\}\mathcal{M}_c \end{aligned}\quad (45)$$

$$\partial_t \mathcal{G}^a = [\textbf{Original Terms}] + \delta^{ab}\{(1/2)\lambda_{\tilde{\gamma}}\partial_b\Delta + 2\lambda_{\tilde{\Gamma}}\partial_b\}\mathcal{H} \quad (46)$$

$$\partial_t \mathcal{A} = [\textbf{Original Terms}] + 2\lambda_{\tilde{A}}\delta^{ij}(\partial_i \mathcal{M}_j) \quad (47)$$

$$\partial_t \mathcal{S} = [\textbf{Original Terms}] + 3\lambda_{\tilde{\gamma}}\Delta\mathcal{H} \quad (48)$$

代数的拘束伝播方程式 (46)-(48) が拡散項を含まなくなる.
$\Rightarrow$ $(C^{BSSN})^2$ は代数的拘束値を含むべきである.

Outline

Test 計量

Polarized Gowdy wave test-bed (Apples-with-Apples test[3] のひとつ)

$$ds^2 = t^{-1/2}e^{\lambda/2}(-dt^2 + dx^2) + t(e^P dy^2 + e^{-P} dz^2) \tag{49}$$

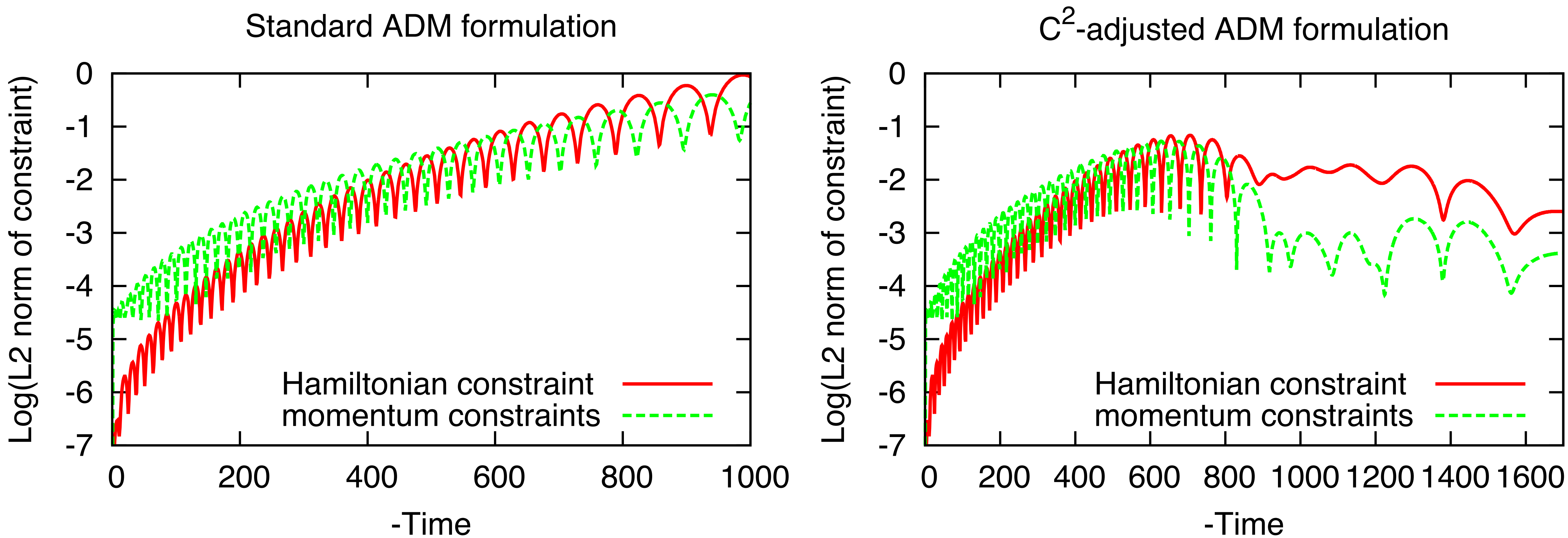
$$P = J_0(2\pi t)\cos(2\pi x) \tag{50}$$

$$\begin{aligned}\lambda = &-2\pi t J_0(2\pi t)J_1(2\pi t)\cos^2(2\pi x) + 2\pi^2 t^2[J_0^2(2\pi t) \\ &+ J_1^2(2\pi t)] - (1/2)\{(2\pi)^2[J_0^2(2\pi) + J_1^2(2\pi)] \\ &- 2\pi J_0(2\pi)J_1(2\pi)\}\end{aligned} \tag{51}$$

ここで, J_n は Bessel 関数.
ほかの Apples-with-Apples テスト (gauge-wave と Linear wave) も行ったが, 今回は Gowdy wave の結果だけを紹介する.

[3] Alcubierre *et al.*, Class. Quant. Grav. **21**, 589 (2004)

Outline

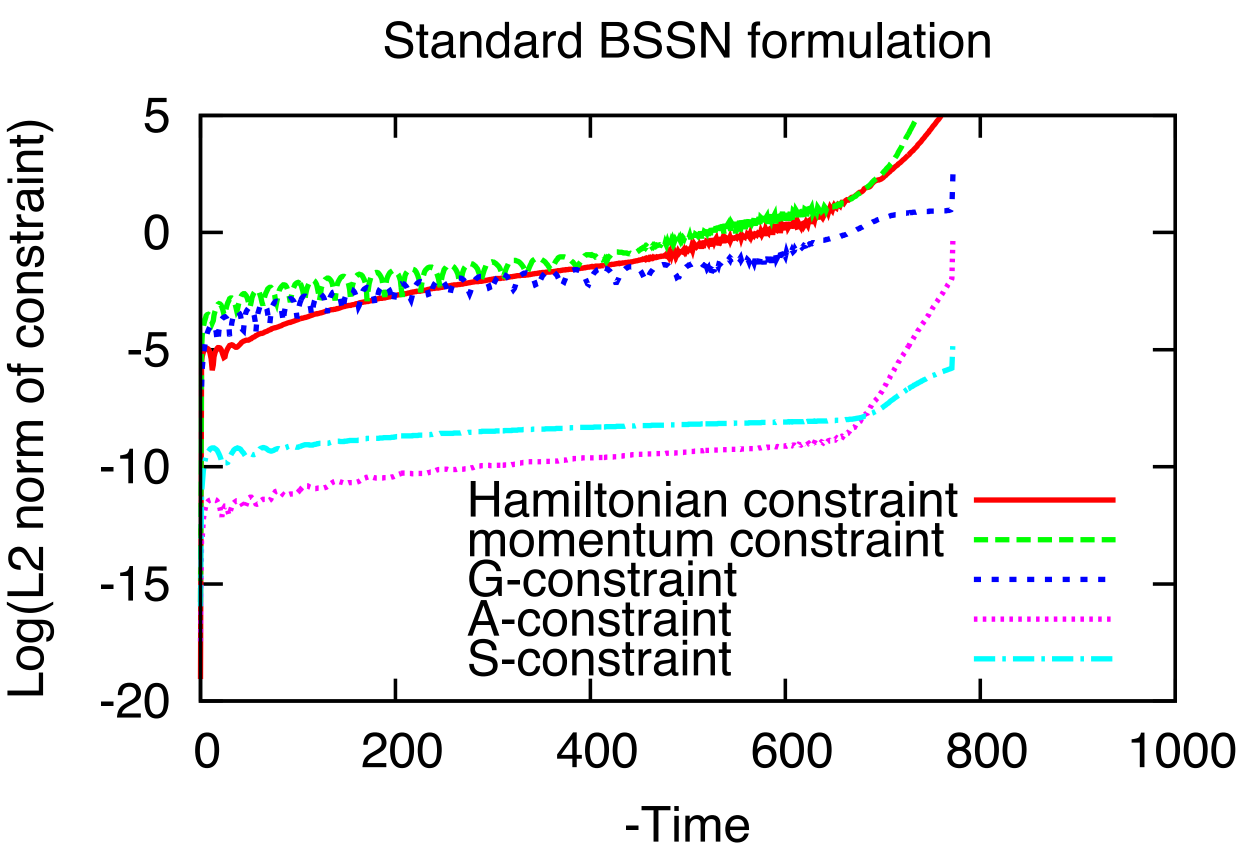


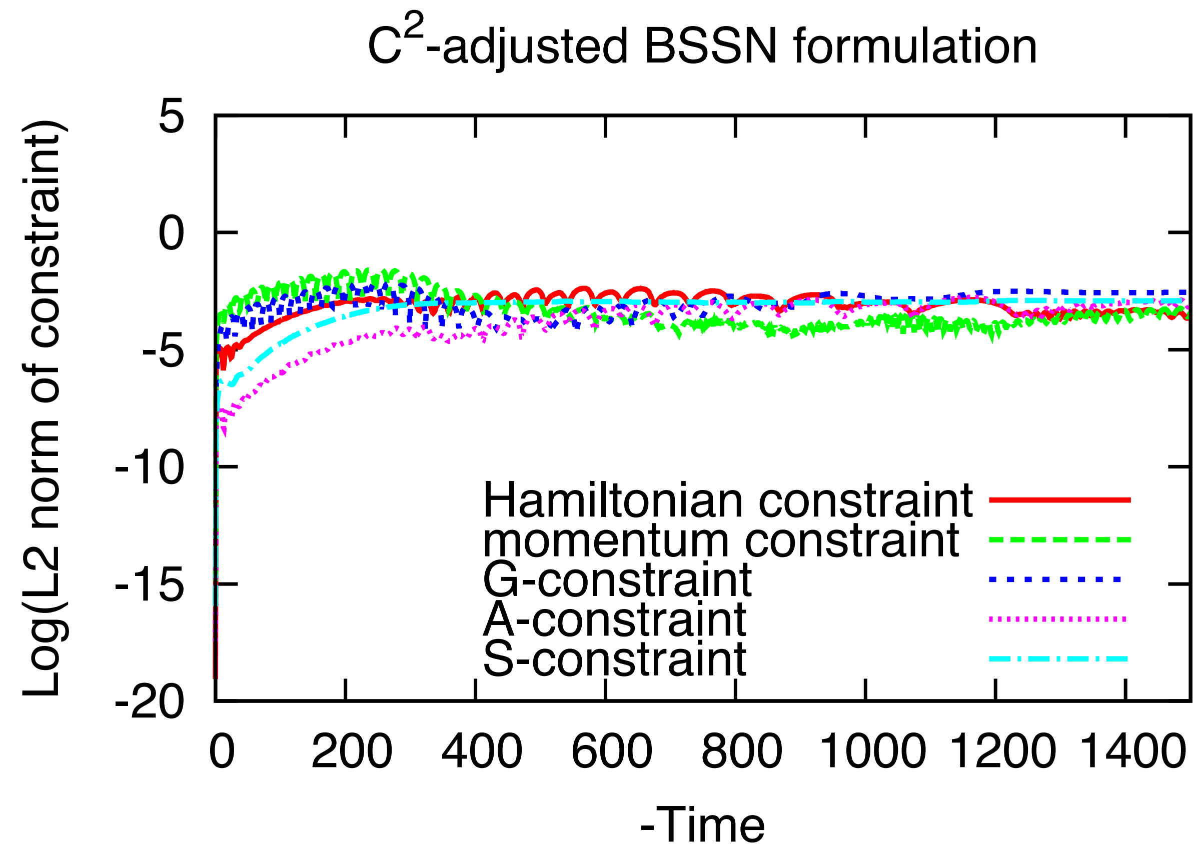
- C^2-adjusted ADM formulation の場合 (右図) のほうが, standard ADM formualtion の場合 (左図) よりも計算時間が約 1.7 倍に伸びた
- C^2-adjusted ADM formulation の拘束値の破れが減少した

(T. Tsuchiya, G. Yoneda, and H. Shinkai, Phys. Rev. D **83**, 064032 (2011))

Outline

数値結果 (BSSN Formulation)





- C^2-adjusted BSSN formulation の場合 (右図) のほうが standard BSSN formulation の場合 (左図) よりも寿命が 2 倍長くなった
- C^2-adjusted BSSN formulation の拘束値の破れが一定になった

(T. Tsuchiya, G. Yoneda, and H. Shinkai, arXiv[gr-qc/1109.5782])

まとめと今後の展望

まとめ

- C^2-adjusted system を ADM formulation と BSSN formulation に適用した.
- C^2-adjusted ADM formulation と C^2-adjusted BSSN formulation の拘束伝播方程式を導出し, damping 項が含まれていることを示した.
- C^2-adjusted BSSN formulation に対して, その拘束伝播方程式から C^2 が代数的拘束値を含むべきであることを示した.
- 実際に C^2-adjusted ADM formulation と C^2-adjusted BSSN formulation を用いて数値計算を行い, その計算時間が延びることを示した.

今後の展望

- first order ADM formulation への適用.
- Lagrange 乗数係数を設定する方法を考案する.