# Boundary Conditions for momentum constraints

Conformally flat case: Bowen's multipole expansion

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While during working in thorn_IVP, I began to think that we should treat boundary conditions for momentum constraint equations more carefully. This is a note of Bowen's expression [GRG14(1982)1183], which was also applied to Baumgarte *et al*, Wilson *et al* directly.

## 1 What Bowen wrote

Under the constant-mean-curvature ($\nabla^i K = 0$) condition and conformally flatness, longitudinal part of the momentum constraints become

$$\Delta W^i + \frac{1}{3}\nabla^i \nabla_j W^j = 8\pi S^i. \tag{1}$$

By introducing a decomposition of $W^i$ into vector and gradient terms

$$W^i = V^i - \frac{1}{4}\nabla^i \theta, \tag{2}$$

we can express (1) as

$$\Delta V^i = 8\pi S^i, \tag{3a}$$
$$\Delta \theta = \nabla_i V^i, \tag{3b}$$

and the longitudinal part of $K_{ij}$ as

$$K_L^{ij} = \nabla^i V^j + \nabla^j V^i - \frac{1}{2}\delta^{ij}\nabla_k V^k - \frac{1}{2}\nabla^i \nabla^j \theta. \tag{4}$$

If the source is of finite extent, then the solutions in the outside vacuum region are expressed as

$$V^i = -2\sum_{l=0}^{\infty} Q^{ij_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}}, \tag{5}$$

$$\begin{aligned}\theta = & -\sum_{l=1}^{\infty} Q^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l-1}} \\ & +\sum_{l=0}^{\infty}\frac{2(l+1)}{(2l+1)(2l+3)} Q_k^{kj_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}} \\ & +\sum_{l=1}^{\infty}\frac{2l-1}{2l+1} M^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l+1}}\end{aligned} \tag{6}$$

where $n^i = x^i r^{-1}$ in the Cartesian cordinate, the multipoles $Q$ and $M$ are defined as

$$Q^{ij_1\cdots j_l} \equiv \frac{(2l-1)!!}{l!}\int S^i(\mathbf{r}) x^{\{j_1}x^{j_2}\cdots x^{j_l\}} dV, \tag{7a}$$
$$M^{ij_1\cdots j_l} \equiv \frac{(2l-1)!!}{l!}\int r^2 S^i(\mathbf{r}) x^{\{j_1}x^{j_2}\cdots x^{j_l\}} dV, \tag{7b}$$

and where brackets denote the completely symmetric trace-free part

$$Z^{\{ij_1\cdots j_l\}} = Z^{(ij_1\cdots j_l)} - \frac{l}{2l+1} Z_k^{k(j_1\cdots j_{l-1}}\delta^{j_l i)} \tag{8}$$

## 2 Which term is the lowest order? (my calculations)

More direct expressions are:

$$
\begin{aligned}
V^i &= -\frac{2}{r}\int S^i dV - \frac{2}{r^3}x^j \int S^i x^j dV - \frac{3}{r^5}x^j x^k \int S^i x^{\{j}x^{k\}}dV \\
&\quad -\frac{5}{r^7}x^j x^k x^l \int S^i x^{\{j}x^k x^{l\}}dV - \cdots \qquad (9a)\\
\theta &= -\frac{x^i}{r}\int S^i dV - \frac{2}{3}\frac{1}{r}\int S^k x^k dV - \frac{x^i x^j}{r^3}\int S^{\{i}x^{j\}}dV - \cdots \qquad (9b)
\end{aligned}
$$

### 2.1 Inspiral Binary of equal mass stars

For the case of inspiral binary (circular motion) of equal mass, which locate on $x$-axis and the orbital rotating vector aligns to $z$-axis, the lowest falling-off behavior is

$$
\begin{aligned}
V^x &= -2\frac{y}{r^3}\int S^x y dV \qquad (10a)\\
V^y &= -2\frac{x}{r^3}\int S^y x dV \qquad (10b)\\
V^z &= -5\frac{xyz}{r^7}\int S^z xyz dV \qquad (10c)\\
\theta &= -\frac{1}{2}\frac{xy}{r^3}[\int S^x y dV + \int S^y x dV] \qquad (10d)
\end{aligned}
$$

### 2.2 Head-on situation of equal mass stars

For the case of head-on two equal mass stars, which move along $x$-axis and with the center of mass is at the origin, then the lowest falling-off behavior is

$$
\begin{aligned}
V^x &= -2\frac{x}{r^3}\int S^x x dV \qquad (11a)\\
V^y &= -5\frac{xyz}{r^7}\int S^y xyz dV \qquad (11b)\\
V^z &= -5\frac{xyz}{r^7}\int S^z xyz dV \qquad (11c)\\
\theta &= -\frac{2}{3}\frac{1}{r}\int S^k x^k dV \qquad (11d)
\end{aligned}
$$

## 3 How we treat these?

I am now coding (9a) and (9b) for general situation as a part of `thorn_outerBC`. It's ready.

## 4 General Case

In general case, longitudinal part of the momentum constraints become

$$\Delta W^i + \frac{1}{3}\nabla^i\nabla_j W^j + R^i{}_j W^j = 8\pi J^i + \frac{2}{3}\psi^6\nabla^i K \quad (\equiv 8\pi S^i). \tag{12}$$

By introducing a decomposition of $W^i$ into vector and gradient terms

$$W^i = V^i - n\nabla^i\theta, \tag{13}$$

(12) becomes

$$\Delta V^i + R^i{}_j W^j + \frac{1}{3}\nabla^i[\nabla_j V^j - 4n\Delta\theta] = 8\pi S^i, \tag{14}$$

which are devided into

$$\begin{aligned} \Delta V^i &= 8\pi S^i - R^i{}_j(V^j - n\nabla^j\theta) \quad (\equiv 8\pi\tilde{S}^i), & \text{(15a)} \\ \Delta\theta &= \frac{1}{4n}\nabla_i V^i. & \text{(15b)} \end{aligned}$$

These equations are not decoupled, but we may apply the similar boundary conditions of Bowen's conformally flat case. If the source is of finite extent, then the solutions in the outside vacuum region are expressed as

$$V^i = -2\sum_{l=0}^{\infty} Q^{ij_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}}, \tag{16}$$

$$\begin{aligned} \theta &= -\sum_{l=1}^{\infty} Q^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l-1}} + \sum_{l=0}^{\infty}\frac{2(l+1)}{(2l+1)(2l+3)} Q_k^{kj_1\cdots j_l} n_{j_1}\cdots n_{j_l}\frac{1}{r^{l+1}} \\ &\quad + \sum_{l=1}^{\infty}\frac{2l-1}{2l+1} M^{\{ij_1\cdots j_{l-1}\}} n_i n_{j_1}\cdots n_{j_{l-1}}\frac{1}{r^{l+1}} \end{aligned} \tag{17}$$

where $n^i = x^i r^{-1}$ in the Cartesian cordinate, the multipoles $Q$ and $M$ are defined as

$$\begin{aligned} Q^{ij_1\cdots j_l} &\equiv \frac{(2l-1)!!}{l!}\int \tilde{S}^i(\mathbf{r}) x^{\{j_1} x^{j_2}\cdots x^{j_l\}} dV, & \text{(18a)} \\ M^{ij_1\cdots j_l} &\equiv \frac{(2l-1)!!}{l!}\int r^2\tilde{S}^i(\mathbf{r}) x^{\{j_1} x^{j_2}\cdots x^{j_l\}} dV, & \text{(18b)} \end{aligned}$$

and where brackets denote the completely symmetric trace-free part

$$Z^{\{ij_1\cdots j_l\}} = Z^{(ij_1\cdots j_l)} - \frac{l}{2l+1} Z_k^{k(j_1\cdots j_{l-1}}\delta^{j_l i)} \tag{19}$$

This routine is also coded up in `thorn_IVP`.