Dirichlet Series Associated with Hyperharmonic Numbers
by
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Abstract

We investigate a complex variable function $h_r(s)$ defined as a Dirichlet series associated with hyperharmonic numbers. This function is a specialization of the non-strict multiple zeta function, and it can be meromorphically continued to the whole complex plane. We represent the function $h_r(s)$ in terms of $h_1(s)$ and the Riemann zeta function, and this result gives information of poles of $h_r(s)$.

keywords: Dirichlet series, harmonic numbers, multiple zeta function

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1. Introduction and main results

Harmonic numbers $H_n \ (n \geq 0)$ are rational numbers defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \ \ (n \geq 1)$$

and $H_0 = 0$. These numbers appear in various areas in mathematics and have been classically investigated. Harmonic numbers have been generalized in many directions. For example, Conway and Guy [4, p. 258] defined hyperharmonic numbers $H_n^{(r)}$ for integers $r \geq 1$ by the following recurrence relations:

$$H_n^{(1)} = H_n,$$

$$H_n^{(r)} = \sum_{i=1}^{n} H_i^{(r-1)} \ (r \geq 2).$$

Benjamin-Gaebler-Gaebler [3] investigated these numbers from a combinatorial point of view. They proved that $H_n^{(r)}$ are expressed in terms of generalized Stirling numbers, called $r$-Stirling numbers. As stated in [4], hyperharmonic numbers can be expressed in terms of the ordinary harmonic numbers:

$$H_n^{(r)} = \binom{n + r - 1}{r - 1} \left( H_{n+r-1} - H_{r-1} \right)$$

for $r \geq 1$. Recently this identity was generalized by Mez˝o-Dil [9, Theorem 1]. They also considered the infinite sum

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m}$$

for an integer $m \geq r + 1$, and gave some identities involving this sum (see [9, Sect. 4 and 6]).

In this paper, as a natural extension of (1.2), we consider the following Dirichlet series associated with hyperharmonic numbers:

$$h_r(s) := \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^s} \quad (s \in \mathbb{C}).$$

By (1.1) and the well-known estimate $H_n \sim \log n$, we have $H_n^{(r)} = O(n^{r-1} \log n)$ as $n$ tends to infinity for a fixed integer $r \geq 1$. Therefore the right-hand side of (1.3) is absolutely convergent for $\Re(s) > r$ and it defines holomorphic function in this region. Since the derivative of the Riemann zeta function $\zeta'(s)$ can be written as $-\sum_{n=1}^{\infty} \log n/n^s$ for $\Re(s) > 1$, the function $h_1(s)$ can be regarded as an analogue of $\zeta'(s)$. Matsuoka [8] proved the meromorphic continuation of $h_1(s)$ to the whole $s$-plane and gave the location of poles and residues (see Theorem 2.2 in the next section). This function has been investigated more precisely by Apostol-Vu [2] and Mez˝o-Dil [9].

As stated in Proposition 2.1 below, the function $h_r(s)$ is a specialization of a multiple zeta function:

$$h_r(s) = \zeta_{r+1}(s, 0, \ldots, 0, 1).$$

Here $\zeta_r(s_1, \ldots, s_r)$ are non-strict multiple zeta functions defined by

$$\zeta_r(s_1, \ldots, s_r) = \sum_{m_1 \geq m_2 \geq \cdots \geq m_r \geq 1} m_1^{-s_1} m_2^{-s_2} \cdots m_r^{-s_r}.$$

When $r = 1$, the function $\zeta_1(s)$ is nothing but the Riemann zeta function $\zeta(s)$. It is known that the right-hand side of (1.4) is absolutely convergent when $\Re(s_1)$ is sufficiently large. Probably strict multiple zeta functions, defined by

$$\zeta_r(s_1, \ldots, s_r) = \sum_{m_1 > m_2 > \cdots > m_r \geq 1} m_1^{-s_1} m_2^{-s_2} \cdots m_r^{-s_r},$$

are better known than non-strict ones. Values of strict multiple zeta functions at positive integers are called Multiple Zeta Values (MZVs) and many relations among MZVs have been given (e.g. see [5].
and references therein). The multiple sum (1.5) is also absolutely convergent when \( \Re(s_1) \) is sufficiently large, and Akiyama-Egami-Tanigawa [1], Zhao [10] and Matsumoto-Tanigawa [7] proved the meromorphic continuation of \( \zeta_r(s_1, \ldots, s_r) \) to the whole \( \mathbb{C}^r \) space by different methods. Since the function \( \zeta_r^* \) can be expressed in terms of \( \zeta_1, \zeta_2, \ldots, \zeta_r \), it can be also meromorphically continued to the whole \( \mathbb{C}^r \) space. As a special case, we obtain the meromorphic continuation of \( h_r(s) \) to the whole \( s \)-plane.

The following is the main result of this paper. We note that the statement (i) gives another proof of the meromorphic continuation of \( h_r(s) \).

**Theorem 1.1.** (i) For a complex number \( s \) with \( \Re(s) > r \), we have

\[
h_r(s) = \frac{1}{r!} \left( \sum_{k=1}^{r} \begin{bmatrix} r \\ k \end{bmatrix} h_1(s-k+1) \right) + \sum_{k=1}^{r} \left( k \begin{bmatrix} r \\ k+1 \end{bmatrix} - \begin{bmatrix} r \\ k \end{bmatrix} H_{r-1} \right) \zeta(s-k+1),
\]

where \( \begin{bmatrix} r \\ k \end{bmatrix} \) are Stirling numbers of the first kind. The function \( h_r(s) \) can be meromorphically continued to the whole \( s \)-plane by this expression.

(ii). The function \( h_r(s) \) has double poles at \( s = 1, 2, \ldots, r \) and has \((\text{possible})\) simple poles at \( s = -a \) \((a = 0, 1, 2, 3, \ldots)\). The residues at each poles are given as follows:

\[
\text{Res}_{s=-a} h_r(s) = \frac{1}{r!} \left( a \begin{bmatrix} r \\ a+1 \end{bmatrix} + \begin{bmatrix} r \\ a \end{bmatrix} (\gamma - H_{r-1}) - \sum_{k=a+1}^{r} \begin{bmatrix} r \\ k \end{bmatrix} \frac{B_{k-a}}{k-a} \right) \quad (a = 1, 2, \ldots r).
\]

\[
\text{Res}_{s=-a} h_r(s) = -\frac{1}{r!} \sum_{k=1}^{r} \begin{bmatrix} r \\ k \end{bmatrix} \frac{B_{a+k}}{a+k} \quad (a = 0, 1, 2, \ldots).
\]

Here \( \gamma \) is the Euler constant and \( B_n \) is the \( n \)-th Bernoulli number defined by

\[
\frac{te^t-1}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.
\]

**Remark 1.2.** It is easy to see that

\[
\begin{bmatrix} r \\ 2 \end{bmatrix} - \begin{bmatrix} r \\ 1 \end{bmatrix} H_{r-1} = 0 \quad (r \geq 1)
\]

(by Lemma 2.4 below, for example). Therefore the term for \( k = 1 \) vanishes in the second sum of (1.6).

**Remark 1.3.** We consider the Barnes type multiple zeta function:

\[
Z_r(s) = \sum_{m_1 \geq 0, \ldots, m_r \geq 0 \atop (m_1, \ldots, m_r) \neq (0, \ldots, 0)} (m_1 + \cdots + m_r)^{-s}
\]

for \( \Re(s) > r \). The function \( Z_r(s) \) can be meromorphically continued to the whole plane, and the special values of \( Z_r(s) \) essentially coincide with residues of \( h_r(s) \):

\[
Z_r(-a) = r \text{Res}_{s=-a} h_r(s) \quad (a \geq 1).
\]
2. Proof of Theorem 1.1

First we show that \( h_r(s) \) is a specialization of the non-strict multiple zeta function:

**Proposition 2.1.** For a complex number \( s \) with \( \Re(s) > r \), we have

\[
h_r(s) = \zeta_{r+1}^s(s, 0, \ldots, 0, 1).
\]

**Proof.** By the repeated use of the equation \( H_n^{(r)} = \sum_{i=1}^{n} H_i^{(r-1)} \), we have

\[
h_r(s) = \sum_{n \geq m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{n^s m_r^{r-1} m_1 \cdots m_{r-2}}
\]

\[
= \zeta_{r+1}^s(s, 0, \ldots, 0, 1).
\]

\[\square\]

To prove Theorem 1.1, we need the following theorem proved by Matsuoka (note that he considered the function \( h(s) = h_1(s) - \zeta(s + 1) \)). Our main theorem (Theorem 1.1) is a generalization of this theorem.

**Theorem 2.2** (Matsuoka [8]). (i). The function \( h_1(s) \) can be meromorphically continued to the whole plane and is holomorphic except for \( s = 1 \), \(-1\), \(-3\), \(-5\), …. The function \( h_1(s) \) has a double pole at \( s = 1 \) and simple poles at \( s = 0, -1, -3, -5, … \).

(ii). The residues of \( h_1(s) \) at each poles are given as follows:

\[
\text{Res}_{s=1} h_1(s) = \gamma, \tag{2.1}
\]

\[
\text{Res}_{s=0} h_1(s) = \frac{1}{2}, \tag{2.2}
\]

\[
\text{Res}_{s=1-2a} h_1(s) = -\frac{B_{2a}}{2a} \quad (a \geq 1). \tag{2.3}
\]

**Remark 2.3.** Equations (2.2) and (2.3) can be expressed as

\[
\text{Res}_{s=1-a} h_1(s) = -\frac{B_a}{a} \quad (a \geq 1).
\]

Since \( B_a = 0 \) for all odd integers \( a \geq 3 \), Equation (2.4) implies that \( h_1(s) \) is holomorphic at \( s = -2, -4, -6, \ldots \).

For integers \( r \geq 1 \) and \( k \geq 0 \), we denote by \( \left[ \begin{array}{c} r \\ k \end{array} \right] \) the number of permutations of \( r \) elements with \( k \) disjoint cycles (e.g. [6, Chapter 6]). These numbers are called Stirling numbers of the first kind and appear in the coefficients of the expansion of the rising factorial:

\[
(n + 1) \cdots (n + r - 1) = \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] n^{k-1} \quad (r \geq 1).
\]

**Lemma 2.4.** The following identities hold:

\[
\left( \begin{array}{c} n + r - 1 \\ r - 1 \end{array} \right) = \frac{1}{(r-1)!} \sum_{k=1}^{r} \left[ \begin{array}{c} r \\ k \end{array} \right] n^{k-1}.
\]


\[
\binom{n + r - 1}{r - 1} \left( \frac{1}{n + 1} + \cdots + \frac{1}{n + r - 1} \right) = \frac{1}{(r - 1)!} \sum_{k=1}^{r-1} \binom{r}{k+1} n^{k-1}.
\]

**Proof.** It is clear that (2.6) follows from (2.5). The left-hand side of (2.7) is equal to
\[
\frac{d}{dx} \left( \frac{x + r - 1}{r - 1} \right) \bigg|_{x=n}.
\]
By (2.6), this equals
\[
\frac{1}{(r - 1)!} \sum_{k=2}^{r} \binom{r}{k} (k - 1) x^{k-2} \bigg|_{x=n}
\]
\[
= \frac{1}{(r - 1)!} \sum_{k=1}^{r-1} \binom{r}{k+1} k n^{k-1}
\]
and (2.7) follows. \(\square\)

**Proof of Theorem 1.1.** (i). By (1.1) and Lemma 2.4, we have
\[
H_{n}^{(r)} = \binom{n + r - 1}{r - 1} \left( H_{n} + \frac{1}{n + 1} + \cdots + \frac{1}{n + r - 1} - H_{r-1} \right)
\]
\[
= \frac{1}{(r - 1)!} \left( \sum_{k=1}^{r} \binom{r}{k} H_{n}^{(1-k)} + \sum_{k=1}^{r-1} \binom{r}{k+1} \frac{k}{n^{1-k}} - \sum_{k=1}^{r} \binom{r}{k} H_{r-1} \right).
\]
Therefore we obtain
\[
h_{r}(s) = \frac{1}{(r - 1)!} \left( \sum_{k=1}^{r} \binom{r}{k} h_{1}(s - k + 1)
\right.
\]
\[
+ \sum_{k=1}^{r} \left( k \left[ \binom{r}{k+1} - \binom{r}{k} H_{r-1} \right] \zeta(s - k + 1) \right)
\]
and this proves (1.6). Since \(h_{1}(s)\) and \(\zeta(s)\) are meromorphically continued to the whole plane, the function \(h_{r}(s)\) can be also continued.

(ii). First, from the first part of (1.6), double poles appear at \(s = 1, 2, \ldots, r\) and simple poles appear at \(s = 0, -1, -2, \ldots\). Next, from the latter part of (1.6), only simple poles appear at \(s = 1, 2, \ldots, r\). By Theorem 2.2 and the fact the Riemann zeta function has only a simple pole at \(s = 1\) with residue 1, we obtain residues of \(h_{r}(s)\) as (1.7) and (1.8). \(\square\)

**References**


