Will hyperbolic formulations help numerical relativity?
– Experiments using Ashtekar's connection variables – [*]

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(November 17, 2000)

Abstract

In order to perform accurate and stable long-term numerical integration of the Einstein equations, several hyperbolic systems have been proposed. We here report our numerical comparisons between weakly hyperbolic, strongly hyperbolic, and symmetric hyperbolic systems based on Ashtekar's connection variables. The primary advantage for using this connection formulation is that we can keep using the same dynamical variables for all levels of hyperbolicity. Our numerical code demonstrates gravitational wave propagation in plane symmetric spacetimes, and we compare the accuracy of the simulation by monitoring the violation of the constraints. By comparing with results obtained from the weakly hyperbolic system, we observe the strongly and symmetric hyperbolic system show better numerical performance (yield less constraint violation), but not so much difference between the latter two.

We also study asymptotically constrained systems for numerical integration of the Einstein equations, which are intended to be robust against perturbative errors for the free evolution of the initial data. First, we examine the previously proposed "λ-system", which introduces artificial flows to constraint surfaces based on the symmetric hyperbolic formulation. We show that this system works as expected for the wave propagation problem in the Maxwell system and in general relativity using Ashtekar's connection formulation. Second, we propose a new mechanism to control the stability, which we call the "adjusted system". This is simply obtained by adding constraint terms in the dynamical equations and adjusting its multipliers. We explain why a particular choice of multiplier reduces the numerical errors from non-positive or pure-imaginary eigenvalues of the adjusted constraint propagation equations. This "adjusted system" is also tested in the Maxwell system and in the Ashtekar's system. This mechanism affects more than the system's symmetric hyperbolicity.

1 Introduction

Numerical relativity – solving the Einstein equations numerically – is now an essential field in gravity research. The current mainstream of numerical relativity is to demonstrate the final phase of compact binary objects related to gravitational wave observations [3], and these efforts are now again shedding light on the mathematical structure of the Einstein equations.

Up to a couple of years ago, the standard Arnowitt-Deser-Misner (ADM) decomposition of the Einstein equations was taken as the standard formulation for numerical relativists. Difficulties in accurate/stable long-term evolutions were supposed to be overcome by choosing proper gauge conditions and boundary conditions. Recently, however, several numerical experiments show that the standard ADM is not the best formulation for numerics, and finding a better formulation has become one of the main research topics.

One direction in the community is to apply conformally decoupled and tracefree re-formulation of ADM system which were first used by Nakamura *et al.* [4]. Although there is an effort to show why this re-formulation is better than ADM, we do not yet know this method is robust for all situations.

[*]Proceedings of *The 10th Workshop on General Relativity and Gravitation* at Osaka, Japan, September 2000. This report is based on our works [1] and [2].

Another alternative approach to ADM is to formulate the Einstein equations to reveal hyperbolicity [5]. A certain kind of hyperbolicity of the dynamical equations is essential to analyze their propagation features mathematically, and are known to be useful in numerical approximations. Several hyperbolic formulations have been proposed to re-express the Einstein equations, with different levels: weakly, strongly and symmetric hyperbolic systems. Several numerical tests were also performed in this direction.

The following questions, therefore, naturally present themselves (cf. [6]):

(1) Does hyperbolicity actually contribute to the numerical accuracy/stability?

(2) If so, which level of hyperbolic formulation is practically useful for numerical applications? (or does the symmetric hyperbolicity solve all the difficulties?)

(3) Are there any other approaches to improve the accuracy/stability of the system?

In this report, we try to answer these questions with our simple numerical experiments. Such comparisons are appropriate when the fundamental equations are cast in the same interface, and that is possible using Ashtekar's connection variables [7, 14]. More precisely, the authors' recent studies showed the following:

(a) the original set of dynamical equations proposed by Ashtekar already forms a weakly hyperbolic system [9],

(b) by requiring additional gauge conditions *or* adding constraints to the dynamical equations, we can obtain a strongly hyperbolic system [9],

(c) by requiring additional gauge conditions *and* adding constraints to the dynamical equations, we can obtain a symmetric hyperbolic system [9, 10], and finally

(d) based on the above symmetric hyperbolic system, we can construct a set of dynamical systems which is robust against perturbative errors for constraints and reality conditions [11] (*aka.* λ-system [12]).

Based on the above results (a)-(c), we developed a numerical code which handles gravitational wave propagation in the plane symmetric spacetime. We performed the time evolutions using the above three levels of Ashtekar's dynamical equations together with the standard ADM equation. We compare these for accuracy and stability by monitoring the violation of the constraints. We also show the demonstrations of our λ-system (above (d)), together with new proposal for controlling the stability.

It is worth remarking that this study is the first one which shows full numerical simulations of Lorentzian spacetime using Ashtekar's connection variables. This research direction was suggested [13] soon after Ashtekar completed his formulation, but has not yet been completed. Historically, an application to numerical relativity of the connection formulation was also suggested [14, 15] using Capovilla-Dell-Jacobson's version of the connection variables [16], which produce an direct relation to Newman-Penrose's Ψs. However here we apply Ashtekar's original formulation, because we know how to treat its reality conditions in detail, and how they form hyperbolicities. We will also describe the basic numerical procedures in this paper.

Due to the limited space, we have to omit details in some part in this report. More complete discussion can be found in our recent articles [1, 2].

2 Field equations to be compared

2.1 Hyperbolic formulations

We begin by providing our definitions of hyperbolic systems.

Definition 1 *We assume a certain set of (complex) variables u_α ($\alpha = 1, \cdots, n$) forms a first-order (quasi-linear) partial differential equation system,*

$$\partial_t u_\alpha = J^{l\beta}{}_\alpha(u)\, \partial_l u_\beta + K_\alpha(u), \tag{1}$$

where J (the characteristic matrix) and K are functions of u but do not include any derivatives of u. We say that the system (1) is:

(I). **weakly hyperbolic**, *if all the eigenvalues of the characteristic matrix are real.*

(II). **strongly (diagonalizable) hyperbolic**, *if the characteristic matrix is diagonalizable and has all real eigenvalues.*

(III). **symmetric hyperbolic**, *if the characteristic matrix is a Hermitian matrix.*

We treat $J^{l\beta}{}_{\alpha}$ as a $n \times n$ matrix when the l-index is fixed. We say λ^l is an eigenvalue of $J^{l\beta}{}_{\alpha}$ when the characteristic equation, $\det(J^{l\beta}{}_{\alpha} - \lambda^l \delta^{\beta}{}_{\alpha}) = 0$, is satisfied. The eigenvectors, p^{α}, are given by solving $J^{l\alpha}{}_{\beta}\, p^{l\beta} = \lambda^l p^{l\alpha}$.

These three classes have the relation (III) $\in$ (II) $\in$ (I). The symmetric hyperbolic system gives us the energy integral inequalities, which are the primary tools for studying well-posedness of the system.

For more concrete descriptions for each systems, please refer our paper [9]. It might be worth remarking that the standard ADM formulation cannot be a first-order hyperbolic form, since there are curvature terms in the r.h.s. of dynamical equations.

2.2 Ashtekar's dynamical equations: three levels of hyperbolicity

The key feature of Ashtekar's formulation of general relativity [7] is the introduction of a self-dual connection as one of the basic dynamical variables. The new basic variables are the densitized inverse triad, $\tilde{E}^i_a$, and the SO(3,C) self-dual connection, $\mathcal{A}^a_i$, where the indices $i, j, \cdots$ indicate the 3-spacetime, and $a, b, \cdots$ are for SO(3) space. The total four-dimensional spacetime is described together with the gauge variables $\underset{\sim}{N}, N^i, \mathcal{A}^a_0$, which we call the densitized lapse function, shift vector and the triad lapse function. The system has three constraint equations,

$$\mathcal{C}^{\rm ASH}_{H} := (i/2)\epsilon^{ab}{}_{c}\, \tilde{E}^i_a \tilde{E}^j_b F^c_{ij} \approx 0, \tag{2}$$

$$\mathcal{C}^{\rm ASH}_{Mi} := -F^a_{ij}\tilde{E}^j_a \approx 0, \tag{3}$$

$$\mathcal{C}^{\rm ASH}_{Ga} := \mathcal{D}_i \tilde{E}^i_a \approx 0, \tag{4}$$

which are called the Hamiltonian, momentum, and Gauss constraints equation, respectively. The dynamical equations for a set of $(\tilde{E}^i_a, \mathcal{A}^a_i)$ are

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_{a}\, \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^{c}\, \tilde{E}^i_c, \tag{5}$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_{c}\, \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0, \tag{6}$$

where $F^a_{ij} := 2\partial_{[i}\mathcal{A}^a_{j]} - i\epsilon^a{}_{bc}\,\mathcal{A}^b_i \mathcal{A}^c_j$ is the curvature 2-form.

We have to consider the reality conditions when we use this formalism to describe the classical Lorentzian spacetime. Fortunately, the metric will remain on its real-valued constraint surface during time evolution automatically if we prepare initial data which satisfies the reality condition. More practically, we further require that triad is real-valued. But again this reality condition appears as a gauge restriction on $\mathcal{A}^a_0$[17], which can be imposed at every time step. In our actual simulation, we prepare our initial data using the standard ADM approach, so that we have no difficulties in maintaining these reality conditions.

The set of dynamical equations (5) and (6) [the *original* equations] does have a weakly hyperbolic form [9], so that we regard the mathematical structure of the original equations as one step advanced from the standard ADM. Further, we can construct higher levels of hyperbolic systems by restricting the gauge condition and/or by adding constraint terms, $\mathcal{C}^{\rm ASH}_{H}, \mathcal{C}^{\rm ASH}_{Mi}$ and $\mathcal{C}^{\rm ASH}_{Ga}$, to the original equations [9]. We extract only the final expressions here.

In order to obtain a symmetric hyperbolic system [1)], we add constraint terms to the right-hand-side of (5) and (6). The adjusted dynamical equations,

$$\partial_t \tilde{E}^i_a = -i\mathcal{D}_j(\epsilon^{cb}{}_{a}\, \underset{\sim}{N} \tilde{E}^j_c \tilde{E}^i_b) + 2\mathcal{D}_j(N^{[j}\tilde{E}^{i]}_a) + i\mathcal{A}^b_0 \epsilon_{ab}{}^{c}\, \tilde{E}^i_c + \kappa_1 P^i{}_{ab}\, \mathcal{C}^{\rm ASH\, b}_{G}, \tag{7}$$

[1)] Iriondo et al [18] presented a symmetric hyperbolic expression in a different form. The differences between ours and theirs are discussed in [9, 10].

$$\text{where} \qquad P^i{}_{ab} \equiv N^i \delta_{ab} + i\underset{\sim}{N} \epsilon_{ab}{}^c \tilde{E}^i_c,$$

$$\partial_t \mathcal{A}^a_i = -i\epsilon^{ab}{}_c \underset{\sim}{N} \tilde{E}^j_b F^c_{ij} + N^j F^a_{ji} + \mathcal{D}_i \mathcal{A}^a_0 + \kappa_2 Q^a_i \mathcal{C}^{\rm ASH}_H + \kappa_3 R_i{}^{ja}\, \mathcal{C}^{\rm ASH}_{Mj}, \tag{8}$$

$$\text{where} \qquad Q^a_i \equiv e^{-2} \underset{\sim}{N} \tilde{E}^a_i, \qquad R_i{}^{ja} \equiv i e^{-2} \underset{\sim}{N} \epsilon^{ac}{}_b \tilde{E}^b_i \tilde{E}^j_c$$

form a symmetric hyperbolicity if we further require $\kappa_1 = \kappa_2 = \kappa_3 = 1$ and the gauge conditions,

$$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i, \qquad \partial_i N = 0. \tag{9}$$

We remark that the adjusted coefficients, $P^i{}_{ab}, Q^a_i, R_i{}^{ja}$, for constructing the symmetric hyperbolic system are uniquely determined, and there are no other additional terms (say, no $\mathcal{C}^{\rm ASH}_H, \mathcal{C}^{\rm ASH}_M$ for $\partial_t \tilde{E}^i_a$, no $\mathcal{C}^{\rm ASH}_G$ for $\partial_t \mathcal{A}^a_i$) [9]. The gauge conditions, (9), are consequences of the consistency with (triad) reality conditions.

We can also construct a strongly hyperbolic system by restricting to a gauge $N^l \neq 0, \pm N\sqrt{\gamma^{ll}}$ (where γ^{ll} is the three-metric and we do not sum indices here) for the original equations (5), (6). Or we can also construct from the adjusted equations, (7) and (8), together with the gauge condition

$$\mathcal{A}^a_0 = \mathcal{A}^a_i N^i. \tag{10}$$

As for the strongly hyperbolic system, we hereafter take the latter expression.

	system	variables	Eqs of motion	remark
I	Ashtekar (weakly hyp.)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(5), (6) (original)	"original Ashtekar"
II	Ashtekar (strongly hyp.)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(7), (8) (adjusted with $\kappa = 1$)	(10) required
III	Ashtekar (symmetric hyp.)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(7), (8) (adjusted with $\kappa = 1$)	(9) required
	Ashtekar (lambda)	$(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda\text{s})$	(17), (11) - (13)	
	Ashtekar (adjusted)	$(\tilde{E}^i_a, \mathcal{A}^a_i)$	(7), (8) (adjusted with different κ)	

Table 1: List of systems that we compare in this article.

2.3 Alternative approaches to obtain robust evolution 1: "λ-system"

Based on the symmetric hyperbolic feature of the system, Brodbeck, Frittelli, Hübner and Reula (BFHR) proposed an alternative dynamical system to obtain stable evolutions, which they named "λ-system" [12]. The idea of this approach is to introduce additional variables, λ, which indicates the violation of the constraints, and to construct a symmetric hyperbolic system for both the original variables and λs together with imposing dissipative dynamical equations for λs. BFHR constructed their λ-system based on Frittelli-Reula's symmetric hyperbolic formulation of the Einstein equations [19], and we [11] have also presented the similar system for the Ashtekar's connection formulation based on the above symmetric hyperbolic expression. Here we present our system which evolves the spacetime to the constraint surface, $\mathcal{C}_H \approx \mathcal{C}_{Mi} \approx \mathcal{C}_{Ga} \approx 0$ as the attractor. In [11], we also present a system which controls the perturvative violation of the reality condition.

We introduce new variables $(\lambda, \lambda_i, \lambda_a)$, as they obey the dissipative evolution equations

$$\partial_t \lambda = \alpha_1\, \mathcal{C}_H - \beta_1\, \lambda, \tag{11}$$

$$\partial_t \lambda_i = \alpha_2\, \tilde{\mathcal{C}}_{Mi} - \beta_2\, \lambda_i, \tag{12}$$

$$\partial_t \lambda_a = \alpha_3\, \mathcal{C}_{Ga} - \beta_3\, \lambda_a, \tag{13}$$

where $\alpha_i \neq 0$ (allowed to be complex numbers) and $\beta_i > 0$ (real numbers) are constants.

If we take $u^{(DL)}_\alpha = (\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a)$ as a set of dynamical variables, then the principal part of (11)-(13) can be written as

$$\partial_t \lambda \cong -i\alpha_1 \epsilon^{bcd} \tilde{E}^j_c \tilde{E}^l_d (\partial_l \mathcal{A}^b_j), \tag{14}$$

$$\partial_t \lambda_i \cong \alpha_2[-e\delta^l_i \tilde{E}^j_b + e\delta^j_i \tilde{E}^l_b](\partial_l \mathcal{A}^b_j), \tag{15}$$

$$\partial_t \lambda_a \cong \alpha_3 \partial_l \tilde{E}^l_a. \tag{16}$$

The characteristic matrix of the system $u^{(DL)}_\alpha$ does not form a Hermitian matrix. However, if we modify the right-hand-side of the evolution equation of $(\tilde{E}^i_a, \mathcal{A}^a_i)$, then the set becomes a symmetric hyperbolic system. This is done by adding $\bar{\alpha}_3 \gamma^{il}(\partial_l \lambda_a)$ to the equation of $\partial_t \tilde{E}^i_a$, and by adding $i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d (\partial_l \lambda) + \bar{\alpha}_2(-e\gamma^{lm}\tilde{E}^a_i + e\delta^m_i \tilde{E}^{la})(\partial_l \lambda_m)$ to the equation of $\partial_t \mathcal{A}^a_i$. The final principal part, then, is written as

$$\partial_t \begin{pmatrix} \tilde{E}^i_a \\ \mathcal{A}^a_i \\ \lambda \\ \lambda_i \\ \lambda_a \end{pmatrix} \cong \begin{pmatrix} A^l{}_a{}^{bi}{}_m & 0 & 0 & 0 & \bar{\alpha}_3 \gamma^{il}\delta_a{}^b \\ 0 & D^{la}{}_{ib}{}^m & i\bar{\alpha}_1 \epsilon^a{}_c{}^d \tilde{E}^c_i \tilde{E}^l_d & \bar{\alpha}_2 e(\delta^m_i \tilde{E}^{la} - \gamma^{lm}\tilde{E}^a_i) & 0 \\ 0 & -i\alpha_1 \epsilon_b{}^{cd} \tilde{E}^m_c \tilde{E}^l_d & 0 & 0 & 0 \\ 0 & \alpha_2 e(\delta^m_i \tilde{E}^l_b - \delta^l_i \tilde{E}^m_b) & 0 & 0 & 0 \\ \alpha_3 \delta_a{}^b \delta^l{}_m & 0 & 0 & 0 & 0 \end{pmatrix} \partial_l \begin{pmatrix} \tilde{E}^m_b \\ \mathcal{A}^b_m \\ \lambda \\ \lambda_m \\ \lambda_b \end{pmatrix}. \tag{17}$$

Clearly, the solution $(\tilde{E}^i_a, \mathcal{A}^a_i, \lambda, \lambda_i, \lambda_a) = (\tilde{E}^i_a, \mathcal{A}^a_i, 0, 0, 0)$ represents the original solution of the Ashtekar system. If the λs decay to zero after the evolution, then the solution also describes the original solution of the Ashtekar system in that stage. Since the dynamical system of $u^{(DL)}_\alpha$, (17), constitutes a symmetric hyperbolic form, the solutions to the λ-system are unique. BFHR showed analytically that such a decay of λs can be seen for λs sufficiently close to zero with a choice of appropriate combination of αs and βs, and that statement can be also applied to our system. Therefore, the dynamical system, (17), is useful for stabilizing numerical simulations from the point that it recovers the constraint surface automatically.

2.4 Alternative approaches to obtain robust evolution 2: "adjusted system"

We also try to compare a set of evolution system, which we propose as "adjusted-system". The essential procedures are to add constraint terms in the right-hand-side of the dynamical equations with multipliers, and to choose the multipliers so as to decrease the violation of constraint equation. This second step will be explained by obtaining non-positive (or non-zero) eigenvalues of the adjusted constraint propagation equations. We remark that this eigenvalue criterion is also the core part of the theoretical support of the above λ-system.

The fundamental equations that we will demonstrate in this report, are the same with (7) and (8), but here the real-valued constant multipliers κs are not necessary equals to unity. We set $\kappa \equiv \kappa_1 = \kappa_2 = \kappa_3$ for simplicity. Apparently the set of (7) and (8) becomes the original weakly hyperbolic system if $\kappa = 0$, becomes the symmetric hyperbolic system if $\kappa = 1$ and $N = const.$. The set remains strongly hyperbolic systems for other choices of κ except $\kappa = 1/2$ which only forms weakly hyperbolic system.

3 Comparing numerical performance

3.1 Model and Numerical method

The model we present here is gravitational wave propagation in a planar spacetime under periodic boundary condition. We perform a full numerical simulation using Ashtekar's variables. We prepare two +-mode strong pulse waves initially by solving the ADM Hamiltonian constraint equation, using York-O'Murchadha's conformal approach. Then we transform the initial Cauchy data (3-metric and extrinsic curvature) into the connection variables, $(\tilde{E}^i_a, \mathcal{A}^a_i)$, and evolve them using the dynamical equations. For the presentation in this article, we apply the geodesic slicing condition (ADM lapse $N = 1$ or densitized lapse $\underset{\sim}{N} = 1$, with zero shift and zero triad lapse). We have used both the Brailovskaya integration scheme, which is a second order predictor-corrector method, and the so-called iterative Crank-Nicholson integration scheme for numerical time evolutions. The details of the numerical method are described in [1], where we also described how our code shows second order convergence behaviour.

More specifically, we set our initial guess 3-metric as

$$\hat{\gamma}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ sym. & 1 + K(e^{-(x-L)^2} + e^{-(x+L)^2}) & 0 \\ sym. & sym. & 1 - K(e^{-(x-L)^2} + e^{-(x+L)^2}) \end{pmatrix}, \tag{18}$$

in the periodically bounded region $x = [-5, +5]$. Here K and L are constants and we set $K = 0.3$ and $L = 2.5$ for the plots.

In order to show the expected "stabilization behaviour" clearly, we artificially add an error in the middle of the time evolution. We added an artificial inconsistent rescaling once at time $t = 6$ for the $\mathcal{A}^2_y$ component as $\mathcal{A}^2_y \to \mathcal{A}^2_y(1 + \text{error})$.

3.2 Differences between three levels of hyperbolicity

We have performed comparisons of stability and/or accuracy between weakly and strongly hyperbolic systems, and between weakly and symmetric hyperbolic systems[1]. (We can not compare strongly and symmetric hyperbolic systems directly, because these two requires different gauge conditions.)

We omit figures in this report, but one can see a part of results in Fig.1 and Fig.2. We may conclude that higher level hyperbolic system gives us slightly accurate evolutions. However, if we evaluate the magnitude of L2 norms, then we also conclude that there is no measurable differences between strongly and symmetric hyperbolicities. This last fact will be supported more affirmatively in the next experiments.

3.3 Demonstrating "λ-system"

Next, we show a result of the "λ-system" [2]. Fig.1 (a) shows how the violation of the Hamiltonian constraint equation, $\mathcal{C}_H$, become worse depending on the term error. The oscillation of the L2 norm $\mathcal{C}_H$ in the figure due to the pulse waves collide periodically in the numerical region. We, then, fix the error term as a 20% spike, and try to evolve the same data in different equations of motion, i.e., the original Ashtekar's equation [solid line in Fig.1 (b)], strongly hyperbolic version of Ashtekar's equation (dotted line) and the above λ-system equation (other lines) with different βs but the same α. As we expected, all the λ-system cases result in reducing the Hamiltonian constraint errors.

3.4 Demonstrating "adjusted system"

We here examine how the adjusted multipliers contribute to the system's stability [2]. In Fig.2, we show the results of this experiment. We plot the violation of the constraint equations both $\mathcal{C}_H$ and $\mathcal{C}_{Mx}$. An artificial error term was added in the same way as above. The solid line is the case of $\kappa = 0$, that is the case of "no adjusted" original Ashtekar equation (weakly hyperbolic system). The dotted line is for $\kappa = 1$, equivalent to the symmetric hyperbolic system. We see other line ($\kappa = 2.0$) shows better performance than the symmetric hyperbolic case.

4 Why adjusted equations have better performance?

The remaining question is: why we can get the better performance by adding constraint terms in the dynamical equations? The added terms are basically *error* terms during the evolution for its original dynamical equations. Nevertheless, these terms improve the accuracy of the evolution. This question applies to both higher level of hyperbolicities and also for our proposing "adjusted systems". Here we introduce our plausible explanation schematically. The detail explanations and numerical experiments are in [2].

Suppose we have constraint equations, $\mathcal{C}_1 \approx 0$, $\mathcal{C}_2 \approx 0, \cdots$, in a system. We, normally, monitor the error of the evolution by evaluating these constraint equations on the each constant-time hypersurface. Such monitoring, on the other hand, can be performed also by checking the evolution equations of the constraint, which we denote constraint propagation equations (cf. [21]). If this set of constraint

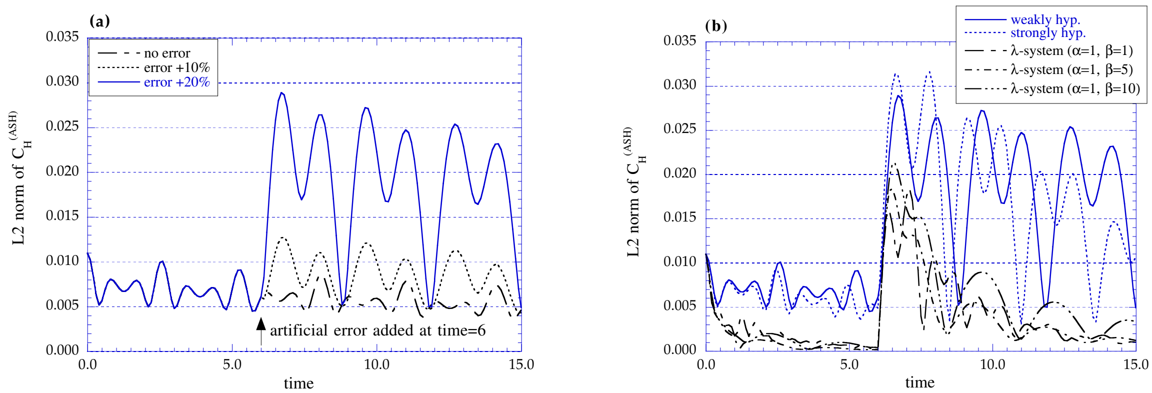

Figure 1: Demonstration of the λ-system in the Ashtekar equation. We plot the violation of the constraint (L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$) for the cases of plane wave propagation under the periodic boundary. To see the effect more clearly, we added artificial error at $t = 6$. Fig. (a) shows how the system goes bad depending on the amplitude of artificial error. The error was of the form $\mathcal{A}_y^2 \to \mathcal{A}_y^2(1+ \text{ error})$. All the lines are of the evolution of Ashtekar's original equation (no λ-system). Fig. (b) shows the effect of λ-system. All the lines are 20% error amplitude, but shows the difference of evolution equations. The solid line is for Ashtekar's original equation (the same as in Fig.(a)), the dotted line is for the strongly hyperbolic Ashtekar's equation. Other lines are of λ-systems, which produces better performance than that of the strongly hyperbolic system.

propagation equations could be written in a first order form, then we may predict the evolution behavior by its characteristic matrix, M, which is expressed by

$$\partial_t \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \vdots \end{pmatrix} \simeq M \partial_i \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \vdots \end{pmatrix}, \tag{19}$$

where $\simeq$ means we are extracting only the characteristic part.

The idea here is to estimate the eigenvalues of the characteristic matrix, M, after we took the Fourier transformation on the both sides of (19).

1. Clearly, if all the real part of the eigenvalues are negative, then all constraints decays to zero along to the system's evolution.
2. Alternatively, if all the eigenvalues are pure-imaginary, then all constraints evolve similar to wave equations, and diffusion due to differenciation may preserve from its unstable evolution.

These guidelines are supported by von Neumann's stability analysis. For the adjusted Ashtekar equations, the eigenvalues of the amplicication matrix in von Neumann's method for the set of $(\mathcal{C}_H, \mathcal{C}_{Mi}, \mathcal{C}_{Ga})$ are $\{1, 1 - \nu^2 \kappa \sin\theta \pm i\nu\kappa|\sin\theta|(\text{multiplicity 2 each}), 1 - \nu^2(1-2\kappa)^2 \sin^2\theta \pm i\nu|1-2\kappa||\sin\theta|\}$, where $\nu = \Delta t/\Delta x$, θ is the parameter for the frequency in grid spacing domain, and κ is the adjusted parameter (§2.4). We observe that von Neumann's necessary condition for the stability can be obtained in larger non-zero κ.

In [2], we show that such a case can be obtained by adding 'adjusted terms' both for Ashtekar's and Maxwell's systems. There we also show examples of unstable evolution by choosing adjusted terms which produce positive eigenvalues of M.

In this point, we can say that adjusted terms are responsible for obtaining the stable and/or accurate evolution system, and this is a way to control the stability of simulation, which effects more than the

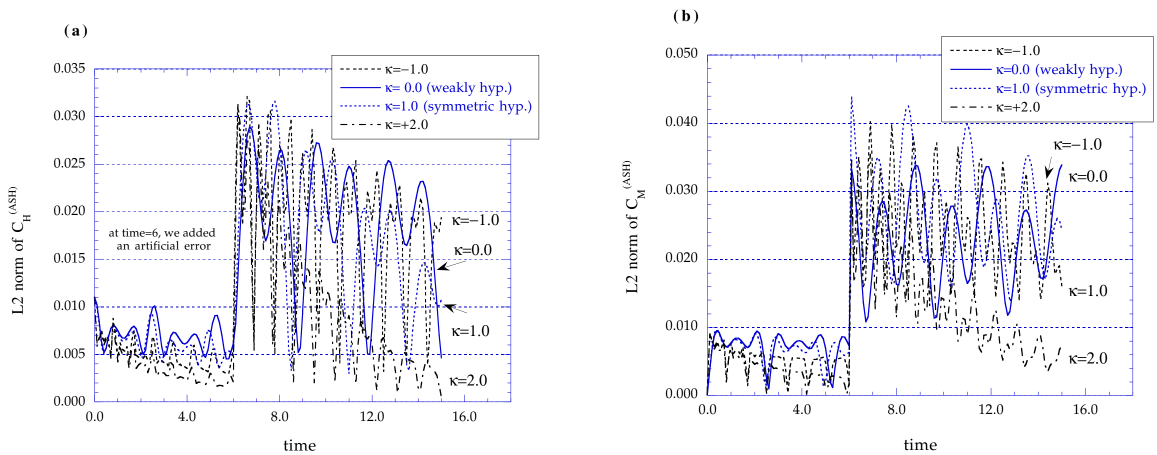

Figure 2: Demonstration of the adjusted system in the Ashtekar equation. An artificial error term was added at $t = 6$, in the form of $\mathcal{A}^2_y \to \mathcal{A}^2_y(1+$ error), where error is +20% as before. Fig. (a) and (b) are L2 norm of the Hamiltonian constraint equation, $\mathcal{C}_H$, and momentum constraint equation, $\mathcal{C}_{Mx}$, respectively. The solid line is the case of $\kappa = 0$, that is the case of "no adjusted" original Ashtekar equation (weakly hyperbolic system). The dotted line is for $\kappa = 1$, equivalent to the symmetric hyperbolic system. We see other line ($\kappa = 2.0$) shows better performance than the symmetric hyperbolic case.

system's hyperbolicity.

5 Concluding remarks

We presented numerical comparisons on the accuracy/stability of the systems, mainly between three mathematical levels of hyperbolicity: weakly hyperbolic, strongly hyperbolic, and symmetric hyperbolic systems. We apply Ashtekar's connection formulation, because this is the only known system in which we can compare three hyperbolic levels with the same interface. We also tested two another approaches with a purpose of constructing an "asymptotically constrained" system, "λ-system" and "adjusted system", which can be robust against perturbative errors for the free evolution of the initial data.

We may conclude that higher levels of hyperbolic formulations help the numerics more, though its differences are small [22]. Two other approaches work as desired. These show quite good performance than those of the original and its symmetric hyperbolic form. This indicates, in turn, that the symmetric hyperbolic system is not always the best for controlling accuracy or stability.

The reason why higher hyperbolicity shows better stability may be explained by the eigenvalue analysis of the propagation equation of the constraints. Our two guidelines, negative-real eigenvalues or pure-imaginary eigenvalues, give us clear indications whether the constraints decay (i.e. stable system) or not for perturbative errors, at least to all our numerical experiments.

We think these results open a new direction to numerical relativists for future treatment of the Einstein equations.

To conclude, we are glad to announce that Ashtekar's connection variables have finally been applied in numerical simulations. This new approach, we hope, will contribute to understanding further of gravitational physics, and will open a new window for peeling off interesting non-linear natures together with a step to numerical treatment of quantum gravity.

Acknowledgements

HS appreciates helpful comments by Abhay Ashtekar, Jorge Pullin, Douglas Arnold and L. Samuel Finn, and the hospitality of the CGPG group. Numerical computations were performed using machines at CGPG. This work was supported in part by the NSF grants PHYS98-00973, and the Everly research funds of Penn State. HS was supported by the Japan Society for the Promotion of Science as a research fellow abroad. HS appreciates the travel support from the LOC of the JGRG workshop.

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