Initial data for Neutron Star Binary Simulations

A review + α

From Newtonian, Post-Newtonian, towards General Relativistic

January 99, Hisa-aki Shinkai (Washington Univ.)

Outline

- What are current research interests in this field?
- What are good directions in this field?

Contents

1 Our targets: Binary Neutron Star Problem

Reviews: É.Flanagan, gr-qc/9804024; B.F.Schutz, gr-qc/9802020; K.S.Thorne, gr-qc/9706079

1.1 Laser Interferometers

LIGO	USA	4Km	$\sim$ 2001	http://www.ligo.caltech.edu
VIRGO	France/Italy	3Km	$\sim$ 2001	http://www.pi.infn.it/virgo/virgoHome.html
GEO	Germany/UK	600m	$\sim$ 2000	http://www.geo600.uni-hannover.de/
TAMA	Japan	300m	$\sim$ 1999	http://tamago.mtk.nao.ac.jp/

1.2 Sources of gravitational waves

1.3 Neutron Star Binary Coalesence

Event Rate gr-qc/9812054 (Nakamura-Oohara)

- Estimation from known binary systems: PSR 1913+16, 1534+12, $\cdots$
  $\underline{10^{-6} \sim 2 \times 10^{-5} \text{ events yr}^{-1} \text{ galaxy}^{-1}}$

  E.S.Phinney, ApJ. 380(1991)L17 ; R.Narayan, T. Piran, and A. Shemi, ApJ.379(1991)L17; E.P.J. Van den Heuvel, and D.R. Lorimer, MNRAS 283(1996)L37

- Formation theories of binary neutron stars,
  $\underline{2 \times 10^{-5} \sim 3 \times 10^{-4} \text{ events yr}^{-1} \text{ galaxy}^{-1}}$

  Yungelson et al., MNRAS 268(1994)871; astro-ph/9801127 V.M. Lipunov et al. MNRAS 288(1997) 245;

- the number density of galaxies is $\underline{\sim 10^{-2}/\text{Mpc}^3}$.
- therefore $\underline{\sim 2 \times 10^{-7} \text{events } /\text{Mpc}^3}$

$$\boxed{\text{several coalescing events/year within 200Mpc}}$$

What Information can we extract from Gravitational Waveform of Binary Neutron Star Coalescence?

1. $\boxed{\text{INSPIRAL phase [}\sim 3\text{ mins.]}}$ (Post-Newtonian Approx.)

$$
\begin{array}{lll}
\text{'chirps' } \dfrac{df}{dt} & \Rightarrow & \text{'chirp mass' } M_c \equiv \dfrac{(M_1 M_2)^{3/5}}{(M_1+M_2)^{1/5}} \\
\text{amplitude (incleasing)} & \Rightarrow & M_c, \text{ distance} \\
\text{amplitude } (h_+/h_\times) & \Rightarrow & \text{inclination} \\
\text{waveform} & \Rightarrow & \text{eccentricity} \\
\text{moduration} & \Rightarrow & \text{spin}, \cdots
\end{array}
$$

2. $\boxed{\text{ISCO phase}}$ (Post-Newtonian & Numerical Relativity)

$$
\begin{array}{rll}
\text{frequency} & \Rightarrow & \text{Mass-Radius relation} \\
 & \Rightarrow & \text{Equation of States}
\end{array}
$$

3. $\boxed{\text{COALESCE phase [}\sim 3\text{ millisecs.]}}$ (Numerical Relativity)

$$
\begin{array}{lll}
\text{waveform} \cdots & \Rightarrow & ? \\
 & \Rightarrow & \text{BH parameters } (m, a), \cdots \\
 & \Rightarrow & \text{GR test} \\
\text{other elements?} & \Rightarrow & \gamma\text{-ray burst ?} \\
 & \Rightarrow & \text{r-process elements?}
\end{array}
$$

4. $\boxed{\text{BLACK HOLE formation phase [}\sim 10\text{ msecs.]}}$ (Perturbation)

$$
\text{Quasi-Normal Modes} \quad \Rightarrow \quad \text{Black Hole formation}
$$

5. $\boxed{\text{STATISTICS}}$

$$
\begin{array}{lll}
\text{with optical identification} & \Rightarrow & \text{Hubble parameter} \\
\text{statics} & \Rightarrow & \text{cosmological parameters}
\end{array}
$$

2 Innermost Stable Circular Orbit problem

- Where the ISCO is located?
- What physics determine it?
- What observable effects ensue?

2.1 ISCO? – No definit definition

Brief Review by D.Eardley, *Matter of Gravity 9* (Feb/97)

ISCO in GR: test particle around a BH

- $r = 6M$ for a non-rotating BH
- $r = M$ for a maximally rotating BH

ISCO: post-Newtonian analytic approach

Kidder-Will-Wiseman, CQG9(92)L125
Kidder-Will-Wiseman, PRD47(93)3281
Lai-Wiseman, PRD54(96)3958
Will-Wiseman, PRD54(96)4813

- If one ignores radiation, the ISCO is well defined.
- If radiation reaction is included, the ISCO is "reasonably well defined"

2.2 ISCO in Newtonian Gravity

2.2.1 Two Point Particles in Newtonian Gravity

Each star's mass	M_1 and M_2	
Total mass	M	$M = M_1 + M_2$
Reduced mass	μ	$\mu = M_1 M_2/(M_1 + M_2)$
circular orbit radius	r_0	
respective distances from CoM	r_1 and r_2	
orbital ang. vel.	Ω	
specific ang. mom.	l	$l = r_0 v = r_0^2 \Omega$

Circular Kepler Motion

From def. of the center of mass:

$$M_1 r_1 = M_2 r_2 = \mu r_0 \tag{1}$$

The Kepler's III law gives us

$$\Omega^2 = \frac{GM}{r_0^3} \tag{2}$$

Therefore

$$r_0 = \frac{l^2}{GM} \tag{3}$$

Energy conservation equation

$$\frac{\mu}{2}\left(\frac{dr}{dt}\right)^2 = E - V(r); \quad V(r) = \frac{\mu l^2}{2r^2} - \frac{GM\mu}{r}, \tag{4}$$

$$\frac{dV}{dr}(r = r_0) = 0, \quad \frac{d^2V}{dr^2}(r = r_0) > 0$$

$$\boxed{\text{There is no ISCO.}}$$

2.2.2 Finite extension with tidal deformation

$$
\begin{aligned}
\frac{\mu}{2}\left(\frac{dr}{dt}\right)^2 &= E - V(r); \\
V(r) &= \frac{\mu l^2}{2r^2} - \frac{GM\mu}{r} - \frac{3GM}{2r^3}\not{I}_{11}(t),
\end{aligned}
\tag{5}
$$

where the $\not{I}_{11}$ quadrupole moment as

$$\not{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\sum_{k=1}^{3} I_{kk} \ \sim O(r^{-3})$$

where

$$I_{ijk\cdots} = \int \rho x_i x_j x_k \cdots d^3x.$$

Pioneering Works:

Chandrasekhar "Ellipsoidal Figures of Equillibrium" (1969)

- incompressible matter

Lai-Rasio-Shapiro ApJ.Suppl.88(93)205, ApJ.420(94)811

- compressible matter, ellipsoidal approx.
- synchronized, hydrodynamical effect is included.
  - ISCO exists for stiff EOS If the mass and the radius of each star are fixed, the angular velocity Ω_{ISCO} is larger for softer EOS.
  - dynamical instabilities always occur later along equilibrium sequences than secular instabilities.

Uryu-Eriguchi ApJ.Suppl.118(98)563

- compressible matter, numerical
- irrotational
  - ISCO exists for stiff EOS

Ellipsoidal Figures of Equilibrium

S. Chandrasekhar (Yale Univ. Press., 1969).

single body

- Maclaurin Spheroids (1742)
  - homogeneous bodies rotate with a uniform angular velocity
  - symmetric to the rotating axis
  - no fluid motion inside body when in the rotating frame together with body
- Jacobi Ellipsoids (1834)
  - similar to Maclaurin Spheoids, but no symmetry, three different axes
- Dedekind Ellipsoids (1860)
  - no rotation in the inertial frame, but the internal fluid rotates.
  - Dedekind's theorem
    For any motion which preserve the configuration of homogeneous ellipsoid, there exists the adjoint motion which forms the same configuration of ellipsoid.
  - the same sequence with Jacobi Ellipsoids
- Riemann Ellipsoids (1860)
  - generalization of Jacobi/Dedekind Ellipsoids
  - homogeneously rotating with the angular momentum Ω in the inertial frame, and also has the uniform vorticity ζ in the rotating frame with Ω

two bodies

the primary $\cdots$ the body in which we are interested
the secondary $\cdots$ the body which is the origin of the tidal effects on the primary

- Roche's problem (1850)
  - the equilibrium and the stability of a homogeneous body (primary) rotating about the secondary in such a manner that their relative dispositions remain the same.
  - the primary has finite volume, while the secondary is a point particle.
  - the secondary affects its tidal force to the primary, but no reaction from the primary.
- Darwin's problem (1906)
  - both the primary and the secondary have finite volume
  - allow for the mutual tidal interactions between the primary and the secondary
- Roche-Riemann's problem

2.3 ISCO in Post-Newtonian Gravity

1. self-gravity
   - self-gravity is stronger than Newtonian gravity
   - each star becomes compact $\Rightarrow$ the tidal effect is less effective
   - $\underline{\Omega_{\rm ISCO} \text{ tends to be increased.}}$
2. gravity acting between two stars
   - destabilize the binary orbit
   - $\underline{\Omega_{\rm ISCO} \text{ tends to be decreased}}$

These features are more remarkable for for softer EOS if the mass and radius are fixed.

Full calculation:
Shibata PRD55(1997)6019

- numerical study, synchronized binary, $\Gamma = 2$
- $\boxed{\text{self-gravity effect} > \text{mutual gravity effect}}$

2.4 ISCO in 2nd Post-Newtonian Gravity?

Full calculations are not yet performed.

2.5 ISCO in GR?

- *'Quasi-Equilibrium'*, **synchronized** binary
  Baumgarte *et al*, PRD57(98)7299
  - ISCO exists for stiff EOS
- *'Quasi-Equilibrium'*, **irrotational** binary
  Bonazzola *et al*, PRL82(99)892

Studies of Equilibrium Binary NS Config. with hydro

February 2, 1999 H. Shinkai

					existence of ISCO
N	ana.	incompr.	Chandrasekhar	(1969)	
N	ana.	compr.	Lai-Ratio-Shapiro	ApJS 88(93)205	ellip.model, $\exists$ for stiff Γ
		compr.		ApJ 420(94)811	ellip.model, $\exists$ for stiff Γ
Synchronized Binary					
N	num.	compr.	Hachisu	ApJS 62(86)461	
N	num.	compr.	New-Tohline	ApJ490(97)311	$\exists$ for stiff Γ
PN	ana.	incompr.	Taniguchi-Shibata	PRD56(97)798	$\exists$
PN	ana.	incompr.	Taniguchi-Shibata	PRD58(98)084012	
PN	num.	compr.	Shibata	PRD55(97)6019	$\exists$ for $\Gamma = 2$
PN	num.	compr.	Shibata-Taniguchi	PRD56(97)811	ellip.model, $\exists$ for all Γ
2PN	formulation		Asada-Shibata	PRD54(96)4944	29 Poissons
GR	num.	compr.	Baumgarte *et. al.*	PRL79(97)1182	
				PRD57(98) 6181	
				PRD57(98) 7299	$\exists$ for $\Gamma = 2$, $\nexists$ for $\Gamma > 3$
Irrotational Binary					
N	semi		Kochanek	ApJ398(92)234	affine model
N	ana.		Bildsten-Cutler	ApJ400(92)175	
			Lai	MNRAS270(94)611	
N	num	incompr.	Uryu-Eriguchi	MNRAS 299(98) 575	$\exists$
N	num	compr.		ApJS 118(98) 563	$\exists$ for $\Gamma > 2.8$
PN	num.	compr.	Lombardi *et. al*	PRD56(97)3416	truncated
PN	num.	incompr.	Taniguchi *et. al*	PTP100(98)703	$\exists$
			Taniguchi	gr-qc/9901048	$\exists$, ellip.model OK
GR	formulation		Bonazzola *et. al*	PRD56(97)7740	
GR	formulation		Asada	PRD57(98)7292	correct Bonazzola *et. al*
GR	formulation		Teukolsky	ApJ504(98)442	another method
GR	formulation		Shibata	PRD58 (98)024012	another method
GR	formulation		Gourgoulhon	gr-qc/9804054	comparison
GR	num.	com.	Bonazzola *et. al*	PRL82(99)892	

Note

```
synchronized binary     <=> fluid velocity = 0 in the co-orbiting frame
irrotational binary     <=> vorticity = 0 in the inertial frame
counter rotating binary <=> orbital angular vel. = - spin angular vel.

In the spacetime with rotational Killing vector
   the counter rotation <=> the irrotation  (cf. Asada).
```

Simulations of Binary Neutron Stars

February 2, 1999

	group	ref	initial data	sep	EOS
N	Nakamura-Oohara	PTP 82(89)535		$2R$	$\Gamma = 2$
N+RR		PTP 82(89)1066		$2R$	$\Gamma = 2$
N+RR		PTP 83(90)906	equili.	$2R$	$\Gamma = 2$
N+RR		PTP 86(91)73	equili.	$3R$	$\Gamma = 2$
PN	Oohara-Nakamura	PTP 88(92)307		$2R$	$\Gamma = 2$
N+RR	Shibata *et al*	PTP 88(92)1079	spin/equil.	$3R$	$\Gamma = 2$
N+RR	Shibata *et al*	PTP 89(93)809	spin/headon	$3R$	$\Gamma = 2$
N	Rasio-Shapiro	ApJ401(92)226	(SPH)equil./headon	$\sim 3R$	$\Gamma = 2$
N		ApJ432(94)242	(SPH)equil./diffM	$\sim 3R$	$\Gamma = 5/3 \sim 3$
N		ApJ438(95)887	(SPH)equil./diffM	$2 - 3R$	$\Gamma = 5/3$
N+RR	Centrella-McMillan	ApJ416(93)719	(SPH) offcenter	$4R$	$\Gamma = 2, 5/3$
N+RRsw	Zhuge *et al*	PRD50(94)6247	(SPH)	$4R$	$\Gamma = 2, 3$
N+RRsw	Zhuge *et al*	PRD54(96)7261	(SPH) spin/diffM	$4R$	$\Gamma = 3, 2, 5/3$
N+RR	Ruffert *et al*	AA311(96)532	equil./spin/ν	$3R$	LS
N+RR		AA319(97)122	equil./spin/ν	$3R$	LS
N+RR		AA321(97)991	vs Shibata/Zhuge	$3R$	LS/Γ
N+RR	Ruffert-Janka	astro-ph/9804132	headon/offcenter/ν	$3R$	LS
N	New-Tohline	ApJ490(97)311	equili.	$3R$	Γ,F,FPS,L
N+RRsw	Davis *et al*	ApJ431(94)742	(SPH) spin	$5R$	$\Gamma = 2.4$/LS
N+RR	Rosswog *et al*	astro-ph/9811367	(SPH) equili/spin/ν	$5R$	LS
PN	Shibata *et al*	PTP98(97)1081	equil.	$2.5R$	$\Gamma = 3$
PN$\sim$GR	Shibata *et al*	PRD58(98)023002	spin	$3R$	$\Gamma = 1.4$
GR	Wilson-Mathews	PRL75(95)4161	conf.flat.evo.	$4 - 5R$	WM
GR	Wilson *et al*	PRD54(96)1317	conf.flat.evo.	$4 - 5R$	WM
GR	Wilson-Mathews	ApJ482(97)929	conf.flat.evo.	$4 - 7R$	WM
GR	Mathews *et al*	PRD58(98)043003			
GR	Marronetti *et al*	PRD58(98)107503			
GR	Oohara *et al*	PTPS128(97)183	conf.flat	$3R$	$\Gamma = 2$

RR with Radiation reaction term
RRsw Radiation reaction term switched off after merging

LS Lattimer-Swesty, Nucl.Phys.A535(91)331
F Arponen, Nucl.Phys.A191(72)257
FPS Lorenz-Ravenhall-Pathick, Phys.Rev.Lett.70(93)379
L Pandharipande-Smith, Nucl.Phys.A237(75)507
WM Wilson-Mayle, Phys.Rep.227(93)97

A Newtonian Fluid Basic Equations

ρ	coordinate rest-mass	
ε	internal energy density	
P	pressure	
$\dfrac{DF}{Dt}$	Euler derivative	$\dfrac{DF}{Dt} = \dfrac{\partial F}{\partial t} + (\mathbf{v} \cdot \nabla) F$

The continuity eq. (mass conservation)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

If the fluid is incompressive,

$$\frac{D\rho}{Dt} = 0 \quad \Leftrightarrow \quad \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho = -\rho(\nabla \cdot \mathbf{v}) = 0.$$

The equation of motion (perfect fluid)

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = -\nabla P - \rho \nabla \psi \tag{2}$$

The momentum conservation

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial \pi_{ik}}{\partial x_k} = -\rho \partial_i \psi, \quad \text{where} \quad \pi_{ik} = P\delta_{ik} + \rho v_i v_k. \tag{3}$$

The energy conservation

Define enthalpy, h,

$$\begin{aligned} h &= \frac{P}{\rho} + \varepsilon \\ dh &= \frac{dP}{\rho} + T ds \quad \rightarrow \quad \frac{1}{\rho}\frac{DP}{Dt} = \frac{Dh}{Dt} - T\frac{Ds}{Dt} = \frac{Dh}{Dt} \end{aligned}$$

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + \varepsilon + \psi\right)\right] + \nabla \cdot \left[\rho \mathbf{v}\left(\frac{1}{2}v^2 + h + \psi\right)\right] = 0$$

Therefore

$$\frac{\partial}{\partial t}[e] + \nabla \cdot [\mathbf{v}(e + P)] = 0, \quad \text{where} \quad e = \rho\left(\frac{1}{2}v^2 + \varepsilon + \phi\right) \tag{4}$$

Equation of State

We assume the adiabatic process (isentropic),

$$\frac{Ds}{Dt} = 0,$$

which can be written in the case of perfect fluid as

$$\frac{Ds}{Dt}\left(\ln\frac{P}{\rho^{\gamma}}\right) = 0, \quad or \quad \frac{\partial}{\partial t}\left(\frac{P}{\rho^{\gamma}}\right) + \mathbf{v}\cdot\nabla\left(\frac{P}{\rho^{\gamma}}\right) = 0. \tag{5}$$

where $\gamma = c_p/c_v$.

A.1 The Equations to solve – Newtonian Fluid –

EOM, continuity, conservation law.

$$\begin{aligned}
\partial_t\rho + \partial_j(\rho v^j) &= 0, && (6a)\\
\partial_t(\rho v^i) + \partial_j(\rho v^i v^j) &= -\partial_i P - \rho\partial_i\psi, && (6b)\\
\partial_t(\rho\varepsilon) + \partial_j(\rho\varepsilon v^j) &= -P\partial_j v^j, && (6c)
\end{aligned}$$

Poisson eq.

$$\Delta\psi = 4\pi G\rho. \tag{7}$$

EOS (normally assumed as polytropic type)

$$P = (\gamma - 1)\rho\varepsilon. \tag{8}$$

B Newtonian with Radiation Reaction ($Newton.{+}P^{2.5}N$)

ρ	coordinate rest-mass	
ε	internal energy density	
P	pressure	
w^i	momentum per unit rest-mass	$v^i = w^i + \dfrac{4}{5}\dfrac{G}{c^5}Q^{[3]}_{ij}w^j$

EOM, continuity, conservation law

$$\begin{aligned}
\partial_t\rho + \partial_j(\rho v^j) &= 0, & (1a)\\
\partial_t(\rho w^i) + \partial_j(\rho w^i v^j) &= -\partial_i P - \rho\partial_i(\psi + \psi_{\mathrm{react}}), & (1b)\\
\partial_t(\rho\varepsilon) + \partial_j(\rho\varepsilon v^j) &= -P\partial_j v^j, & (1c)
\end{aligned}$$

Radiation Reaction potential ψ_{react}

$$\psi_{\mathrm{react}} = \frac{2}{5}\frac{G}{c^5}\left\{R - Q^{[3]}_{ij}x^i\partial_j\psi\right\}, \tag{2}$$

Poisson equations

$$\begin{aligned}
\Delta\psi &= 4\pi G\rho, & (3a)\\
\Delta\psi_t &= -4\pi G\partial_j(\rho v^j), & (3b)\\
\Delta R &= 4\pi G Q^{[3]}_{ij}x^i\partial_j\rho & (3c)
\end{aligned}$$

EOS

$$P = (\gamma - 1)\rho\varepsilon. \tag{4}$$

Quadrupole moment

$$\begin{aligned}
Q_{ij} &= STF\left\{\int \rho x^i x^j dV\right\} = \int \rho(x^i x^j - \frac{1}{3}\delta_{ij}x^k x^k)dV,\\
Q^{[1]}_{ij} &= STF\left\{\int 2\rho x^i v^j dV\right\} = \int \rho(x^i v^j + v^i x^j - \frac{2}{3}\delta_{ij}v^k x^k)dV,\\
Q^{[2]}_{ij} &= STF\left\{\int 2[\rho v^i v^j - 2\delta_{ij}P - \rho x^i\partial_j(\psi + \psi_{react})]dV\right\}\\
&= STF\left\{\int 2[\rho v^i v^j - \rho x^i\partial_j(\psi + \psi_{react})]dV\right\},\\
Q^{[3]}_{ij} &= STF\left\{2\int[2P\partial_j v_i - 2\rho v^i\partial_j\psi + x^i\partial_j\psi\partial_k(\rho v^k) - \rho x_i\partial_j\psi_t]dV\right\}
\end{aligned}$$

where the notation STF means

$$STF\{A_{ij}\} = \frac{1}{2}(A_{ij} + A_{ji}) - \frac{1}{3}\delta_{ij}A_{kk}.$$

Quadrupole Formula in Numerical Simulations

L.S.Finn and C.R.Evans, ApJ 351(1990)588

- Standard Quadrupole Formulat (SQF)

$$h_{ij}^{TT} = \frac{2}{r}\frac{d^2}{dt^2}I\!\!\!-_{ij}^{TT} \equiv \frac{2}{r}\frac{d^2}{dt^2}\left[\int d^3x\ \rho r^2\left(n_i n_j - \frac{1}{3}\eta_{ij}\right)\right]^{TT} \tag{5}$$

second derivative in time
large-moment arm of the quadrupole moment

$\rightarrow$ results in high-frequency noise

- Alternative (QF2)
  Apply the continuity eq. or the conservation of energy,

$$\frac{d}{dt}I\!\!\!-_{ij} = -\int d^3x\ r^2\left(n_i n_j - \frac{1}{3}\eta_{ij}\right)\nabla_j(v^j\rho) \tag{6}$$

- Alternative (QF1)
  Similarly, using the identity,

$$x^i x^j \tau^{tt}_{,t} = -(x^i x^j \tau^{tl})_{,l} + x^j\tau^{ti} + x^i\tau^{tj},$$

$$\frac{d}{dt}I\!\!\!-_{ij} = 2\int d^3x\ \rho\left[v_{(i}x_{j)} - \frac{1}{3}\eta_{ij}v_k x^k\right] \tag{7}$$

- Alternative (Steress Tensor)

$$\begin{aligned}\frac{d^2}{dt^2}I\!\!\!-_{ij} &= \int d^3x\ \rho\left(v_i v_j - \frac{1}{3}\eta_{ij}v_k v^k\right)\\ &\quad + \left(\sigma_{ij} - \frac{1}{3}\eta_{ij}\sigma^k_k\right) + \left(\Phi_{,i}\Phi_{,j} - \frac{1}{3}\eta_{ij}\Phi^{,k}\Phi_{,k}\right)\end{aligned} \tag{8}$$

$$\boxed{\text{Experiments with rotating stellar core collapse.}}$$

C Newtonian Equilibrium Configuration (Synchronized)

C.1 Newtonian basic equations

Roberts, ApJ 137 (1963) 1129

The equilibrium configuration of a uniformly rotating object is given by minimizing the total energy,

$$\begin{aligned} E_{total} &= E_{rotational} + E_{internal} + E_{potential} \\ &= \frac{1}{2}\Omega^2 \int \rho R^2 dV + \int \rho\varepsilon dV + \frac{1}{2}\int \rho\psi dV \end{aligned}$$

under the assumptions of keeping the mass M and angular momentum $\mathcal{M}$:

$$M = \int \rho dV, \quad \mathcal{M} = \Omega \int \rho R^2 dV.$$

where $\Omega = \frac{d\phi}{dt}$ is angular velocity, $R^2 = x^2 + y^2$, ε is the internal energy density, where the energy density ϵ is given by

$$\epsilon = \rho c^2 + \rho\varepsilon = \text{material energy} + \text{internal energy}$$

If we specify EOS as polytropic form

$$p = K\rho^{1+(1/n)} = K\rho^{\Gamma} = (\Gamma - 1)\rho\varepsilon$$

then the variation of E gives us

$$\begin{aligned} \delta E &= \int \delta\rho \; dV[\varepsilon + \frac{p}{\rho} + \psi - \frac{1}{2}\Omega^2 R^2] \\ &= \int \delta\rho \; dV[K(n+1)\rho^{1/n} + \psi - \frac{1}{2}\Omega^2 R^2] \\ &= \int \delta\rho \; dV[K\frac{\Gamma}{\Gamma - 1}\rho^{\Gamma-1} + \psi - \frac{1}{2}\Omega^2 R^2], \end{aligned}$$

where we used Poisson equation

$$\boxed{\Delta\psi = 4\pi G\rho,}$$

$$\Delta\delta\psi = 4\pi G\delta\rho.$$

Thus to get the equilibrium data, we need to solve

$$\boxed{K\frac{\Gamma}{\Gamma-1}\rho^{\Gamma-1} + \psi - \frac{1}{2}\Omega^2 R^2 = C = const.}$$

together with Poisson eq.

C.2 Newtonian, Synchronized – Numerical method

Nakamura-Oohara, PTP83(1991) 906, PTP86(1992)307

The equations to solve are Poisson eq. and

$$\begin{aligned}\tilde{K}_1\rho^{\Gamma_1-1}+\psi-\frac{1}{2}[(x-x_c)^2+y^2]\Omega^2 &= c_1\\ \tilde{K}_2\rho^{\Gamma_2-1}+\psi-\frac{1}{2}[(x-x_c)^2+y^2]\Omega^2 &= c_2\end{aligned}$$

where $\tilde{K}_i = K\frac{\Gamma_i}{\Gamma_i-1}$, and x_c is center of mass $(x_c,0,0)$. Unknown constants are c_1, c_2, x_c, Ω.

Let's locate each NS of EOS $p = K_i\rho^{\Gamma_i}$ $(i=1,2)$, at $(x_i,0,0)$ with its central density ρ_i. Denote also the star surface on y axis as $(x_{0i},0,0)$ for i-th star. Then we have four equations

$$\begin{aligned}\tilde{K}_1\rho_1^{\Gamma_1-1}+\psi(x_1,0,0)-\frac{1}{2}(x_1-x_c)^2\Omega^2 &= c_1\\ \psi(x_{01},0,0)-\frac{1}{2}(x_{01}-x_c)^2\Omega^2 &= c_1\\ \tilde{K}_2\rho_2^{\Gamma_2-1}+\psi(x_2,0,0)-\frac{1}{2}(x_2-x_c)^2\Omega^2 &= c_2\\ \psi(x_{02},0,0)-\frac{1}{2}(x_{02}-x_c)^2\Omega^2 &= c_2\end{aligned}$$

From these relations, we can get

$$\begin{aligned}x_c &= \frac{A_2(x_1^2-x_{01}^2)-A_1(x_2^2-x_{02}^2)}{2[A_2(x_1-x_{01})-A_1(x_2-x_{02})]}\\ \Omega^2 &= \frac{2A_1}{(x_1+x_{01}-2x_c)(x_1-x_{01})}\\ c_i &= \psi(x_{0i},0,0)-\frac{1}{2}(x_{0i}-x_c)^2\Omega^2\end{aligned}$$

where

$$A_i = \psi(x_i,0,0)-\psi(x_{0i},0,0)+\tilde{K}_i\rho_i^{\Gamma_i-1}$$

First give a distribution of ρ, solve the Poisson equation and get ψ. Then get c_1, c_2, x_c, Ω from above expressions and re-define ρ as

$$\rho = \left(\frac{1}{\tilde{K}_i}[c_i-\psi+\frac{1}{2}\{(x-x_c)^2+y^2\}\Omega^2]\right)^{\frac{1}{\Gamma_i-1}}.$$

Equilibrium configuration will be given from the converged ρ from iterating these procedures; say

$$\frac{\int(\rho^{(n)}-\rho^{(n-1)})^2dV}{\int(\rho^{(n)})^2dV} \ll 1$$

D Post-Newtonian Equilibrium Configuration (Synchronized)

D.1 Post-Newtonian – basic equations

Chandrasekhar, ApJ 148 (1967) 621

The equations to solve are

$$\boxed{H_i - \frac{1}{2c^2}(H_i)^2 + \psi - \frac{1}{2}R^2\Omega^2 - \frac{1}{c^2}(-X - 2\psi R^2\Omega^2 + \hat{\beta}_\varphi\Omega^2 + \frac{1}{4}R^4\Omega^4) = const.}$$

and Poisson equations

$$\Delta\psi = 4\pi G\rho \tag{1a}$$

$$\Delta X = 4\pi G\rho\left(\varepsilon + 2v^2 - 2\psi + \frac{3P}{\rho}\right) \tag{1b}$$

$$\Delta P_1 = -4\pi G\rho(x - x_c) \tag{1c}$$

$$\Delta P_2 = -4\pi G\rho(y - y_c) \tag{1d}$$

where $H_i = \tilde{K}_i\rho^{\Gamma_i - 1}$, $\tilde{K}_i = K\frac{\Gamma_i}{\Gamma_i - 1}$, $\Omega = d\varphi/dt$, $R^2 = (x - x_c)^2 + (y - y_c)^2$, and $\hat{\beta}_\varphi$ is given by

$$\begin{aligned}\hat{\beta}_\varphi = & -[\frac{7}{2}((x - x_c)P_1 + (y - y_c)P_2) \\ & +\frac{1}{2}((x - x_c)^2P_{2,y} + (y - y_c)^2P_{1,x} - (x - x_c)(y - y_c)P_{1,y} - (x - x_c)(y - y_c)P_{2,x})]\end{aligned}$$

D.2 Post-Newtonian – Numerical method

Shibata, PRD 55 (1997) 6019

We have better split X as

$$X = X_0 + X_\Omega\Omega^2 \tag{2}$$

and separate (1b) into

$$\Delta X_0 = 4\pi G\rho\left(\varepsilon - 2\psi + \frac{3P}{\rho}\right) \tag{3a}$$

$$\Delta X_\Omega = 8\pi G\rho R^2 \tag{3b}$$

where we used $v = R\Omega$.

Boundary conditions are:

$$
\begin{aligned}
\psi &\to -\frac{G}{r}\int \rho d^3x + O(r^{-3}) && \text{(4a)}\\
X_0 &\to -\frac{G}{r}\int \rho\left(\varepsilon - 2\psi + \frac{3P}{\rho}\right) d^3x + O(r^{-3}) && \text{(4b)}\\
X_\Omega &\to -\frac{G}{r}\int 2\rho R^2 d^3x + O(r^{-3}) && \text{(4c)}\\
P_1 &\to \frac{G(x-x_c)}{r^3}\int \rho (x-x_c)^2 d^3x + O(r^{-4}) && \text{(4d)}\\
P_2 &\to \frac{G(y-y_c)}{r^3}\int \rho (y-y_c)^2 d^3x + O(r^{-4}) && \text{(4e)}
\end{aligned}
$$

Bernoulli equations (for both star 1 and 2):

$$
\begin{aligned}
& H_1|_{x1} - \frac{1}{2c^2}(H_1|_{x1})^2 + \psi|_{x_1} - \frac{1}{2}(x_1-x_c)^2\Omega^2 \\
-\frac{1}{c^2}\left(-X_0|_{x_1} - X_\Omega|_{x_1}\Omega^2 - 2(x_1-x_c)^2\Omega^2\psi|_{x_1} + \frac{1}{4}(x_1-x_c)^4\Omega^4 + \Omega^2\hat{\beta}_\varphi|_{x_1}\right) &= c_1 \\
& \psi|_{x_{01}} - \frac{1}{2}(x_{01}-x_c)^2\Omega^2 \\
-\frac{1}{c^2}\left(-X_0|_{x_{01}} - X_\Omega|_{x_{01}}\Omega^2 - 2(x_{01}-x_c)^2\Omega^2\psi|_{x_{01}} + \frac{1}{4}(x_{01}-x_c)^4\Omega^4 + \Omega^2\hat{\beta}_\varphi|_{x_{01}}\right) &= c_1
\end{aligned}
$$

By eliminating c_1, we get

$$
A\Omega^4 + B\Omega^2 + D = 0 \tag{5}
$$

where

$$
\begin{aligned}
A &= \frac{1}{4c^2}[(x_{01}-x_c)^2 + (x_1-x_c)^2](x_{01}+x_1-2x_c)(x_1-x_{01}) \\
B &= +\frac{1}{2}[(x_{01}-x_c)^2 - (x_1-x_c)^2] + \frac{1}{c^2}[X_\Omega|_{x_1} - X_\Omega|_{x_{01}}] \\
&\quad + \frac{1}{c^2}2[(x_1-x_c)^2\psi|_{x_1} - (x_{01}-x_c)^2\psi|_{x_{01}}] + \frac{1}{c^2}[\hat{\beta}_\varphi|_{x_{01}} - \hat{\beta}_\varphi|_{x_1}] \\
D &= H_1|_{x1} - \frac{1}{2c^2}(H_1|_{x1})^2 + \psi|_{x_1} - \psi|_{x_{01}} + \frac{1}{c^2}[X_0|_{x_1} - X_0|_{x_{01}}]
\end{aligned}
$$

From this equation, we get

$$
\Omega^2 = \frac{-B + \sqrt{B^2 - 4AD}}{2A} \quad \text{and} \quad c_1 = \ldots
$$

For x_c, We can get x_c also from comparing $\Omega^2_{star\ 1} - \Omega^2_{star\ 2} < O(\epsilon)$, by trial x_c.

The density update equation, then, given by (5)

$$\frac{1}{2c^2}H^2 - H + F = 0 \tag{6}$$

where

$$F = c_i - \psi + \frac{1}{2}R^2\Omega^2 + \frac{1}{c^2}\left[-X_0 - X_\Omega\Omega^2 - 2R^2\psi\Omega^2 + \hat{\beta}_\varphi\Omega^2 + \frac{1}{4}R^4\Omega^4\right]$$

From this equation, we get

$$H = \frac{1 \pm \sqrt{1 - 2F/c^2}}{1/c^2},$$

and if the EOS is polytropic,

$$\rho = \left[\frac{1}{\tilde{K}_i}H\right]^{1/(\Gamma_i - 1)}$$

Equilibrium configuration will be given from the converged ρ from iterating these procedures; say

$$\frac{\int(\rho^{(n)} - \rho^{(n-1)})^2 dV}{\int(\rho^{(n)})^2 dV} \ll 1 \tag{7}$$

E Newtonian Equilibrium Configuration (irrotational)

$\mathbf{\Omega} = (0, 0, \Omega)$	orbital angular velocity	constant, origin=CM
$\mathbf{u} = d\mathbf{r}/dt$	velocity in the inertial frame	
$\mathbf{v} = d\mathbf{r}/dt'$	velocity in the rotating frame	

$$\begin{aligned} \mathbf{u} &= \mathbf{v} + \mathbf{\Omega} \times \mathbf{r} \\ \frac{d}{dt}\mathbf{u} &= \frac{d}{dt'}\mathbf{u} + \mathbf{\Omega} \times \mathbf{u} = \frac{d}{dt'}\mathbf{v} + 2\mathbf{\Omega} \times \mathbf{v} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}), \end{aligned}$$

Euler equation (in a uniformly rotating frame)

$$\frac{\partial \mathbf{v}}{\partial t'} + (\mathbf{v} \cdot \nabla)\mathbf{v} + 2\mathbf{\Omega} \times \mathbf{v} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\nabla P - \nabla\psi. \tag{8}$$

assume stationarity in the rotating frame: $\partial/\partial t' \to 0$.

$$\nabla(\frac{1}{2}v^2 + h_N + \phi + \phi_c) + [(\nabla \times \mathbf{v}) + 2\mathbf{\Omega}] \times \mathbf{v} = 0. \tag{9}$$

or in terms of the inertial velocity:

$$\nabla(\frac{1}{2}u^2 - (\mathbf{\Omega} \times \mathbf{r}) \cdot \mathbf{u} + h_N + \phi) + (\nabla \times \mathbf{u}) \times (\mathbf{u} - \mathbf{\Omega} \times \mathbf{r}) = 0. \tag{10}$$

h_N	enthalpy per unit mass	$h_N = \int dP/\rho = u + P/\rho$
u	internal energy per unit mass	
ψ_c	centrifugal potential	$\psi_c = -\frac{1}{2}(\mathbf{\Omega} \times \mathbf{r})^2$

Continuity equation

$\frac{d\rho}{dt} + \nabla \cdot (\rho\mathbf{v}) = 0$, with $\partial/\partial t' \to 0$, becomes

$$\nabla \cdot \mathbf{v} = -\mathbf{v} \cdot \frac{\nabla\rho}{\rho}, \tag{11}$$

or

$$\nabla \cdot \mathbf{u} = -(\mathbf{u} - \mathbf{\Omega} \times \mathbf{r}) \cdot \frac{\nabla\rho}{\rho}. \tag{12}$$

E.1 corotating case

The corotating case corresponds to $\boxed{\mathbf{v} = 0}$.

- Euler eq. (9) is just the Bernoulli equation:

$$h_N + \psi + \psi_c = \text{const.}$$

- continuity eq. (11) is trivially satisfied.

E.2 irrotational case

The irrotational case corresponds to no vorticity in the inertial frame,

$$\boxed{\nabla \times \mathbf{u} = 0} \quad \text{implies} \quad \boxed{\mathbf{u} = \nabla\phi.}$$

- Euler eq. (10)

$$\frac{1}{2}(\nabla\phi)^2 - (\boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla\phi + h_N + \psi = \text{const}$$

- continuity eq. (12)

$$\nabla^2\phi = -(\nabla\phi - \boldsymbol{\Omega} \times \mathbf{r}) \cdot \frac{\nabla\rho}{\rho}.$$

  This must be solved subject to the BC at the surface $\rho = 0$.

$$(\nabla\phi - \boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla\rho|_{\text{surf}} = 0,$$

Alternatively,

- curl of $\mathbf{v}$: $\nabla \times \mathbf{v} = -2\boldsymbol{\Omega}$, is solved by $\mathbf{v} = -\boldsymbol{\Omega} \times \mathbf{r} + \nabla\phi$
- divergence of $\mathbf{v}$ is given by the continuity eq. (11)

$$\nabla \cdot \mathbf{v} = -\mathbf{v} \cdot \frac{\nabla\rho}{\rho},$$

- The Bernoulli integral is

$$\frac{1}{2}v^2 + h_N + \psi + \psi_c = \text{const.}$$