Quasi-Spherical Approximation for Rotating Black Holes

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ABSTRACT

We numerically implement a quasi-spherical approximation scheme for computing gravitational waveforms for coalescing black holes, testing it against angular momentum by applying it to Kerr black holes. As error measures, we take the conformal strain and specific energy due to spurious gravitational radiation. The strain is found to be monotonic rather than wavelike. The specific energy is found to be at least an order of magnitude smaller than the 1% level expected from typical black-hole collisions, for angular momentum up to about 60% to 70% of the maximum, for an initial surface as close as $r = 3m$.

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1 Introduction

Quasi-Spherical Approximation scheme

S.A.Hayward, Phys. Rev. **D 61**, 101503 (2000).

The scheme truncates the Einstein equations by removing second-order terms which would vanish in a spherically symmetric space-time.

- dual-null formulation is adapted to radiation extraction.
- only ordinary differential equations need be solved
- providing a computationally inexpensive estimate of the gravitational waveforms produced by a black-hole collision, given a full numerical simulation up to (or close to) coalescence, or an analytical model thereof.
- No prescribed background is required and that arbitrarily rapid dynamical processes (close to spherical symmetry) are allowed.

How well the scheme handles deviations from spherical symmetry?

against angular momentum? (cf. coalescing black holes)

2 Formulation and Approximation

2.1 Dual-null formulation (Hayward, 1993)

In a coordinate basis $(u_+, u_-; e_i)$ such that $u_\pm = \partial/\partial x^\pm$, where $e_i = \partial/\partial x^i$ is a basis for the transverse surfaces, the metric takes the form

$$g \;=\; h_{ij}(dx^i + s^i_+ dx^+ + s^i_- dx^-) \otimes (dx^j + s^j_+ dx^+ + s^j_- dx^-) - 2e^{-f} dx^+ \otimes dx^-. \qquad (1)$$

Then $(h, f, s_\pm)$ are configuration fields and the independent momentum fields are found to be linear combinations of

$$\begin{aligned}
\text{expansions} \quad \theta_\pm &= *L_\pm{*1}, & (2)\\
\text{shears} \quad \sigma_\pm &= \bot L_\pm h - \theta_\pm h, & (3)\\
\text{inaffinities} \quad \nu_\pm &= L_\pm f, & (4)\\
\text{twist} \quad \omega &= \tfrac{1}{2} e^f h([l_-, l_+]), & (5)
\end{aligned}$$

where $*$ is the Hodge operator of h and $L_\pm$ is shorthand for the Lie derivative along the null normal vectors

$$l_\pm = u_\pm - s_\pm = e^{-f} g^{-1}(n^\mp). \qquad (6)$$

The fields $(\theta_\pm, \sigma_\pm, \nu_\pm, \omega)$ encode the extrinsic curvature of the dual-null foliation.

2.2 Quasi-spherical approximation

Of the dynamical fields and operators introduced above,

- $(s_\pm, \sigma_\pm, \omega, D)$ vanish in spherical symmetry $\quad$ (D=covariant derivative of h)
- while $(h, f, \theta_\pm, \nu_\pm, \Delta_\pm)$ generally do not.

The quasi-spherical approximation consists of linearizing in $(s_\pm, \sigma_\pm, \omega, D)$, i.e. setting to zero any second-order terms in these quantities.

This yields

- a greatly simplified truncation of the full field equations, the first-order dual-null form of the vacuum Einstein system.
- the truncated equations decouple into a three-level hierarchy, the last level being irrelevant to determining the gravitational waveforms.

We also introduce a conformal factor Ω and related quantities by

$$h_{ij} = \Omega^{-2} k_{ij}, \qquad \vartheta_\pm = \Omega^{-1} \theta_\pm, \qquad \varsigma_\pm = \Omega \sigma_\pm, \tag{7}$$

then the remaining equations are the quasi-spherical equations

$$\Delta_\pm \Omega = -\tfrac{1}{2}\Omega^2 \vartheta_\pm, \tag{8a}$$
$$\Delta_\pm f = \nu_\pm, \tag{8b}$$
$$\Delta_\pm \vartheta_\pm = -\nu_\pm \vartheta_\pm, \tag{8c}$$
$$\Delta_\pm \vartheta_\mp = -\Omega(\tfrac{1}{2}\vartheta_+ \vartheta_- + e^{-f}), \tag{8d}$$
$$\Delta_\pm \nu_\mp = -\Omega^2(\tfrac{1}{2}\vartheta_+ \vartheta_- + e^{-f}), \tag{8e}$$

and the linearized equations

$$\Delta_\pm k = \Omega \varsigma_\pm, \tag{9}$$
$$\Delta_\pm \varsigma_\mp = -\tfrac{1}{2}\Omega \vartheta_\mp \varsigma_\pm. \tag{10}$$

These are all ordinary differential equations; no transverse D derivatives occur. Thus we have an effectively two-dimensional system to be integrated independently at each angle of the sphere.

The initial-data formulation is based on a spatial surface S orthogonal to $l_\pm$ and the null hypersurfaces $\Sigma_\pm$ generated from S by $l_\pm$, assumed future-pointing. The initial data for the above equations are $(\Omega, f, k, \vartheta_\pm)$ on S and $(\varsigma_\pm, \nu_\pm)$ on $\Sigma_\pm$.

2.3 Strain

The variables are directly related to physically measurable quantities. In particular,

$$\epsilon = \frac{1}{2}\int_\gamma \dot{k} d\tau \tag{9}$$

is the transverse strain tensor measured along a worldline γ normal to the transverse surfaces, where $\dot{k} = \perp L_\lambda k$ in terms of a vector $\lambda = \partial/\partial\tau$ tangent to γ. Writing $\lambda = a^+ l_+ + a^- l_-$ yields

$$\epsilon = \frac{1}{2}\int_\gamma \Omega(a^+ \varsigma_+ + a^- \varsigma_-) d\tau. \tag{10}$$

It is convenient to use the conformal strain tensor

$$\varepsilon_+ = \frac{1}{2}\int \varsigma_{-\theta\theta} dx^-, \qquad \varepsilon_\times = \frac{1}{2}\int \varsigma_{-\theta\phi} dx^-, \tag{11}$$

where the integal is at constant x^+.

In order to compare with observational results, the displacements to be measured by an interferometer are

$$\frac{\delta\ell}{\ell} = \epsilon(e, e), \qquad \text{where} \qquad \epsilon = \frac{\varepsilon}{R}, \tag{12}$$

where R is the distance between the source and the detector, the unit vector e is the direction of displacement. In a weak gravitational field, one may use Newtonian physics, where ϵ reduces to the Newtonian strain tensor.

2.4 Energy

We define the energy flux ϕ of the gravitational waves, or more conveniently, the conformal energy flux $\varphi = \Omega^{-2}\phi$, as the 1-form $\varphi = \varphi_+ dx^+ + \varphi_- dx^-$, where

$$\varphi_\pm = -\frac{e^f \vartheta_\mp k^{ab} k^{cd} \varsigma_{\pm ac} \varsigma_{\pm bd}}{64\pi}. \tag{13}$$

These expressions have the same form as those for the conformal Bondi flux at $\Im^\mp$, but we propose using them locally. Then ϕ_- is the outgoing flux and ϕ_+ is the ingoing flux.

The corresponding energy E of the gravitational waves is then given by

$$\Delta_\pm E = \oint \varphi_\pm \sin\theta d\theta \wedge d\phi \tag{14}$$

with the initial condition $E|_S = 0$.

Thus the Bondi energy at $\Im^+$ is $E + E_0$, where E_0 is the Bondi energy at the intersection with Σ_+, which in the Kerr case will be just the ADM mass m. This is the fraction of the original mass-energy which has radiated away.

3 Model and Numerical Procedures

3.1 Model: Kerr black hole

$$ds^2 = -\frac{\Delta}{\Sigma}[dt - a\sin^2\theta d\phi]^2 + \frac{\sin^2\theta}{\Sigma}[(r^2+a^2)d\phi - adt]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \tag{15}$$

$$\Delta = r^2 - 2mr + a^2, \qquad \Sigma = r^2 + a^2\cos^2\theta$$

3.2 Initial data

An explicit dual-null form of the Kerr metric is not known as the Kruskal form. However, we can find the initial data analytically as functions of (r, θ), then convert to the required functions of $(x^\pm, \theta)$.

The null normal vectors are initially given by

$$l_\pm|_{\Sigma_\pm} = \left(\frac{\Sigma}{2(\Delta - a^2\sin^2\theta)}\right)^{1/2} \partial_t \pm \left(\frac{\Delta}{2\Sigma}\right)^{1/2} \partial_r \tag{16}$$

where the normalization is such that $f|_S = 0$. We also fix

$$\nu_\pm|_{\Sigma_\pm} = 0, \tag{17}$$

which means that $x^\pm|_{\Sigma_\pm}$ are affine parameters. This implies $f|_{\Sigma_\pm} = 0$, fixing $l_\mp|_{\Sigma_\pm}$ and therefore locally determining the dual-null foliation.

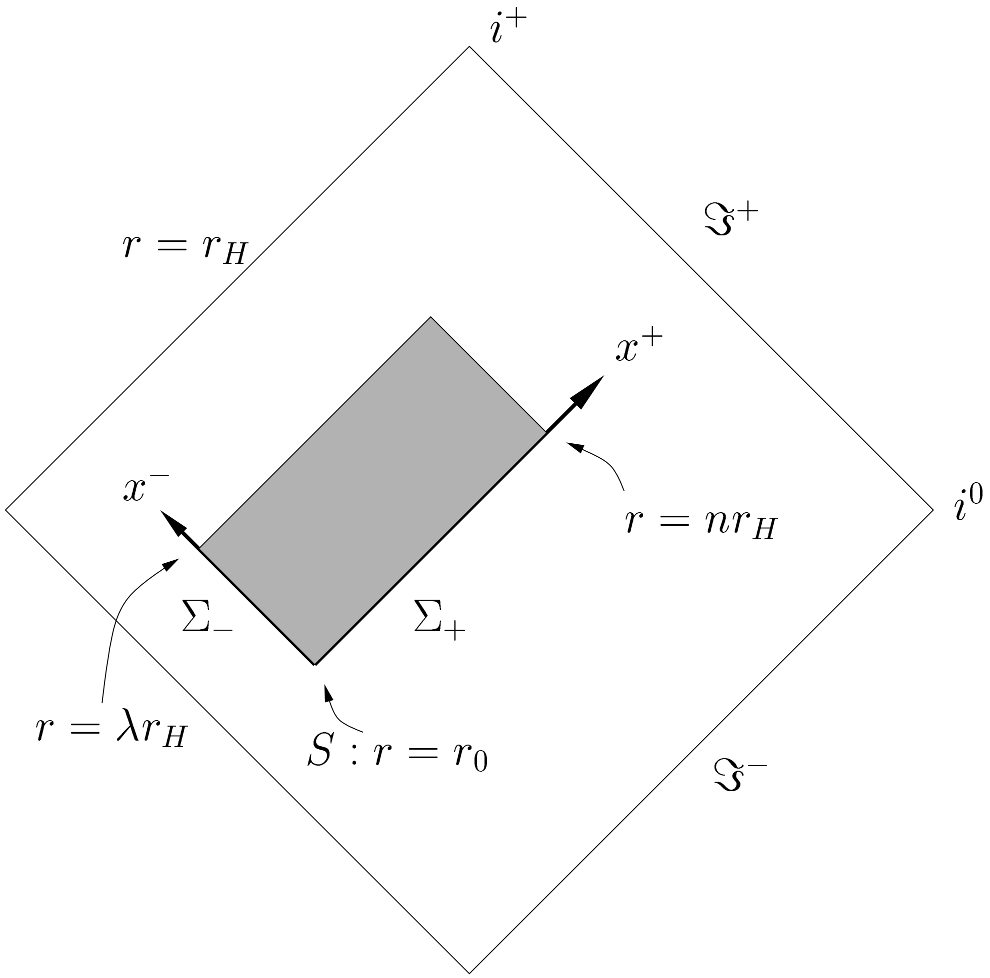

Figure 1: The region of numerical integration is shown as the shaded region in the picture. Initial data is prescribed on a spatial surface S of constant Boyer-Lindquist $r = r_0$ and t, and the null hypersurfaces $\Sigma_\pm$ generated from it. On Σ_- (Σ_+), the x^- (x^+) coordinate is set so as to cover the region $\lambda r_H \leq r \leq r_0$ ($r_0 \leq r \leq nr_H$), where $1 < \lambda \approx 1$ and $n \gg 1$ are constants to be set by hand.

The quasi-spherical conformal factor is

$$\Omega = \left((r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right)^{-1/4} \tag{18}$$

which is real and positive. Then we obtain the conformal metric

$$k = \Omega^2 \Sigma d\theta^2 + \frac{\sin^2\theta}{\Omega^2\Sigma} d\phi^2, \tag{19}$$

and the conformally rescaled expansions and shears

$$\vartheta_\pm|_S = \pm\sqrt{\frac{\Delta}{2\Sigma}}\Omega^3 U, \tag{20}$$

$$\varsigma_\pm|_{\Sigma_\pm} = \pm\sqrt{\frac{\Delta}{2\Sigma}}\Omega^5 V a^2 \sin^2\theta \left(d\theta^2 - \frac{\sin^2\theta}{\Omega^4\Sigma^2} d\phi^2\right), \tag{21}$$

where $U = 2r(r^2+a^2) - (r-m)a^2\sin^2\theta, \quad V = r^3 + 3mr^2 + a^2(r-m)\cos^2\theta.$

To complete the initial data construction, we need to know the initial data on $\Sigma_\pm$ as functions of $(x^\pm, \theta)$. So we need to know $r|_{\Sigma_\pm}$ as functions of $(x^\pm, \theta)$. This gives

$$x_\pm|_{\Sigma_\pm} = \pm\int_{r_0}^{r} \left(\frac{2\Sigma}{\Delta}\right)^{1/2} dr', \tag{22}$$

where the integral is along a curve of constant (θ, ϕ). Note that in the equatorial plane $\theta = \pi/2$:

$$x^\pm|_{\Sigma_\pm} = \pm\sqrt{2}\left(\sqrt{\Delta} + m\ln(r - m + \sqrt{\Delta})\right) \mp c. \tag{23}$$

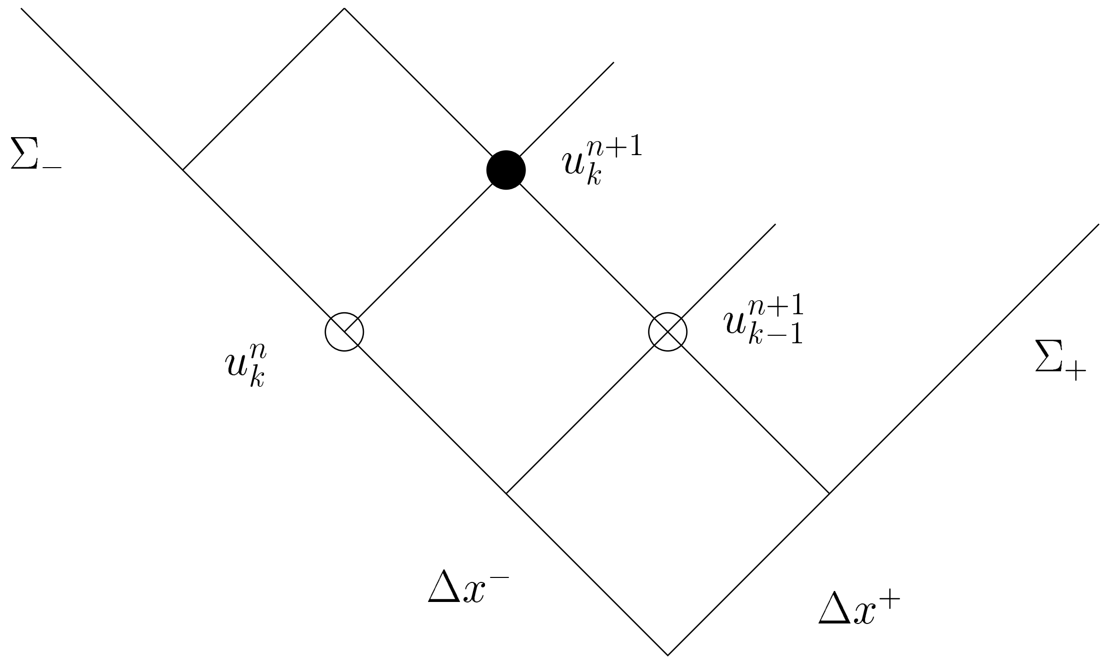

Figure 2: The dual-null integration scheme. In order to obtain the data at grid u_k^{n+1}, we need both u_k^n and u_{k-1}^{n+1}.

4 Numerical Results

4.1 Asymptotic behavior: Specific energy

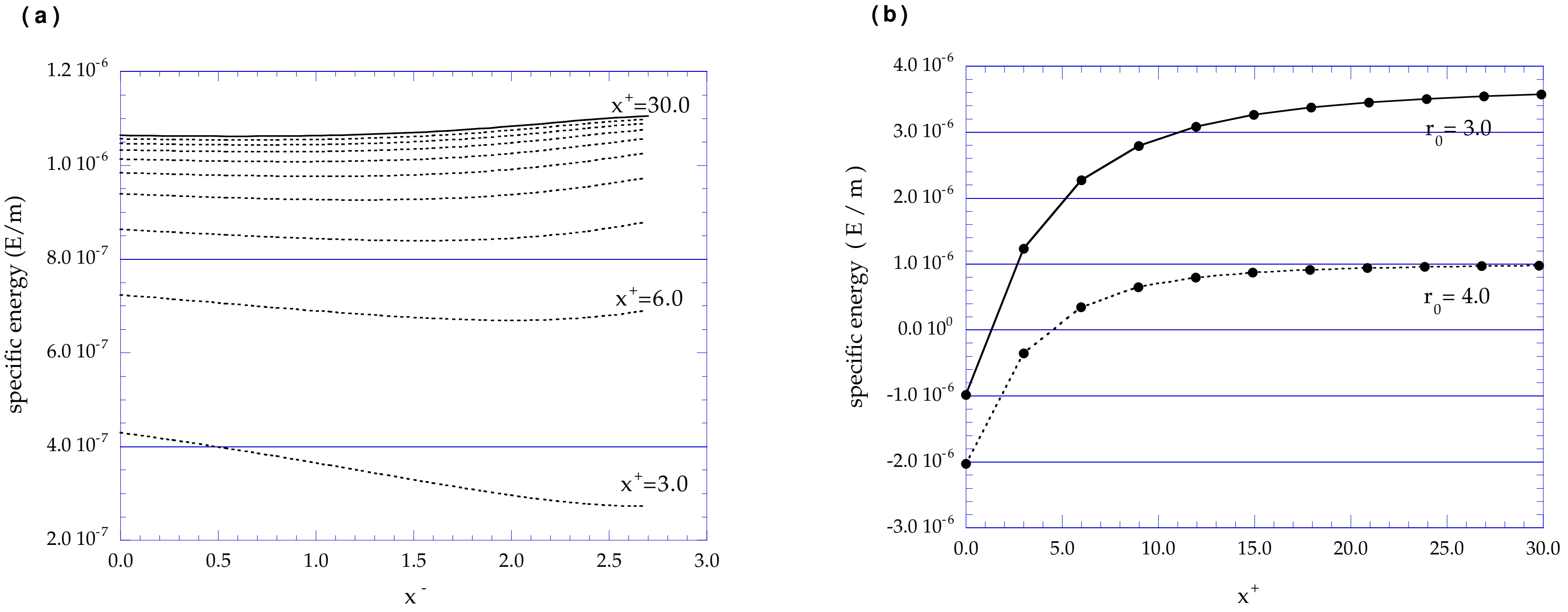

Figure 3: Specific energy E/m for $a/m = 0.1$. (a) E/m is plotted as a function of the ingoing null coordinate x^- for each constant outgoing null coordinate $x^+ = 0.0, 3.0, \cdots, 30.0$. We set $r_0 = 4.0$ for this plot. (b) The integrated E/m over the ingoing null coordinate x^- is shown as a function of the outgoing null coordinate x^+. We show both $r_0 = 3.0$ and 4.0 cases. We see that the specific energy converges to a particular positive value in the x^+ direction, as expected.

4.2 Asymptotic behavior: Conformal strain ε_+

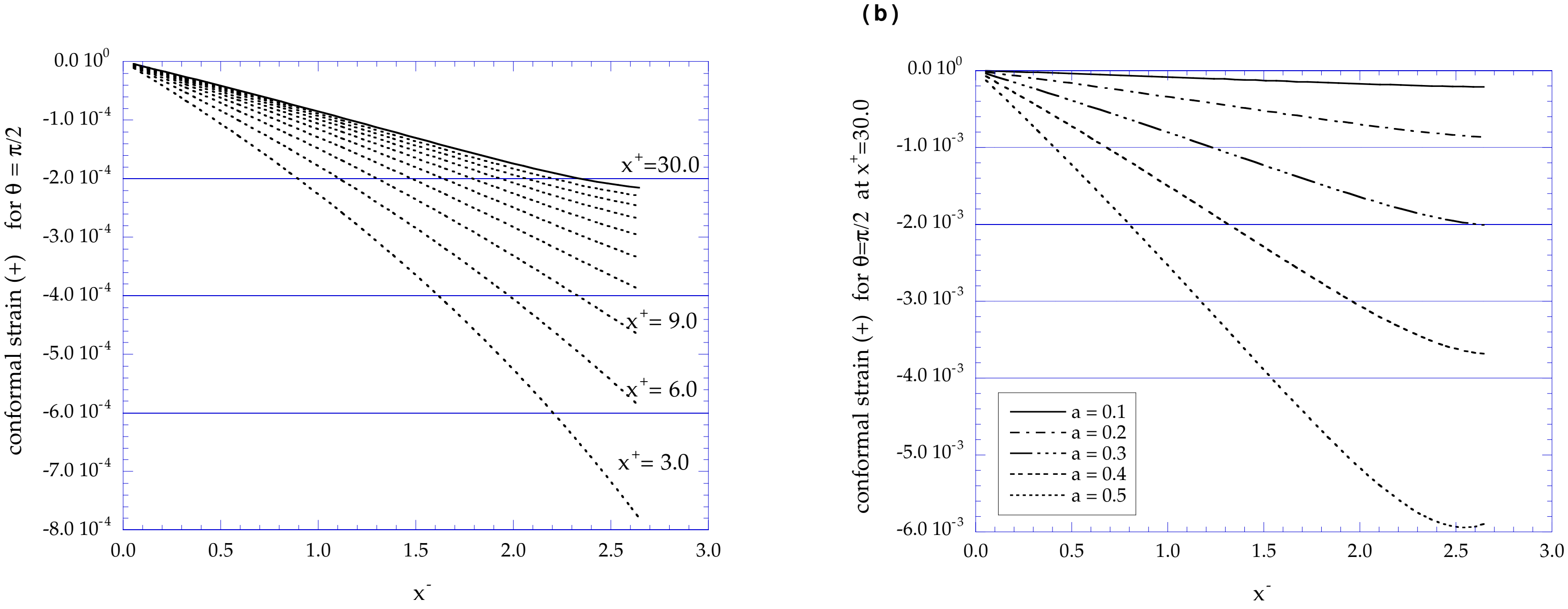

Figure 4: (a) Conformal strain ε_+ for $a/m = 0.1$ and $r_0 = 4.0$ for the equatorial plane $\theta = \pi/2$, showing the convergence of these lines in the x^+ direction. Lines are of $x^+ = 3.0, 6.0, 9.0, \cdots$, and 30.0. We remark that these lines are not wave-like. (b) The dependence of ε_+ on a/m. The lines are for the data at $x^+ = 30.0$ for $\theta = \pi/2$.

4.3 Total specific energy E/m due to spurious radiation

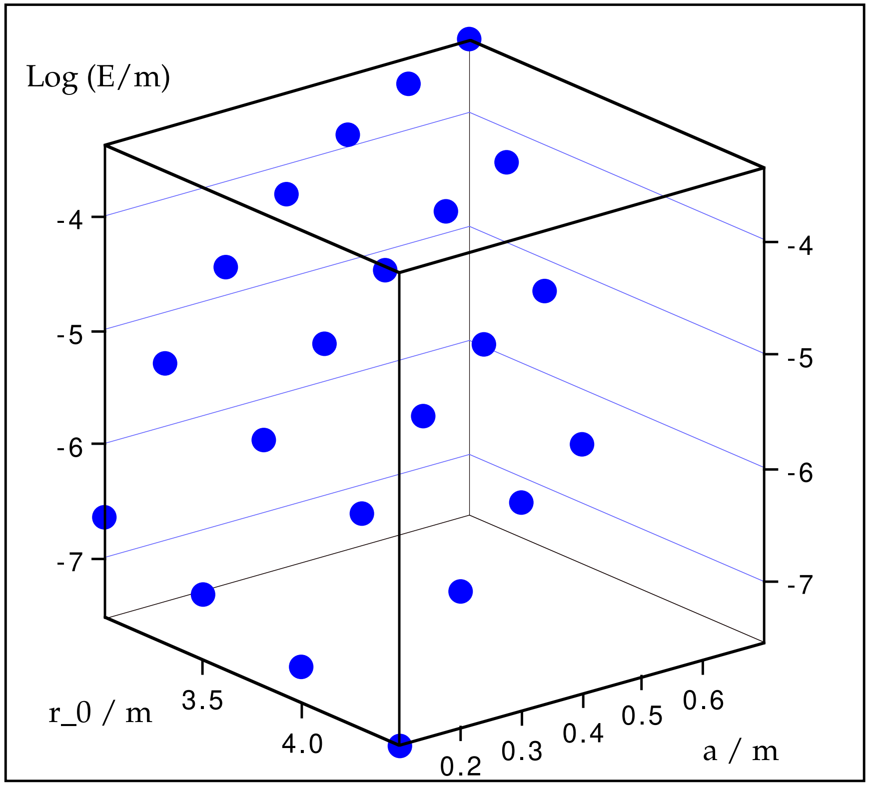

Figure 5: Logarithmic plot of specific energy E/m due to spurious radiation, as a function of a/m and r_0/m. Energy is measured at $x^+ = 30$, and the plotted range is $r_0/m \in [3.0, 5.0]$ and $a/m \in [0.1, 0.7]$.

5 Concluding Remarks

- We tested the quasi-spherical approximation by applying it to Kerr black holes. We numerically calculated the strain and energy flux of the spurious gravitational radiation produced from this approximation.
- We showed
    1. it converges quickly due to our conformal variables
    2. it does not behave like wave-like oscillations
    3. the total radiated energy is at least an order of magnitude less than the gravitational radiation emission estimated from coalescing binary black holes, according to the close-limit approximation.
- These results suggest that the spurious radiation does not fatally affect the gravitational waveform estimation. It might not affect the waveform estimation at all, and we might extract its effect to the total energy by subtracting the amount we showed in Fig.5.

These facts directly encourage the robustness of the quasi-spherical approximation.