Asymptotically constrained and real-valued system based on Ashtekar's variables

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We present a set of dynamical equations based on Ashtekar's extension of the Einstein equation. The system forces the space-time to evolve to the manifold that satisfies the constraint equations or the reality conditions or both as the attractor against perturbative errors. This is an application of the idea by Brodbeck, Frittelli, Hübner and Reula who constructed an asymptotically stable (i.e., constrained) system for the Einstein equation, adding dissipative forces in the extended space. The obtained systems may be useful for future numerical studies using Ashtekar's variables.

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I. INTRODUCTION

Ashtekar's extended formulation of general relativity [1] has many advantages. By using his special pair of variables, the constraint equations which appear in the formulation become low-order polynomials, and the theory has the correct form for gauge theoretical interpretation. These features suggest the possibility for developing a nonperturbative quantum description of gravity. When we apply his formulation to classical dynamics, however, we have to impose the reality condition additionally to the system, since Ashtekar's variables will not produce real-valued metric in general.

Fortunately, it was shown that the secondary condition of the reality of metric will be automatically preserved during the evolution, if the initial data satisfies both primary and secondary metric reality conditions [2]. If we impose the reality condition on the triad (triad reality condition), then we have additional conditions that can be controlled by a part of gauge variable, triad lapse $\mathcal{A}^a_0$ (defined later) [3]. Therefore the reality conditions are controllable, and we think that applying the Ashtekar formulation to dynamics is quite attractive, and broadens our possibilities to attack dynamical issues.

It is, however, also the fact that preliminary numerical simulations of the spacetime using Ashtekar's variables show that the system will not recover real-valued spacetime normally if we relax the metric reality condition locally during the evolution [4]. Therefore, we expect a system that is robust for controlling both the reality conditions and the constraint equations for stable long-term integration.

In this article, we propose new dynamical systems based on Ashtekar's variables, which satisfy this expectation. That is, even if a numerical data gives us a truncated solution which violates the constraints or the reality conditions during steps of time evolution, the system forces the spacetime to evolve to the manifold that satisfies the constraint equations and the reality conditions against these small deviations. This is an extension of the idea by Brodbeck, Frittelli, Hübner and Reula (BFHR) [5] who constructed an asymptotically *stable* system (i.e., approaches to the constraint surface) for the Einstein equation. (The similar effort can be found also in [6].) BFHR introduced the additional dynamical variables, λs, which obey the dissipative dynamical equations and which evolve the spacetime to the constraint surface of general relativity as the attractor in the extended spacetime.

Recently, we have obtained a set of equations of motions for Ashtekar's canonical variables $(\tilde{E}^i_a, \mathcal{A}^a_i)$ in a symmetric hyperbolic form [7] (see also [8,9]). We, here, present first that a set of constraint equations, also forms a symmetric hyperbolic system in its evolution equation. This is the Ashtekar version of the work by Frittelli [10], and is already shown by Iriondo *et al.* [9]. Our version, however, is based on the fixed inner product throughout evolution. We, then, show that a dissipative system for the combination of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ and constraints also forms a symmetric hyperbolic system, following the procedure of *λ-system* by BFHR. We expect that this system has the attractor in the constraint surfaces.

We, next, extend this system also to the one that has the attractor in the real-valued surfaces. Since the dynamical system that we have obtained in a symmetric hyperbolic form requires the triad reality condition [7], our purpose is to construct a system which asymptotically evolves into the triad-real-valued manifold. We show this is available by applying the same technique.

In this article, we discuss only the case of the vacuum spacetime, but including matter is straightforward.

II. ASHTEKAR FORMULATION

We start by giving a brief review of the Ashtekar formulation and the way of handling reality conditions. We also describe a symmetric hyperbolic system that was obtained in [7].

A. Variables and Equations

The key feature of Ashtekar's formulation of general relativity [1] is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write the metric $g_{\mu\nu}$ using the tetrad E^I_μ as $g_{\mu\nu} = E^I_\mu E^J_\nu \eta_{IJ}$ [1]. Define its inverse, E^μ_I, by $E^\mu_I := E^J_\nu g^{\mu\nu}\eta_{IJ}$ and we impose $E^0_a = 0$ as the gauge condition. We define SO(3,C) self-dual and anti self-dual connections ${}^{\pm}\mathcal{A}^a_\mu := \omega^{0a}_\mu \mp (i/2)\epsilon^a{}_{bc}\,\omega^{bc}_\mu$, where ω^{IJ}_μ is a spin connection 1-form (Ricci connection), $\omega^{IJ}_\mu := E^{I\nu}\nabla_\mu E^J_\nu$. Ashtekar's plan is to use only a self-dual part of the connection ${}^+\mathcal{A}^a_\mu$ and to use its spatial part ${}^+\mathcal{A}^a_i$ as a dynamical variable. Hereafter, we simply denote ${}^+\mathcal{A}^a_\mu$ as $\mathcal{A}^a_\mu$.

The lapse function, N, and shift vector, N^i, both of which we treat as real-valued functions, are expressed as $E^\mu_0 = (1/N, -N^i/N)$. This allows us to think of E^μ_0 as a normal vector field to Σ spanned by the condition $t = x^0$ =const., which plays the same role as that of Arnowitt-Deser-Misner (ADM) formulation. Ashtekar treated the set ($\tilde{E}^i_a$, $\mathcal{A}^a_i$) as basic dynamical variables, where $\tilde{E}^i_a$ is an inverse of the densitized triad defined by $\tilde{E}^i_a := eE^i_a$, where $e := \det E^a_i$ is a density. This pair forms the canonical set.

In the case of pure gravitational spacetime, the Hilbert action takes the form

$$\begin{aligned} S = \int \mathrm{d}^4x[(\partial_t\mathcal{A}^a_i)\tilde{E}^i_a + (i/2)\underset{\sim}{N}\tilde{E}^i_a\tilde{E}^j_b F^c_{ij}\epsilon^{ab}{}_c \\ -e^2\Lambda\underset{\sim}{N} - N^i F^a_{ij}\tilde{E}^j_a + \mathcal{A}^a_0\,\mathcal{D}_i\tilde{E}^i_a], \end{aligned} \tag{2.1}$$

where $\underset{\sim}{N} := e^{-1}N$, $F^a_{\mu\nu} := 2\partial_{[\mu}\mathcal{A}^a_{\nu]} - i\epsilon^a{}_{bc}\,\mathcal{A}^b_\mu\mathcal{A}^c_\nu$ is the curvature 2-form, Λ is the cosmological constant, $\mathcal{D}_i\tilde{E}^j_a := \partial_i\tilde{E}^j_a - i\epsilon_{ab}{}^c\,\mathcal{A}^b_i\tilde{E}^j_c$, and $e^2 = \det\tilde{E}^i_a = (\det E^a_i)^2$ is defined to be $\det\tilde{E}^i_a = (1/6)\epsilon^{abc}\underset{\sim}{\epsilon}_{ijk}\tilde{E}^i_a\tilde{E}^j_b\tilde{E}^k_c$, where $\epsilon_{ijk} := \epsilon_{abc}E^a_iE^b_jE^c_k$ and $\underset{\sim}{\epsilon}_{ijk} := e^{-1}\epsilon_{ijk}$ [2].

Varying the action with respect to the non-dynamical variables $\underset{\sim}{N}$, N^i and $\mathcal{A}^a_0$ yields the constraint equations,

$$\mathcal{C}_H := (i/2)\epsilon^{ab}{}_c\,\tilde{E}^i_a\tilde{E}^j_b F^c_{ij} - \Lambda\,\det\tilde{E} \approx 0, \tag{2.2}$$
$$\mathcal{C}_{Mi} := -F^a_{ij}\tilde{E}^j_a \approx 0, \tag{2.3}$$
$$\mathcal{C}_{Ga} := \mathcal{D}_i\tilde{E}^i_a \approx 0. \tag{2.4}$$

The equations of motion for the dynamical variables ($\tilde{E}^i_a$ and $\mathcal{A}^a_i$) are

$$\begin{aligned} \partial_t\tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_a\,\underset{\sim}{N}\tilde{E}^j_c\tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) \\ +i\mathcal{A}^b_0\epsilon_{ab}{}^c\,\tilde{E}^i_c, \end{aligned} \tag{2.5}$$
$$\partial_t\mathcal{A}^a_i = -i\epsilon^{ab}{}_c\,\underset{\sim}{N}\tilde{E}^j_b F^c_{ij} + N^jF^a_{ji} + \mathcal{D}_i\mathcal{A}^a_0 + \Lambda\underset{\sim}{N}\tilde{E}^a_i, \tag{2.6}$$

where $\mathcal{D}_jX^{ji}_a := \partial_jX^{ji}_a - i\epsilon_{ab}{}^c\mathcal{A}^b_jX^{ji}_c$, for $X^{ij}_a + X^{ji}_a = 0$.

B. Reality conditions

In order to construct metric from the variables $(\tilde{E}^i_a, \mathcal{A}^a_i, \underset{\sim}{N}, N^i)$, we first prepare tetrad E^μ_I as $E^\mu_0 = (1/e\underset{\sim}{N}, -N^i/e\underset{\sim}{N})$ and $E^\mu_a = (0, \tilde{E}^i_a/e)$. Using them, we obtain metric $g^{\mu\nu}$ such that $g^{\mu\nu} := E^\mu_I E^\nu_J\eta^{IJ}$.

This metric, in general, is not real-valued in the Ashtekar formulation. To ensure that the metric is real-valued, we need to impose real lapse and shift vectors together with two *metric reality* conditions;

$$\mathrm{Im}(\tilde{E}^i_a\tilde{E}^{ja}) = 0, \tag{2.7}$$
$$W^{ij} := \mathrm{Re}(\epsilon^{abc}\tilde{E}^k_a\tilde{E}^{(i}_b\mathcal{D}_k\tilde{E}^{j)}_c) = 0, \tag{2.8}$$

where the latter comes from the secondary condition of reality of the metric $\mathrm{Im}\{\partial_t(\tilde{E}^i_a\tilde{E}^{ja})\} = 0$ [2], and we assume $\det\tilde{E} > 0$ (see [3]).

For later convenience, we also prepare stronger reality conditions, *triad reality* conditions. The primary and secondary conditions are written respectively as

$$U^i_a := \mathrm{Im}(\tilde{E}^i_a) = 0, \tag{2.9}$$
$$\text{and}\ \ \mathrm{Im}(\partial_t\tilde{E}^i_a) = 0. \tag{2.10}$$

Using the equations of motion of $\tilde{E}^i_a$, the gauge constraint (2.4), the metric reality conditions (2.7), (2.8) and the primary condition (2.9), we see that (2.10) is equivalent to [3]

$$\begin{aligned} \mathrm{Re}(\mathcal{A}^a_0) = \partial_i(\underset{\sim}{N})\tilde{E}^{ia} + (1/2e)E^b_i\underset{\sim}{N}\tilde{E}^{ja}\partial_j\tilde{E}^i_b \\ +N^i\mathrm{Re}(\mathcal{A}^a_i), \end{aligned} \tag{2.11}$$

or with un-densitized variables,

$$\mathrm{Re}(\mathcal{A}^a_0) = \partial_i(N)E^{ia} + N^i\,\mathrm{Re}(\mathcal{A}^a_i). \tag{2.12}$$

From this expression we see that the secondary triad reality condition restricts the three components of "triad lapse" vector $\mathcal{A}^a_0$. Therefore (2.11) is not a restriction on

[1] We use $\mu,\nu = 0,\cdots,3$ and $i,j = 1,\cdots,3$ as spacetime indices, while $I,J = (0),\cdots,(3)$ and $a,b = (1),\cdots,(3)$ are $SO(1,3)$, $SO(3)$ indices respectively. We raise and lower $\mu,\nu,\cdots$ by $g^{\mu\nu}$ and $g_{\mu\nu}$ (the Lorentzian metric); $I,J,\cdots$ by $\eta^{IJ} = \mathrm{diag}(-1,1,1,1)$ and η_{IJ}; $i,j,\cdots$ by γ^{ij} and γ_{ij} (the 3-metric); $a,b,\cdots$ by δ^{ab} and δ_{ab}. We also use volume forms ϵ_{abc}: $\epsilon_{abc}\epsilon^{abc} = 3!$.

[2] When $(i,j,k) = (1,2,3)$, we have $\epsilon_{ijk} = e$, $\underset{\sim}{\epsilon}_{ijk} = 1$, $\epsilon^{ijk} = e^{-1}$, and $\tilde{\epsilon}^{ijk} = 1$.

the dynamical variables ($\tilde{E}^i_a$ and $\mathcal{A}^a_i$) but on the slicing, which we should impose on each hypersurface.

Throughout the discussion in this article, we assume that initial data of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ for evolution are solved so as to satisfy all three constraint equations and metric reality condition (2.7) and (2.8). Practically, this is obtained, for example, by solving ADM constraints and by transforming a set of initial data to Ashtekar's notation.

C. A symmetric hyperbolic form

We say that the system is first-order (quasi-linear) partial differential equation system, if a certain set of (complex) variables u_α ($\alpha = 1, \cdots, n$) forms

$$\partial_t u_\alpha = \mathcal{M}^{l\beta}{}_\alpha(u)\,\partial_l u_\beta + \mathcal{N}_\alpha(u), \tag{2.13}$$

where $\mathcal{M}$ (the characteristic matrix) and $\mathcal{N}$ are functions of u but do not include any derivatives of u. If the characteristic matrix is a Hermitian matrix, then we say (2.13) is a symmetric hyperbolic system.

For a pair of $u^{(D)}_\alpha = (\tilde{E}^i_a, \mathcal{A}^a_i)$, a symmetric hyperbolic system is obtained by modifying the right-hand-side of the dynamical equations using appropriate combinations of the constraint equations. The final form of the system [7] is written as

$$\mathcal{M}(\tilde{E},\tilde{E})^{labij} = i\epsilon^{abc}\underset{\sim}{N}\tilde{E}^l_c\gamma^{ij} + N^l\gamma^{ij}\delta^{ab}, \tag{2.14}$$

$$\mathcal{M}(\tilde{E},\mathcal{A})^{labij} = \mathcal{M}(\mathcal{A},\tilde{E})^{labij} = 0, \tag{2.15}$$

$$\begin{aligned}\mathcal{M}(\mathcal{A},\mathcal{A})^{labij} = {}& i\underset{\sim}{N}(\epsilon^{abc}\tilde{E}^j_c\gamma^{li} - \epsilon^{abc}\tilde{E}^l_c\gamma^{ji} \\ & -e^{-2}\tilde{E}^{ia}\epsilon^{bcd}\tilde{E}^j_c\tilde{E}^l_d - e^{-2}\epsilon^{acd}\tilde{E}^i_d\tilde{E}^l_c\tilde{E}^{jb} \\ & +e^{-2}\epsilon^{acd}\tilde{E}^i_d\tilde{E}^j_c\tilde{E}^{lb}) + N^l\delta^{ab}\gamma^{ij}.\end{aligned} \tag{2.16}$$

where $\mathcal{M}(*,*)$ means a block component of the characteristic matrix as

$$\partial_t \begin{bmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \end{bmatrix} \cong \begin{bmatrix} \mathcal{M}(\tilde{E},\tilde{E})^l{}_a{}^b{}_j{}^i & \mathcal{M}(\tilde{E},\mathcal{A})^l{}_{ab}{}^{ij} \\ \mathcal{M}(\mathcal{A},\tilde{E})^{lab}{}_{ij} & \mathcal{M}(\mathcal{A},\mathcal{A})^{la}{}_{bi}{}^j \end{bmatrix} \partial_l \begin{bmatrix} \tilde{E}^j_b \\ \mathcal{A}^b_j \end{bmatrix}, \tag{2.17}$$

where $\cong$ means that we have extracted only the terms which appear in the principal part of the system. The inner product of a set of the variables is

$$\langle(\tilde{E}^i_a, \mathcal{A}^a_i)|(\tilde{E}^i_a, \mathcal{A}^a_i)\rangle = \gamma_{ij}\delta^{ab}\tilde{E}^i_a\bar{\tilde{E}}^j_b + \gamma^{ij}\delta_{ab}\mathcal{A}^a_i\bar{\mathcal{A}}^b_j. \tag{2.18}$$

We note that this symmetric hyperbolic system is obtained under the assumption of the triad reality condition, together with gauge conditions, $\mathcal{A}^a_0 = \mathcal{A}^a_i N^i$ and $\partial_i N = 0$.

III. ASYMPTOTICALLY CONSTRAINED SYSTEM

Frittelli [10] showed that the propagation of the constraint equations in the standard ADM system of the Einstein equation forms a symmetric hyperbolic system. This fact suggests that a small violation of the constraint equations such as a truncation error in numerical simulation does not behave in the fatal way immediately.

Similarly, we can show a set of constraint equations (2.2), (2.3) and (2.4), forms a symmetric hyperbolic system in its evolution equations. The principal part of the time derivatives of $\mathcal{C}_H$, $\tilde{\mathcal{C}}_{Mi} := e\,\mathcal{C}_{Mi}$ and $\mathcal{C}_{Ga}$ become

$$\partial_t \begin{pmatrix} \mathcal{C}_H \\ \tilde{\mathcal{C}}_{Mi} \\ \mathcal{C}_{Ga} \end{pmatrix} \cong \begin{pmatrix} N^l & -e\underset{\sim}{N}\gamma^{li} & 0 \\ -e\underset{\sim}{N}\delta_i{}^l & N^l\delta_i{}^j + i\underset{\sim}{N}\tilde{\epsilon}^{lj}{}_i & 0 \\ 0 & 0 & i\underset{\sim}{N}\epsilon_a{}^{bc}\tilde{E}^l_c + N^l\delta_a{}^b \end{pmatrix} \partial_l \begin{pmatrix} \mathcal{C}_H \\ \tilde{\mathcal{C}}_{Mj} \\ \mathcal{C}_{Gb} \end{pmatrix}, \tag{3.1}$$

which forms a Hermitian matrix under the inner product rule of

$$\begin{aligned}&\langle(\mathcal{C}_H, \mathcal{C}_{Mi}, \mathcal{C}_{Ga})|(\mathcal{C}_H, \mathcal{C}_{Mi}, \mathcal{C}_{Ga})\rangle := \\ &\mathcal{C}_H\bar{\mathcal{C}}_H + \gamma^{ij}\mathcal{C}_{Mi}\bar{\mathcal{C}}_{Mj} + \delta^{ab}\mathcal{C}_{Ga}\bar{\mathcal{C}}_{Gb}.\end{aligned} \tag{3.2}$$

We note that non-principal parts of the dynamical equations of a set of $u^{(C)}_\alpha = (\mathcal{C}_H, \tilde{\mathcal{C}}_M, \mathcal{C}_G)$ include the terms of $u^{(D)}_\alpha$ and $u^{(C)}_\alpha$. These facts suggest that all constraint equations behave a well-posed feature. Iriondo *et al* [9] present a similar result, but our definition of the inner product does not include any coefficients. We also remark that all other definitions of the inner products throughout this article does not include any coefficients neither (i.e., obey the normal index notation). Thus we omit to express the inner product hereafter.

Following the BFHR's procedure [5], we next construct a dynamical system which evolves the spacetime to the constrained surface, $\mathcal{C}_H \approx \mathcal{C}_{Mi} \approx \mathcal{C}_{Ga} \approx 0$ as the attractor. We introduce new variables $(\lambda, \lambda_i, \lambda_a)$, as they obey the dissipative evolution equations

$$\partial_t\lambda = \alpha_1\,\mathcal{C}_H - \beta_1\,\lambda, \tag{3.3}$$

$$\partial_t\lambda_i = \alpha_2\,\tilde{\mathcal{C}}_{Mi} - \beta_2\,\lambda_i, \tag{3.4}$$

$$\partial_t\lambda_a = \alpha_3\,\mathcal{C}_{Ga} - \beta_3\,\lambda_a, \tag{3.5}$$

where $\alpha_i \neq 0$ (allowed to be complex numbers) and $\beta_i > 0$ (real numbers) are constants.

If we take $u^{(DL)}_\alpha = (\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a)$ as a set of dynamical variables, then the principal part of (3.3)-(3.5) can be written as

$$\partial_t \lambda \cong -i\alpha_1 \epsilon^{bcd} \tilde{E}^j_c \tilde{E}^l_d (\partial_l \mathcal{A}^b_j), \tag{3.6}$$

$$\partial_t \lambda_i \cong \alpha_2 [-e\delta^l_i \tilde{E}^j_b + e\delta^j_i \tilde{E}^l_b](\partial_l \mathcal{A}^b_j), \tag{3.7}$$

$$\partial_t \lambda_a \cong \alpha_3 \partial_l \tilde{E}^l_a. \tag{3.8}$$

The characteristic matrix of the system $u^{(DL)}_\alpha$ does not form a Hermitian matrix. However, if we modify the right-hand-side of the evolution equation of $(\tilde{E}^i_a, \mathcal{A}^a_i)$, then the set becomes a symmetric hyperbolic system. This is done by adding $\bar{\alpha}_3 \gamma^{il}(\partial_l \lambda_a)$ to the equation of $\partial_t \tilde{E}^i_a$, and by adding $i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d (\partial_l \lambda) + \bar{\alpha}_2(-e\gamma^{lm}\tilde{E}^a_i + e\delta^m_i \tilde{E}^{la})(\partial_l \lambda_m)$ to the equation of $\partial_t \mathcal{A}^a_i$. The final principal part, then, is written as

$$\partial_t \begin{pmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \\ \lambda \\ \lambda_i \\ \lambda_a \end{pmatrix} \cong \begin{pmatrix} \mathcal{M}(\tilde{E},\tilde{E})^l{}_a{}^{bi}{}_m & 0 & 0 & 0 & \bar{\alpha}_3 \gamma^{il}\delta_a{}^b \\ 0 & \mathcal{M}(\mathcal{A},\mathcal{A})^{l\,a}{}_{ib}{}^m & i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d & \bar{\alpha}_2 e(\delta^m_i \tilde{E}^{la} - \gamma^{lm}\tilde{E}^a_i) & 0 \\ 0 & -i\alpha_1 \epsilon_b{}^{cd} \tilde{E}^m_c \tilde{E}^l_d & 0 & 0 & 0 \\ 0 & \alpha_2 e(\delta^m_i \tilde{E}^l_b - \delta^l_i \tilde{E}^m_b) & 0 & 0 & 0 \\ \alpha_3 \delta_a{}^b \delta^l{}_m & 0 & 0 & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} \tilde{E}^m_b \\ \mathcal{A}^b_m \\ \lambda \\ \lambda_m \\ \lambda_b \end{pmatrix}. \tag{3.9}$$

Clearly, the solution $(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a) = (\tilde{E}^i_a, \mathcal{A}^a_i, 0, 0, 0)$ represents the original solution of the Ashtekar system. If the λs decay to zero after the evolution, then the solution also describes the original solution of the Ashtekar system in that stage. Since the dynamical system of $u^{(DL)}_\alpha$, (3.9), constitutes a symmetric hyperbolic form, the solutions to the λ-system are unique. BFHR showed analytically that such a decay of λs can be seen for λs sufficiently close to zero with a choice of appropriate combination of αs and βs, and that statement can be also applied to our system. Therefore, the dynamical system, (3.9), is useful for stabilizing numerical simulations from the point that it recovers constraint surface automatically.

IV. ASYMPTOTICALLY REAL-VALUED SYSTEM

We, next, extend the system, (3.9), to the one that also has the attractor in the real-valued surfaces. Since the dynamical system of $u^{(D)}_\alpha$ requires the triad reality condition to form a symmetric hyperbolic system, our purpose is to construct a system which asymptotically evolves into the triad-real-valued manifold.

In order to obtain such a system, we add two more new variables, λ^{ij} and λ^i_a, which satisfy the evolution equations

$$\partial_t \lambda^{ij} = \alpha_4\, W^{ij} - \beta_4\, \lambda^{ij}, \tag{4.1}$$

$$\partial_t \lambda^i_a = \alpha_5\, U^i_a - \beta_5\, \lambda^i_a, \tag{4.2}$$

corresponding to the secondary metric reality conditions, (2.8), and the primary triad reality conditions, (2.9), respectively. The equation (4.1) is necessary to complete this system since the term W^{ij} appears in the non-principal part of the equation $\partial_t U^i_a$. Thus, these two equations will guarantee the consistency with the secondary condition. The principal terms of (4.1) and (4.2) become

$$\partial_t \lambda^{ij} \cong \alpha_4 \mathrm{Re}(\epsilon^{abc} \tilde{E}^k_c \tilde{E}^{(j}_a \partial_k \tilde{E}^{i)}_b), \tag{4.3}$$

$$\partial_t \lambda^i_a \cong 0, \tag{4.4}$$

under the assumption of the triad reality condition.

A set of variables $u^{(DLR)}_\alpha = (\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a, \lambda^{ij}, \lambda^i_a)$, then, forms a symmetric hyperbolic system if we further modify the equations for $\partial_t \tilde{E}^i_a$ similarly to the case of (3.9). The final set of equations in the matrix form can be written as

$$\partial_t \begin{pmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \\ \lambda \\ \lambda_i \\ \lambda_a \\ \lambda^{ij} \\ \lambda^i_a \end{pmatrix} \cong \begin{pmatrix} & & & \bar{\alpha}_4 \mathrm{Re}(\epsilon_a{}^c{}_d \tilde{E}^l_c \tilde{E}^d_{(m}\delta^i_{n)}) & 0 \\ & \mathcal{M}(3.9) & & 0 & \vdots \\ & & & \vdots & \vdots \\ \alpha_4 \mathrm{Re}(\epsilon^{bcd} \tilde{E}^l_c \tilde{E}^{(i}_d \delta^{j)}_m) & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} \tilde{E}^m_b \\ \mathcal{A}^b_m \\ \lambda \\ \lambda_m \\ \lambda_b \\ \lambda^{mn} \\ \lambda^m_b \end{pmatrix}, \tag{4.5}$$

where $\mathcal{M}(3.9)$ denotes the matrix in the right-hand-side of (3.9).

Clearly, the solution $(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a, \lambda^{ij}, \lambda^i_a) = (\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a, 0, 0)$ represents the original solution of (3.9). The same discussion in the previous section can be applied also here. Therefore, we expect that the system (4.5) controls the violation of the triad reality condition during the time integration.

We remark that a reduced set of the variables $u^{(DR)}_\alpha = (\tilde{E}^i_a, \mathcal{A}^a_i, \lambda^{ij}, \lambda^i_a)$ does not work for the purpose of controlling reality condition, since the secondary condition of the reality requires the constraints to be satisfied.

V. CONCLUDING REMARKS

We showed a set of dynamical equations, which has the constraint surface as its attractor, by introducing new additional variables that obey dissipative equations of motion. Casting on BFHR's analytical proof [5], we expect that this set of equations is robust against a perturbative error of the constraint equations. Thus, the system may be useful for future numerical studies with its stability property.

We also showed an advanced set of equations that has its attractor also in the real-valued surface. Since our symmetric hyperbolic system of the original Ashtekar's variables requires the reality condition on the triad, new system is designed as such a way. The same above discussion can be applied to this advanced set, and we expect asymptotically real-valued feature in its evolution.

The problem of these systems might be that they require many additional variables. From the point of numerical applications, this claim would not be so serious problem, as we see the success of the dissipative maximal slicing condition [11] ('K-driver' in the literature). However, in order to find out clear geometrical meanings of the extension of the λ-system, we are now trying to reduce the number of the variables. We will report this effort elsewhere.

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[1] A. Ashtekar, Phys. Rev. Lett. **57**, 2244 (1986); Phys. Rev. **D36**, 1587 (1987); *Lectures on Non-Perturbative Canonical Gravity* (World Scientific, Singapore, 1991).
[2] A. Ashtekar, J. D. Romano and R. S. Tate, Phys. Rev. **D40**, 2572 (1989).
[3] G. Yoneda and H. Shinkai, Class. Quantum Grav. **13**, 783 (1996).
[4] G. Yoneda, H. Shinkai and A. Nakamichi, Phys. Rev. **D56**, 2086 (1997).
[5] O. Brodbeck, S. Frittelli, P. Hübner and O.A. Reula, J. Math. Phys. **40**, 909 (1999).
[6] S. Detweiler, Phys. Rev. **D35**, 1095 (1987).
[7] G. Yoneda and H. Shinkai, Phys. Rev. Lett. **82**, 263 (1999); see also gr-qc/9901053.
[8] M.S. Iriondo, E.O. Leguizamón and O.A. Reula, Phys. Rev. Lett. **79**, 4732 (1997).
[9] M.S. Iriondo, E.O. Leguizamón and O.A. Reula, Adv. Theor. Math. Phys. **2**, 1075 (1998).
[10] S. Frittelli, Phys. Rev. **D55**, 5992 (1997).
[11] J. Balakrishna *et al*, Class. Quant. Grav. **96**, L135 (1997).